Flash Cards

to accompany

A First Course in Linear Algebra

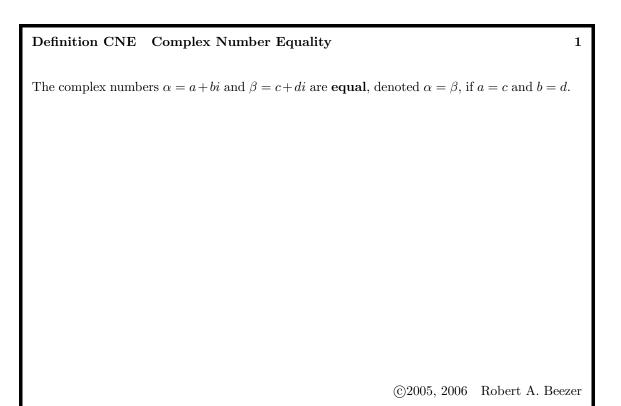
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Version 2.30

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The most recent version of this work can always be found at http://linear.ups.edu.



Definition CNA Complex Number Addition

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The **sum** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is (a + c) + (b + d)i.



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The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is (ac - bd) + (ad + bc)i.

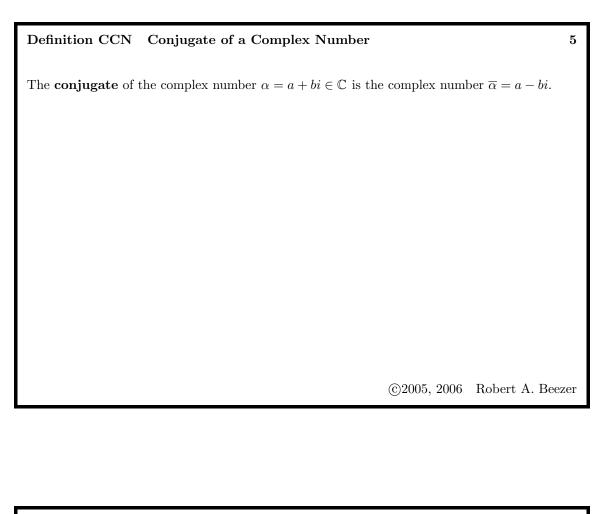
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Theorem PCNA Properties of Complex Number Arithmetic

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The operations of addition and multiplication of complex numbers have the following properties.

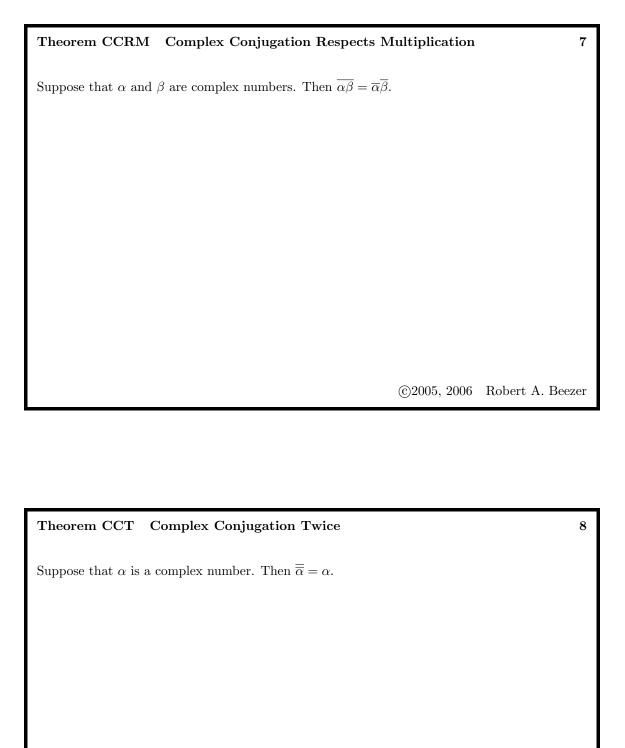
- ACCN Additive Closure, Complex Numbers If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers For any $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers For any $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- **ZCN Zero, Complex Numbers** There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$
- MICN Multiplicative Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

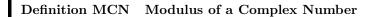


Theorem CCRA Complex Conjugation Respects Addition

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Suppose that α and β are complex numbers. Then $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.





q

The **modulus** of the complex number $\alpha = a + bi \in \mathbb{C}$, is the nonnegative real number

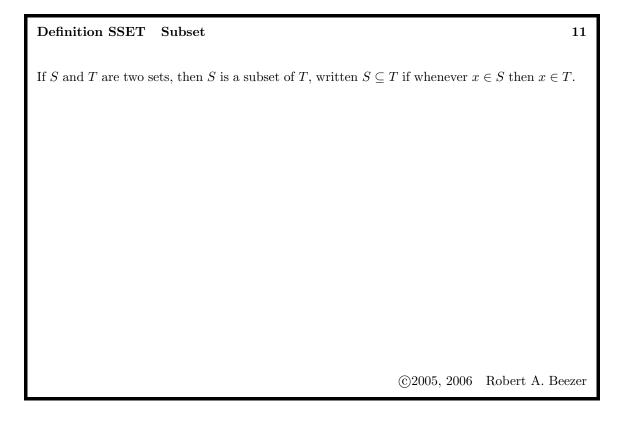
$$|\alpha| = \sqrt{\alpha \overline{\alpha}} = \sqrt{a^2 + b^2}.$$

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Definition SET Set

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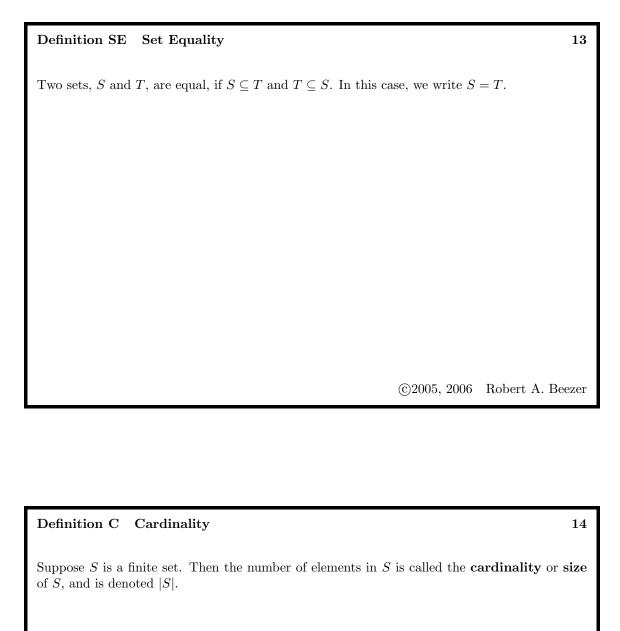
A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.

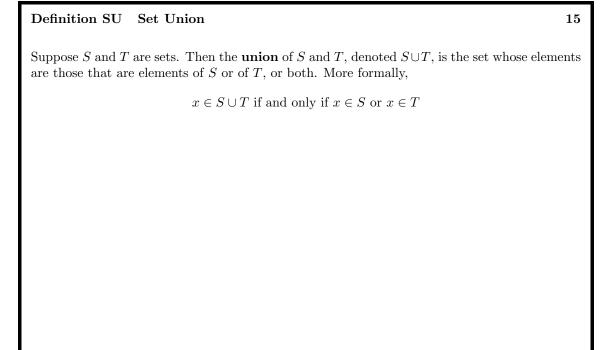


Definition ES Empty Set

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The empty set is the set with no elements. Its is denoted by \emptyset .





Definition SI Set Intersection

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Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$ if and only if $x \in S$ and $x \in T$

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Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$

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Definition SLE System of Linear Equations

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A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

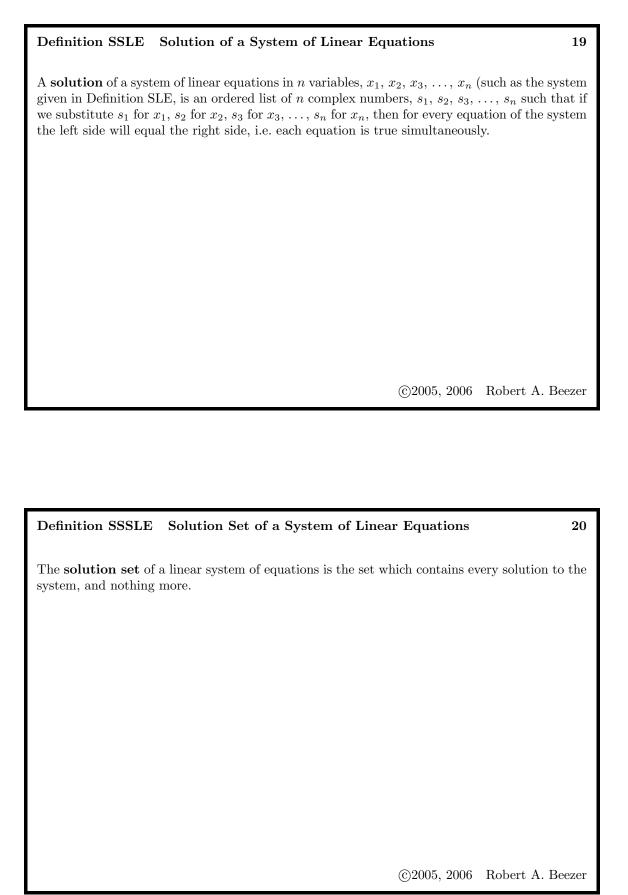
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

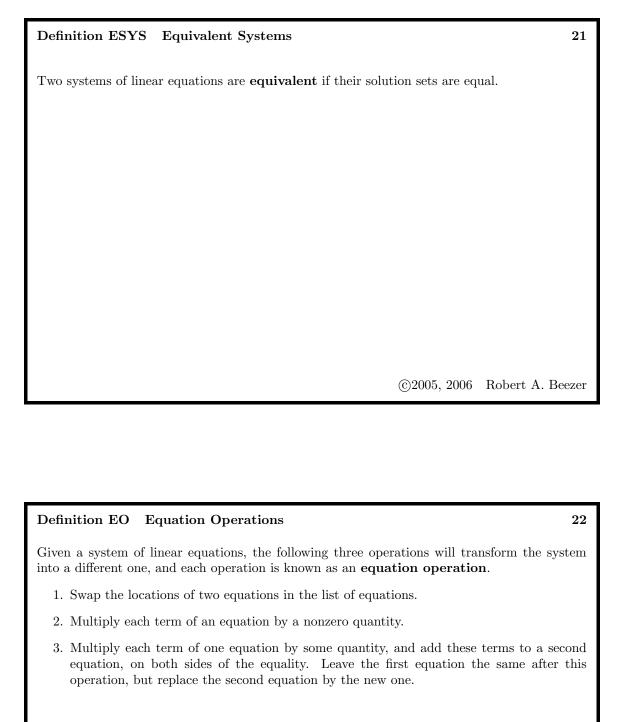
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

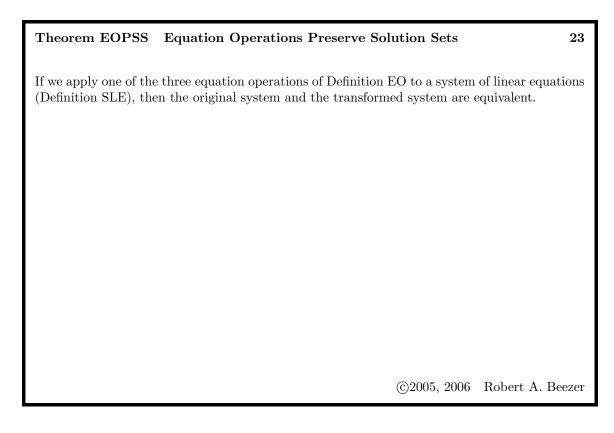
$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .







Definition M Matrix 24

An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, \ldots) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the **entry** or **component** that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$.

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Definition ZCV Zero Column Vector

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The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

Definition CM Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SOLV Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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Definition MRLS Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and \mathbf{b} is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

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Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants \mathbf{b} . Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (number n+1) is the column vector \mathbf{b} . This matrix will be written as $[A \mid \mathbf{b}]$.

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Definition RO Row Operations

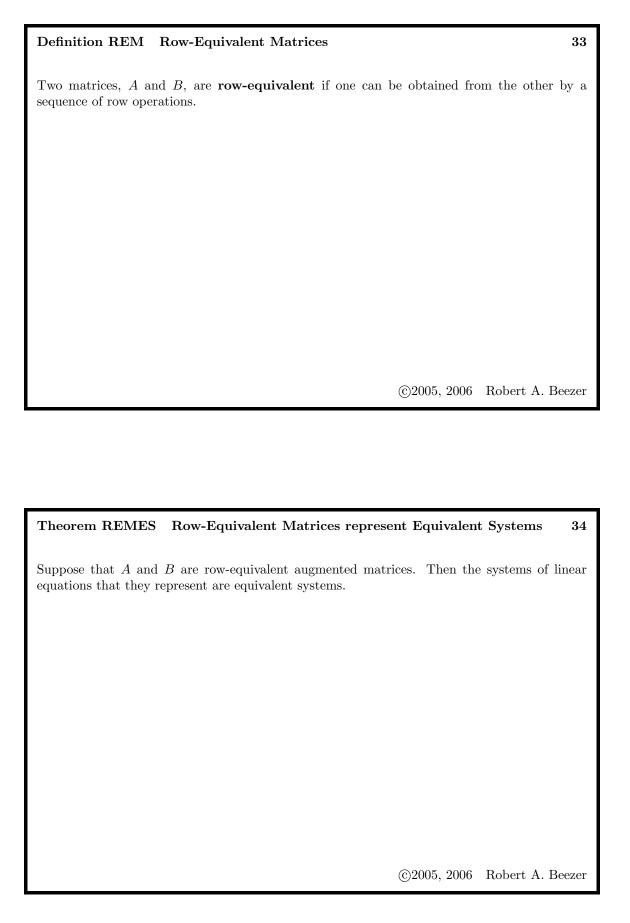
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The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.



Definition RREF Reduced Row-Echelon Form

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A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

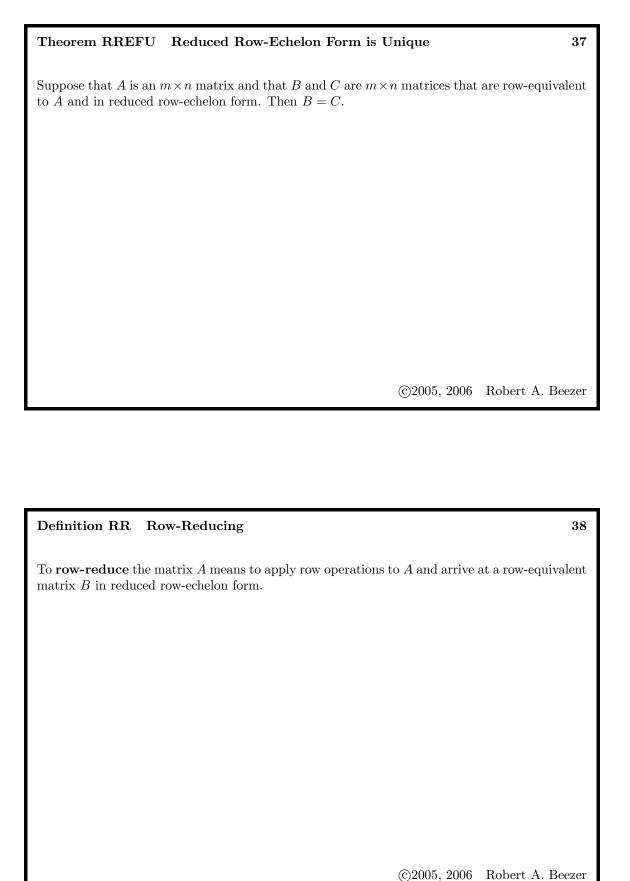
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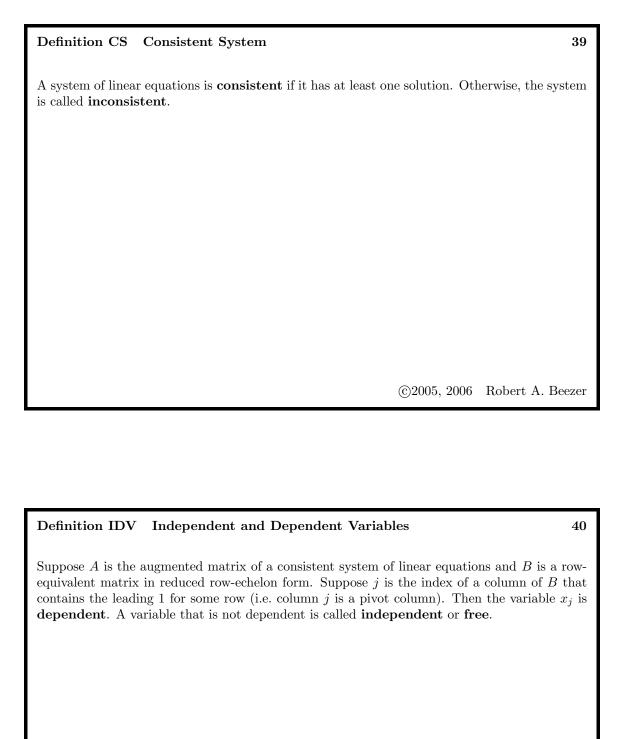
Theorem REMEF Row-Equivalent Matrix in Echelon Form

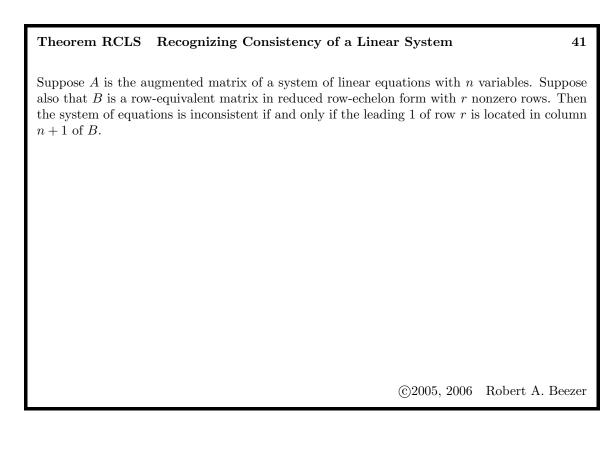
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Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.



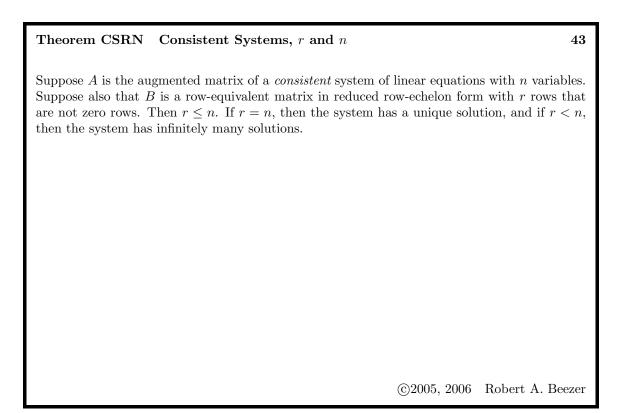




Theorem ISRN $\,$ Inconsistent Systems, r and n

 $\mathbf{42}$

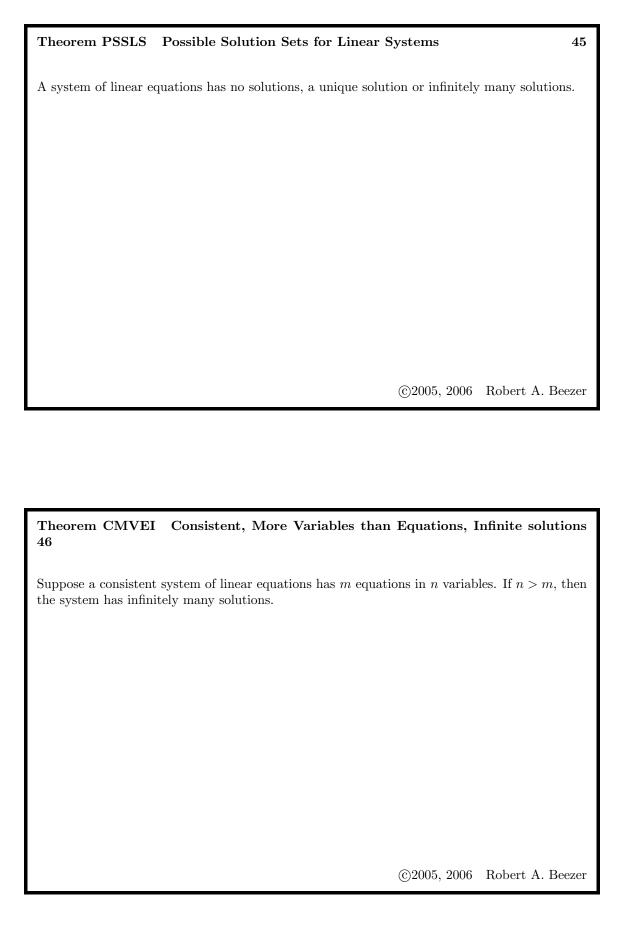
Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

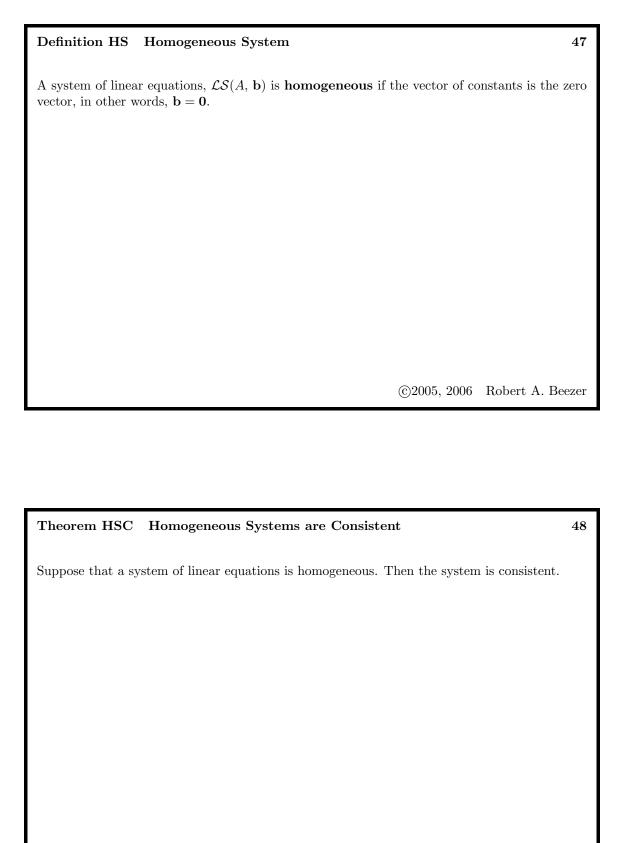


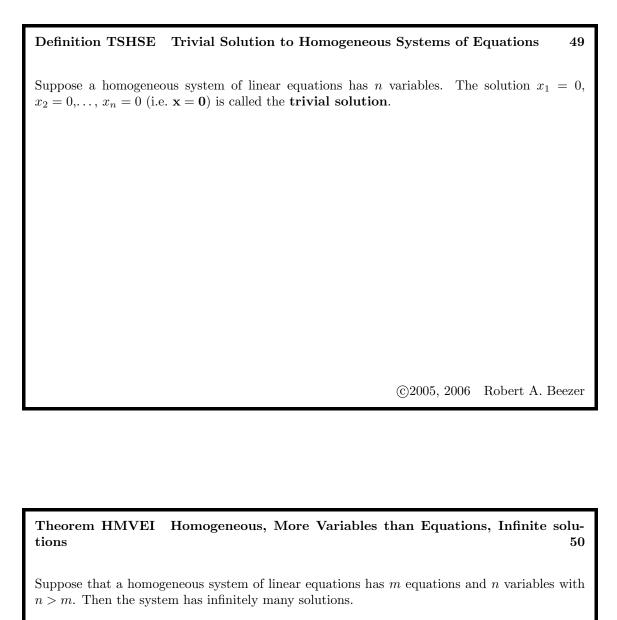
Theorem FVCS Free Variables for Consistent Systems

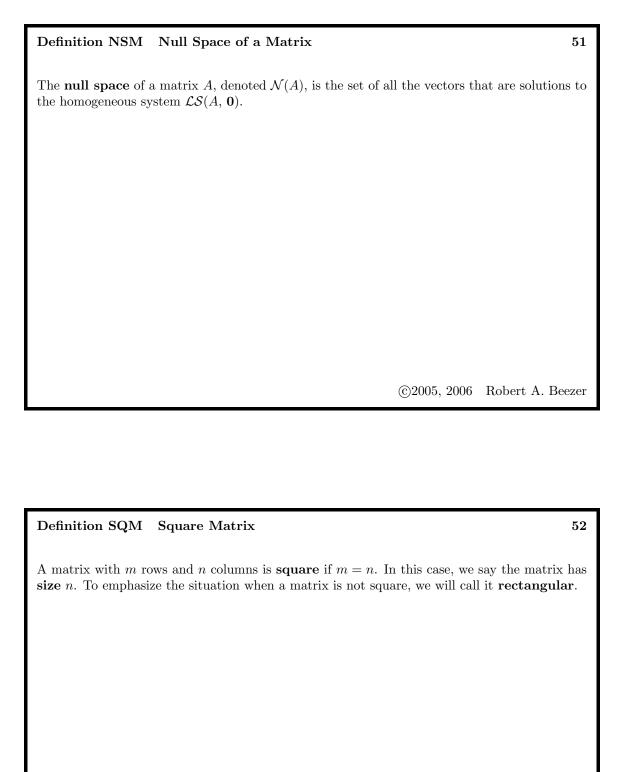
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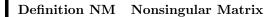
Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.











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Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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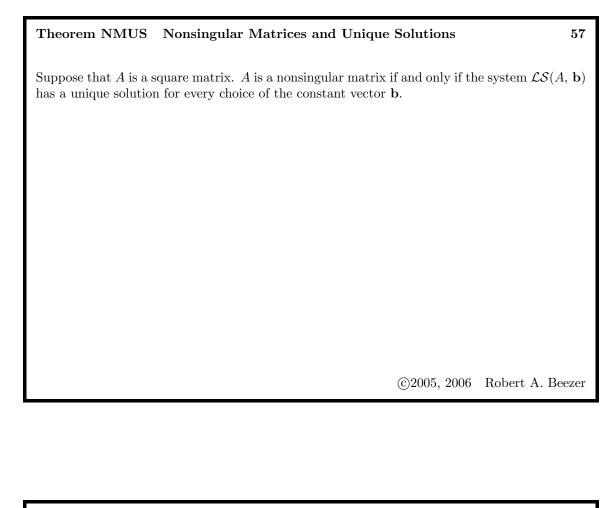
Definition IM Identity Matrix

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The $m \times m$ identity matrix, I_m , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \le i, j \le m$$

Theorem NMRRI	Nonsingular Matrices Row Reduce to the Identity matrix 55
	square matrix and B is a row-equivalent matrix in reduced row-echelon
form. Then A is nons	ingular if and only if B is the identity matrix.
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Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 56
Suppose that A is a s	square matrix. Then A is nonsingular if and only if the null space of A ,
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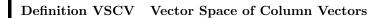


Theorem NME1 Nonsingular Matrix Equivalences, Round 1

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .



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The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} .

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Definition CVE Column Vector Equality

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Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Then \mathbf{u} and \mathbf{v} are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$\left[\mathbf{u}
ight]_i = \left[\mathbf{v}
ight]_i$$

$$1 \leq i \leq m$$

Definition CVA Column Vector Addition

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Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. The **sum** of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \le i \le m$$

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Definition CVSM Column Vector Scalar Multiplication

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Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the scalar multiple of \mathbf{u} by α is the vector $\alpha \mathbf{u}$ defined by

$$[\alpha \mathbf{u}]_i = \alpha \left[\mathbf{u} \right]_i$$

$$1 \leq i \leq m$$

Theorem VSPCV Vector Space Properties of Column Vectors

Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **ZC Zero Vector, Column Vectors** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- \bullet OC One Column Vectors If $u \in \mathbb{C}^m$ then 1u = u

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Definition LCCV Linear Combination of Column Vectors

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Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear** combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$$

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if \mathbf{b} equals the linear combination of the columns of A formed with the entries of \mathbf{x} ,

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

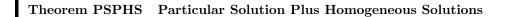
66

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$\begin{aligned} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} \end{aligned}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} | \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$



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Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then **y** is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

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Definition SSCV Span of a Set of Column Vectors

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Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

Theorem SSNS Spanning Sets for Null Spaces

69

70

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle$$

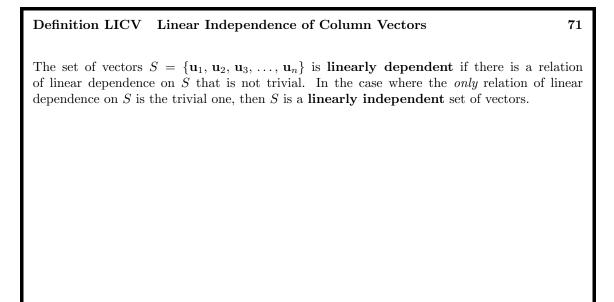
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Definition RLDCV Relation of Linear Dependence for Column Vectors

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

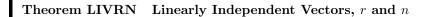
is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.



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Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.



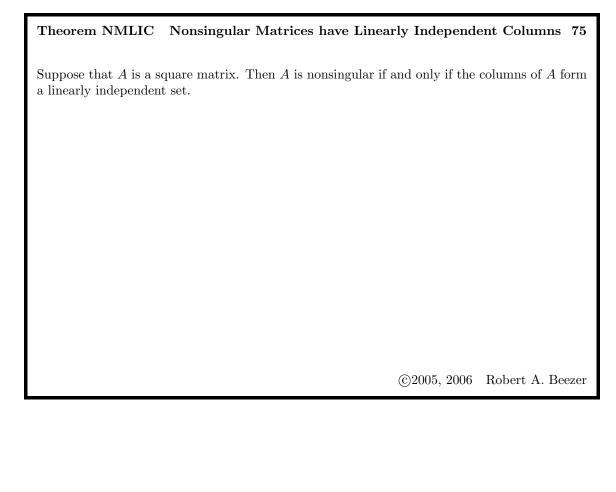
74

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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${\bf Theorem~MVSLD~~More~Vectors~than~Size~implies~Linear~Dependence}$

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that n > m. Then S is a linearly dependent set.



Theorem NME2 Nonsingular Matrix Equivalences, Round 2

76

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces

77

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

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Theorem DLDS Dependency in Linearly Dependent Sets

78

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

Theorem BS Basis of a Span

79

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

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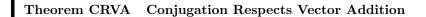
Definition CCCV Complex Conjugate of a Column Vector

80

Suppose that **u** is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \le i \le m$$



Suppose ${\bf x}$ and ${\bf y}$ are two vectors from $\mathbb{C}^m.$ Then

$$\overline{\mathbf{x}+\mathbf{y}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$$

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Theorem CRSM Conjugation Respects Vector Scalar Multiplication

82

Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP Inner Product

83

Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[\mathbf{u} \right]_1 \overline{\left[\mathbf{v} \right]_1} + \left[\mathbf{u} \right]_2 \overline{\left[\mathbf{v} \right]_2} + \left[\mathbf{u} \right]_3 \overline{\left[\mathbf{v} \right]_3} + \dots + \left[\mathbf{u} \right]_m \overline{\left[\mathbf{v} \right]_m} = \sum_{i=1}^m \left[\mathbf{u} \right]_i \overline{\left[\mathbf{v} \right]_i}$$

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Theorem IPVA Inner Product and Vector Addition

84

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1.
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

2.
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

	Theorem IPSM	Inner Product	and Scalar	Multiplication
--	--------------	---------------	------------	----------------

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

- 1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- 2. $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$

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Theorem IPAC Inner Product is Anti-Commutative

86

Suppose that **u** and **v** are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

Definition NV Norm of a Vector

87

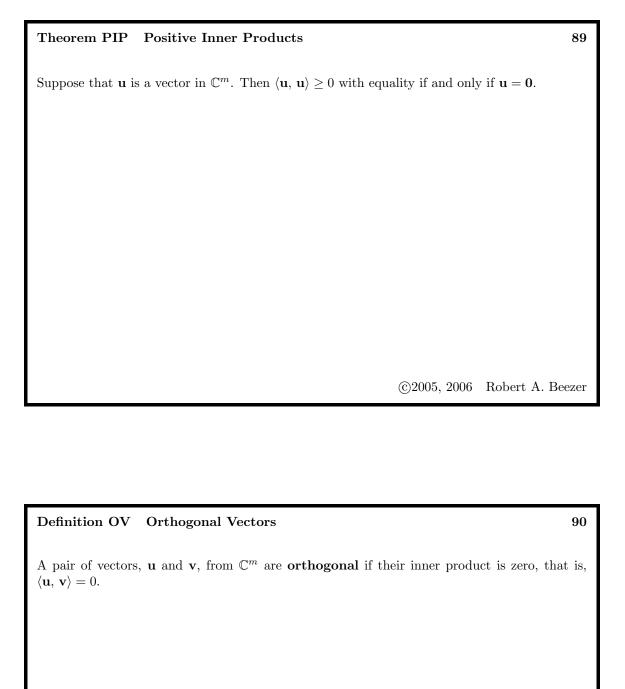
The \bf{norm} of the vector \bf{u} is the scalar quantity in $\mathbb C$

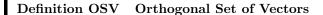
$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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88

Suppose that **u** is a vector in \mathbb{C}^m . Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.





Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then S is an **orthogonal** set if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

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Definition SUV Standard Unit Vectors

92

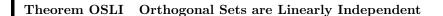
Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$\left[\mathbf{e}_{j}\right]_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .



Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

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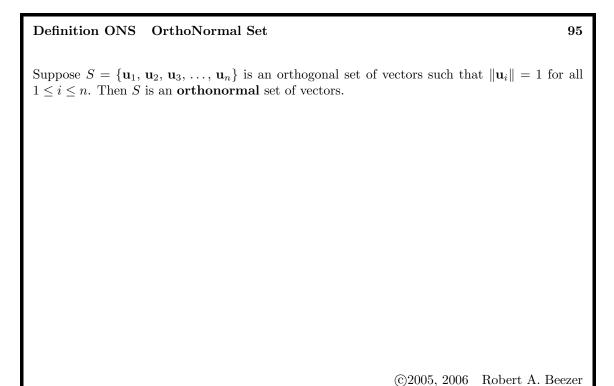
Theorem GSP Gram-Schmidt Procedure

94

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors \mathbf{u}_i , $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

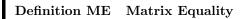
Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.



Definition VSM Vector Space of $m \times n$ Matrices

96

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.



The $m \times n$ matrices A and B are **equal**, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, 1 < j < n.

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Definition MA Matrix Addition

98

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} 1 \le i \le m, \ 1 \le j \le n$$

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Theorem VSPM Vector Space Properties of Matrices

100

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- **ZM Zero Vector, Matrices** There is a matrix, \mathcal{O} , called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One Matrices If $A \in M_{\text{max}}$ then 1A A



The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

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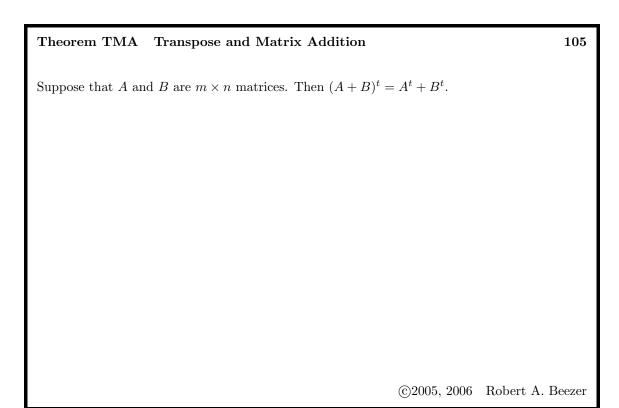
Definition TM Transpose of a Matrix

102

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix		103
The matrix A is symmetric if $A = A^t$.		
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Theorem SMS Symmetric Matrices are Square		104
Suppose that A is a suppose tria matrix. Then A is square		101



Theorem TMSM Transpose and Matrix Scalar Multiplication

106

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.



Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

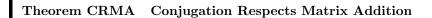
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Definition CCM Complex Conjugate of a Matrix

108

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written \overline{A} is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$



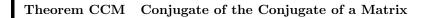
110

Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

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Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.



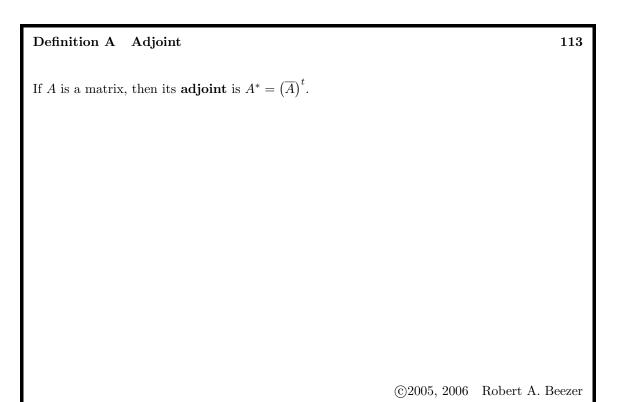
Suppose that A is an $m \times n$ matrix. Then $\overline{\overline{(A)}} = A$.

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Theorem MCT Matrix Conjugation and Transposes

112

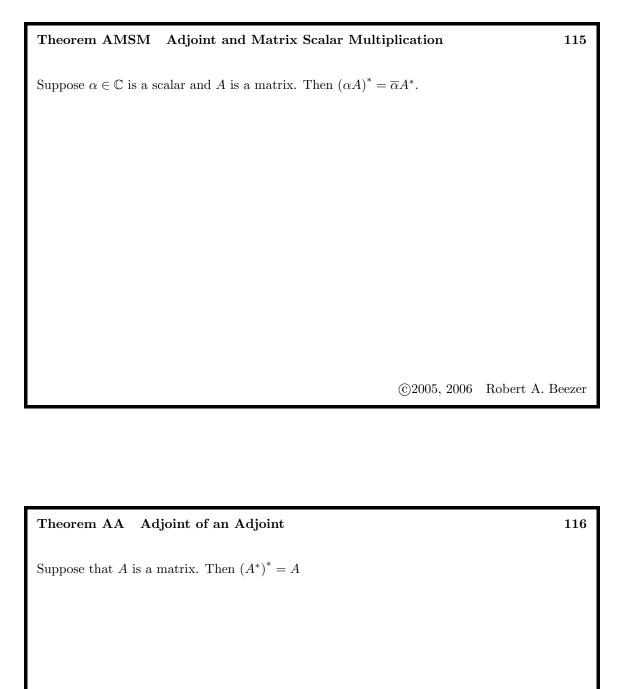
Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.



Theorem AMA Adjoint and Matrix Addition

114

Suppose A and B are matrices of the same size. Then $(A + B)^* = A^* + B^*$.





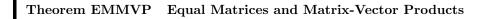
Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

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Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 118

The set of solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ equals the set of solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.



Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then A = B.

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Definition MM Matrix Multiplication

120

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

Theorem EMP **Entries of Matrix Products**

121

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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Theorem MMZM Matrix Multiplication and the Zero Matrix

122

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$ 2. $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

Theorem MMIM Matrix Multiplication and Identity Matrix

123

124

Suppose A is an $m \times n$ matrix. Then

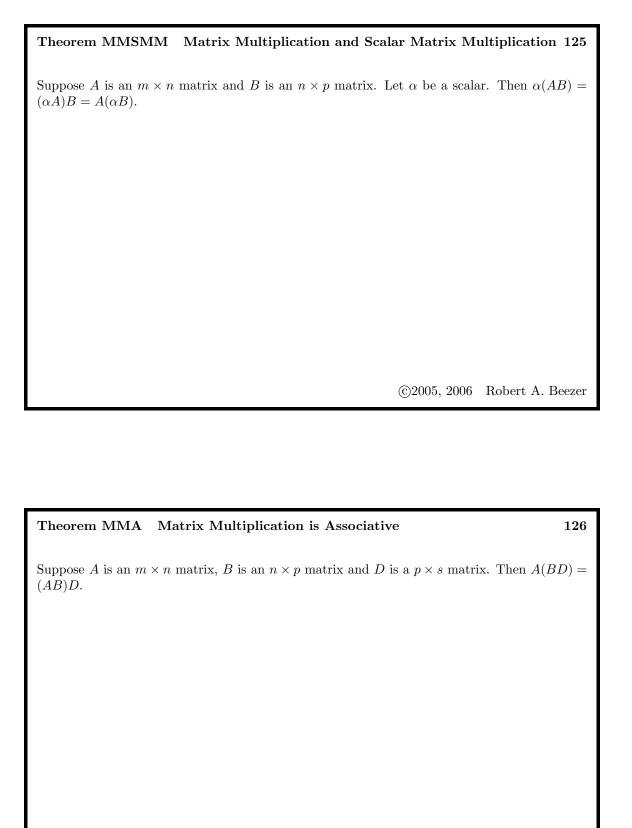
- 1. $AI_n = A$
- $2. \quad I_m A = A$

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Theorem MMDAA Matrix Multiplication Distributes Across Addition

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

- 1. A(B+C) = AB + AC
- $2. \quad (B+C)D = BD + CD$





128

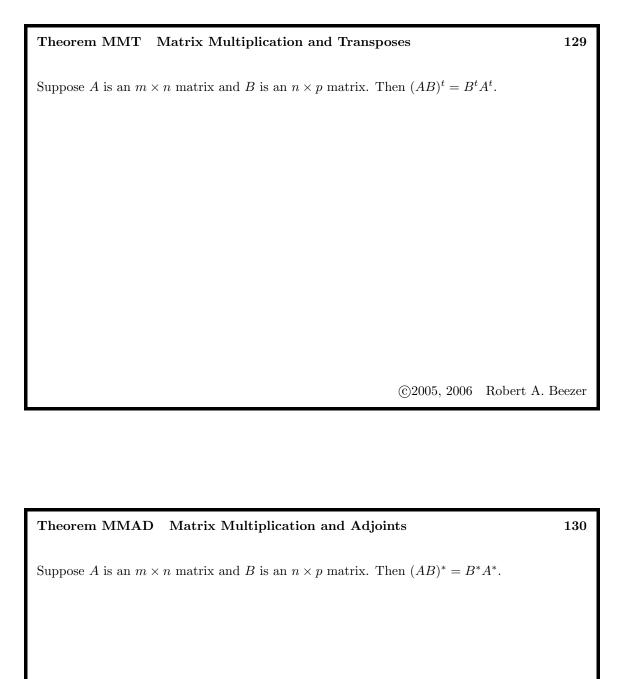
If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

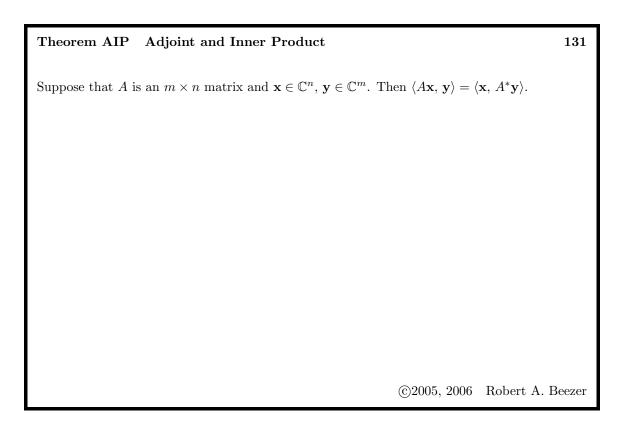
$$\langle \mathbf{u}, \, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

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Theorem MMCC Matrix Multiplication and Complex Conjugation

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A} \overline{B}$.

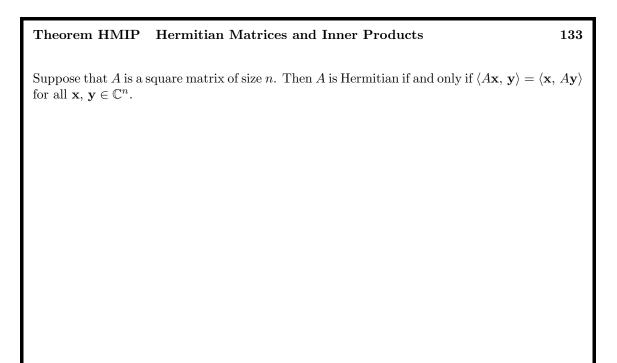




Definition HM Hermitian Matrix

132

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = A^*$.



Definition MI Matrix Inverse

134

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

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Theorem TTMI Two-by-Two Matrix Inverse

135

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Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

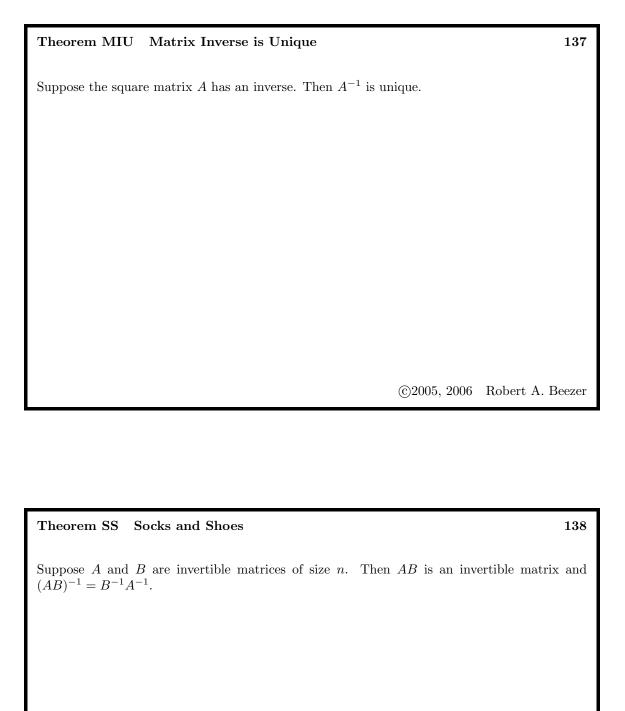
Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

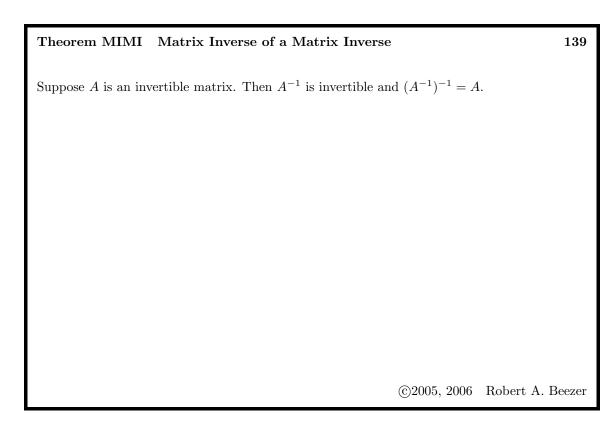
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Theorem CINM Computing the Inverse of a Nonsingular Matrix

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.





Theorem MIT Matrix Inverse of a Transpose

140

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.



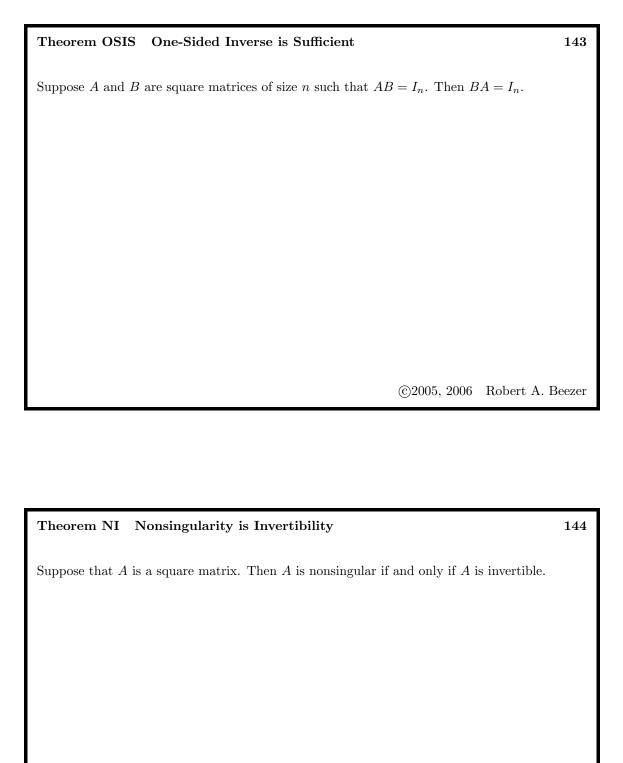
Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

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Theorem NPNT Nonsingular Product has Nonsingular Terms

142

Suppose that A and B are square matrices of size n. The product AB is nonsingular if and only if A and B are both nonsingular.



Theorem NME3 Nonsingular Matrix Equivalences, Round 3

145

Suppose that A is a square matrix of size n. The following are equivalent.

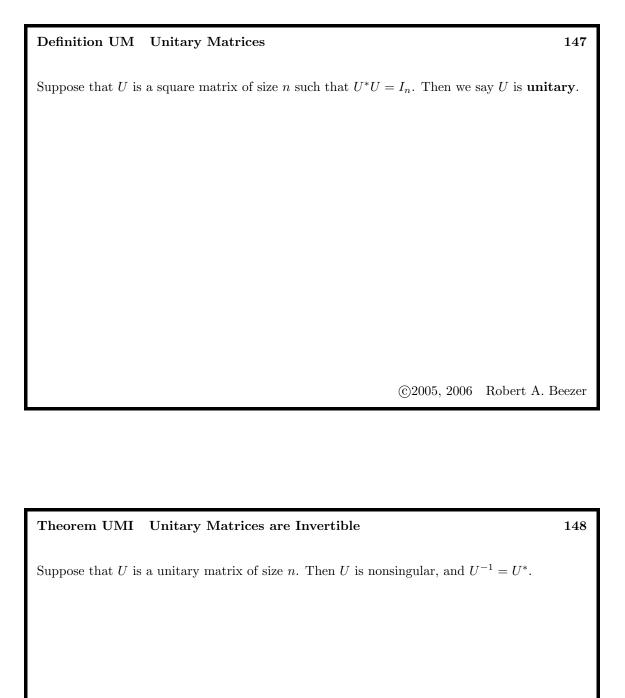
- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

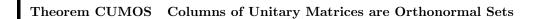
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Theorem SNCM Solution with Nonsingular Coefficient Matrix

146

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.





Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.

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Theorem UMPIP Unitary Matrices Preserve Inner Products

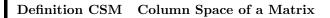
150

Suppose that U is a unitary matrix of size n and \mathbf{u} and \mathbf{v} are two vectors from \mathbb{C}^n . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

and

$$||U\mathbf{v}|| = ||\mathbf{v}||$$



Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A.

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS Column Spaces and Consistent Systems

152

Suppose A is an $m \times n$ matrix and **b** is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Theorem BCS Basis of the Column Space

153

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $C(A) = \langle T \rangle$.

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Theorem CSNM Column Space of a Nonsingular Matrix

154

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Theorem NME4 Nonsingular Matrix Equivalences, Round 4

155

Suppose that A is a square matrix of size n. The following are equivalent.

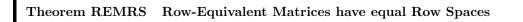
- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

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Definition RSM Row Space of a Matrix

156

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.



Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

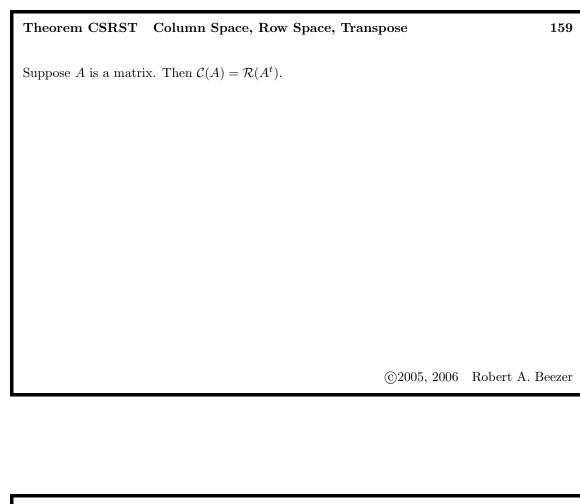
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Theorem BRS Basis for the Row Space

158

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.



Definition LNS Left Null Space

160

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Suppose A is an $m \times n$ matrix. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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Theorem PEEF Properties of Extended Echelon Form

162

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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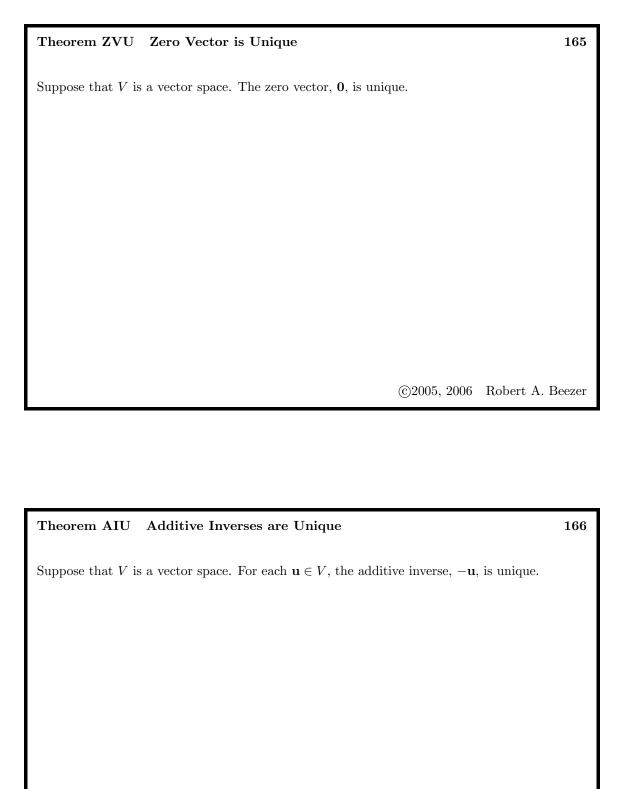
Definition VS Vector Space

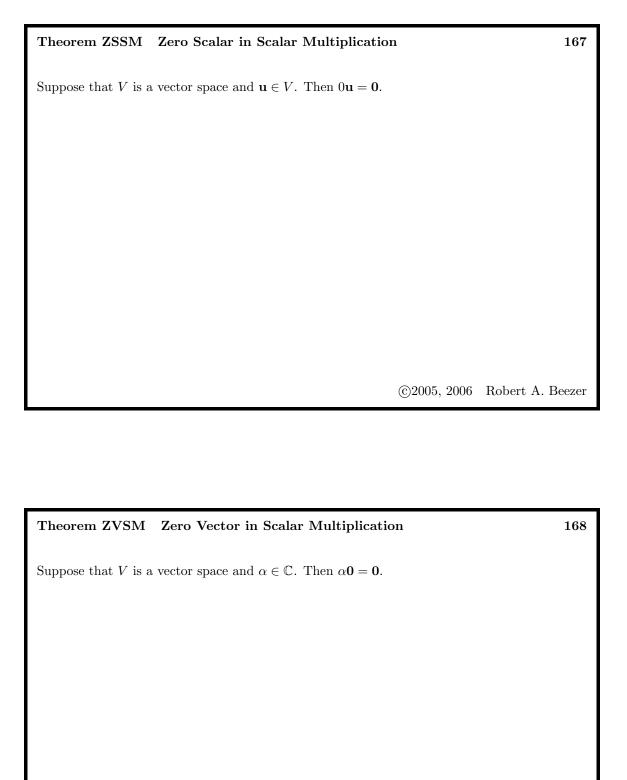
164

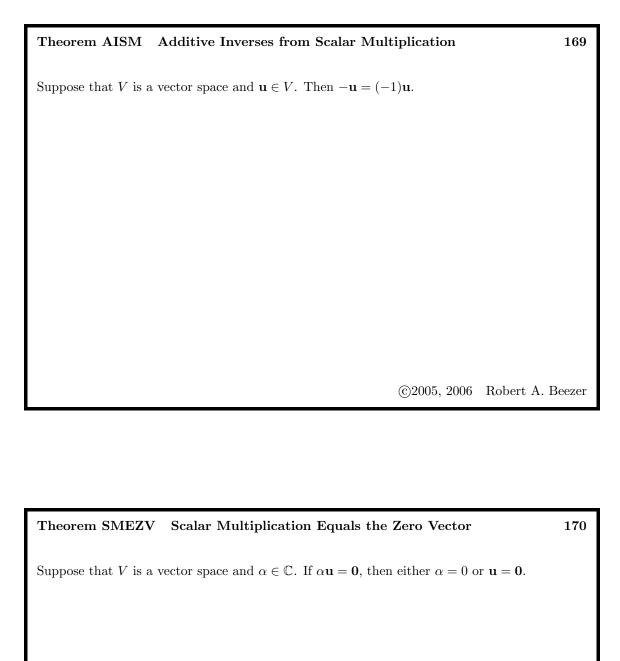
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** over \mathbb{C} if the following ten properties hold.

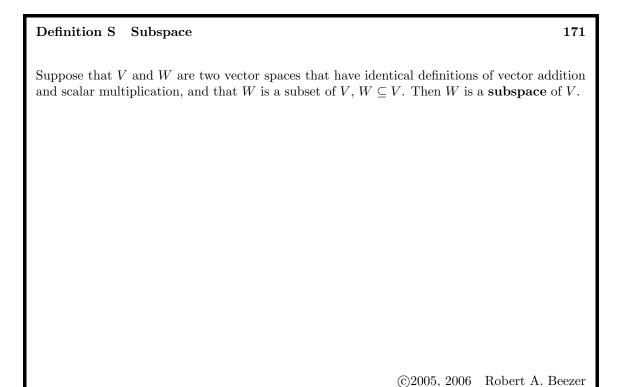
- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- One If $u \in V$ then 1u = u

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.







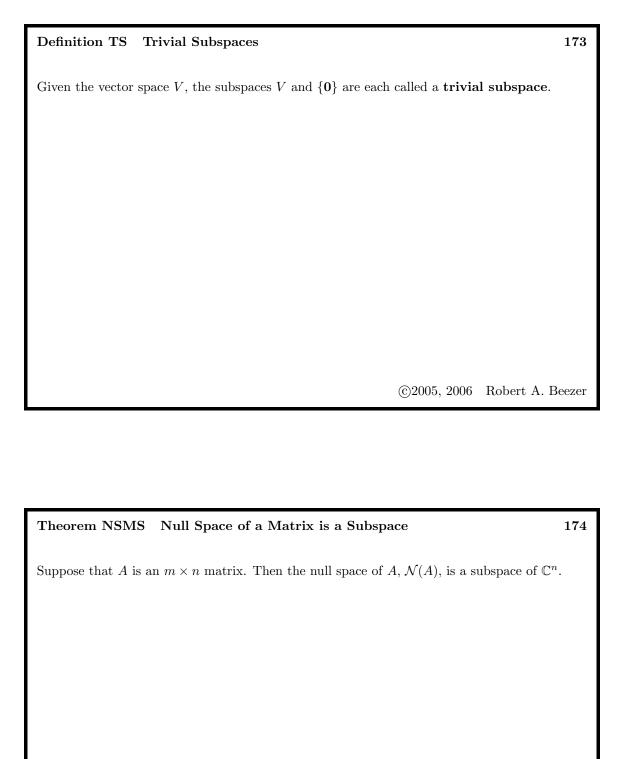


Theorem TSS Testing Subsets for Subspaces

172

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.



Definition LC Linear Combination

175

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

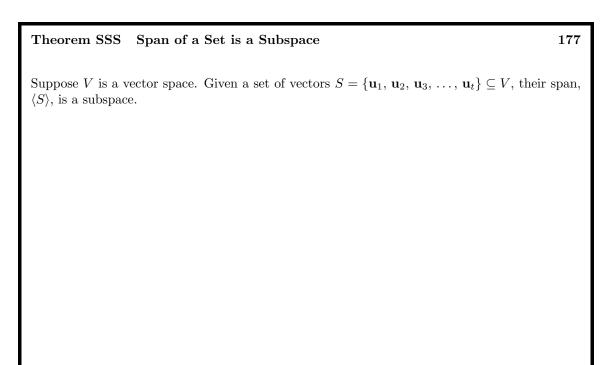
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Definition SS Span of a Set

176

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t | \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$



Theorem CSMS Column Space of a Matrix is a Subspace

178

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .

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Theorem RSMS Row Space of a Matrix is a Subspace	179
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .	
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Theorem LNSMS Left Null Space of a Matrix is a Subspace	180
Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .	

Definition RLD Relation of Linear Dependence

181

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

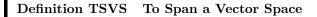
is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is a **trivial relation of linear dependence** on S.

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Definition LI Linear Independence

182

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ from V is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.



Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

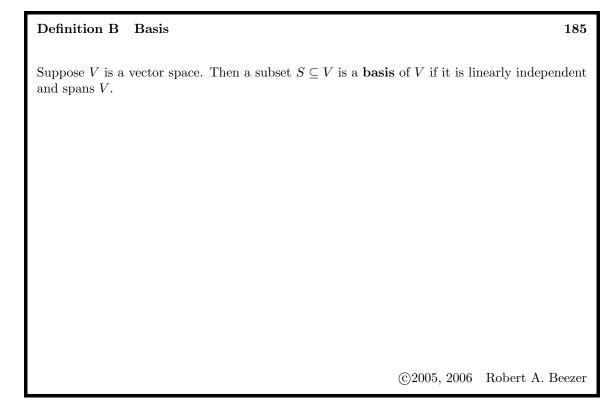
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Theorem VRRB Vector Representation Relative to a Basis

184

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let \mathbf{w} be any vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \dots, a_m$ such that

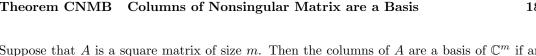
$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$



Theorem SUVB Standard Unit Vectors are a Basis

186

The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i | 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .



Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

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Theorem NME5 Nonsingular Matrix Equivalences, Round 5

188

187

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

Theorem CNMB

- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

Theorem COB Coordinates and Orthonormal Bases

189

Suppose that $B = \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$

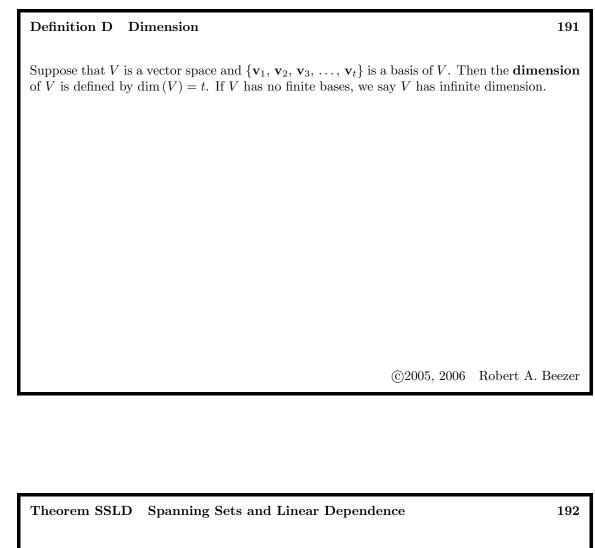
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190

Let A be an $n \times n$ matrix and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ be an orthonormal basis of \mathbb{C}^n . Define

$$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$$

Then A is a unitary matrix if and only if C is an orthonormal basis of \mathbb{C}^n .



Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of t+1 or more vectors from V is linearly dependent.

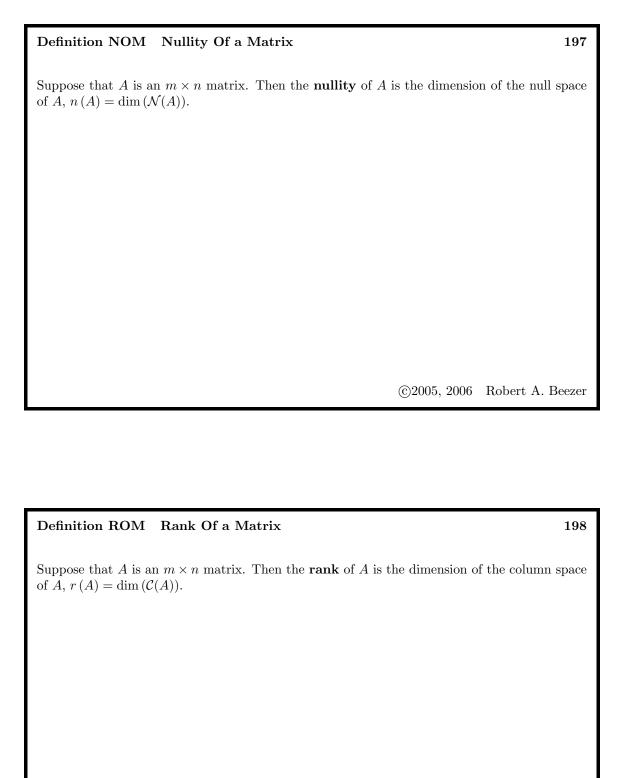
Theorem BIS Bases have Identical Sizes	193
Suppose that V is a vector space with a finite basis B and a second basis C have the same size.	C. Then B and C
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Theorem DCM Dimension of \mathbb{C}^m	194
The dimension of \mathbb{C}^m (Example VSCV) is m .	

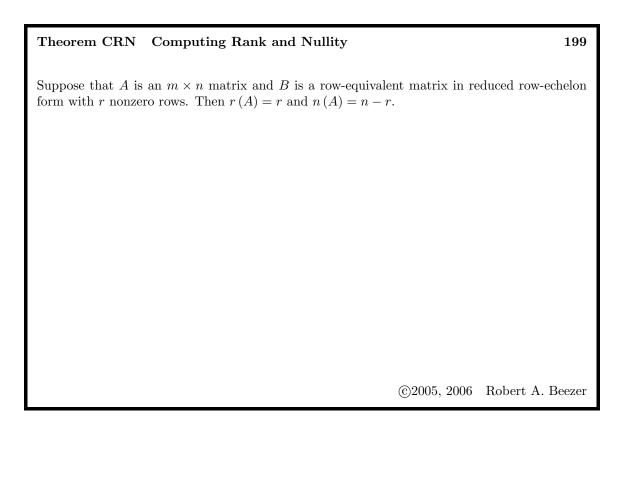
Theorem DP Dimension of P_n	195
The dimension of P_n (Example VSP) is $n+1$.	
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Theorem DM Dimension of M_{mn}

196

The dimension of M_{mn} (Example VSM) is mn.





Theorem RPNC Rank Plus Nullity is Columns

200

Suppose that A is an $m \times n$ matrix. Then r(A) + n(A) = n.

Theorem RNNM Rank and Nullity of a Nonsingular Matrix

201

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

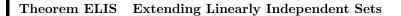
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Theorem NME6 Nonsingular Matrix Equivalences, Round 6

202

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.



Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

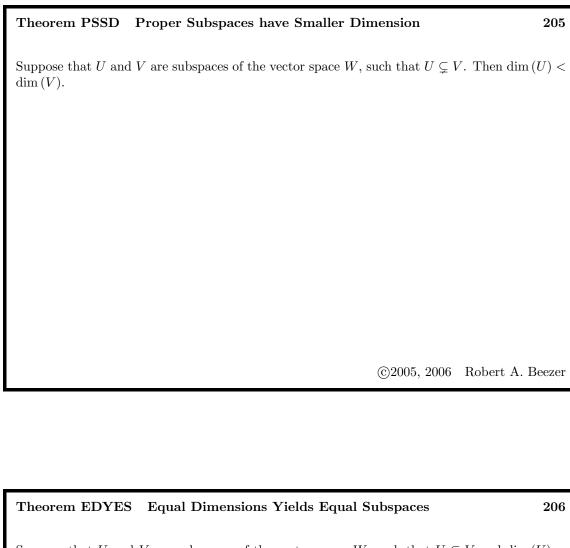
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Theorem G Goldilocks

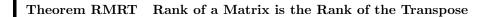
204

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.



Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then U = V.



Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

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Theorem DFS Dimensions of Four Subspaces

208

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

Definition DS Direct Sum

209

Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

- 1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

Then V is the **direct sum** of U and W and we write $V = U \oplus W$.

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Theorem DSFB Direct Sum From a Basis

210

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and $m \leq n$. Define

$$U = \langle \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_m\} \rangle \qquad W = \langle \{\mathbf{v}_{m+1}, \, \mathbf{v}_{m+2}, \, \mathbf{v}_{m+3}, \, \dots, \, \mathbf{v}_n\} \rangle$$

Then $V = U \oplus W$.



Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.

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Theorem DSZV Direct Sums and Zero Vectors

212

Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

- 1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
- 2. Whenever $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$, $\mathbf{w} \in W$ then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

Theorem DSZI Direct Sums and Zero Intersection

213

Suppose U and W are subspaces of the vector space V. Then $V=U\oplus W$ if and only if

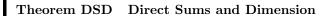
- 1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
- 2. $U \cap W = \{0\}.$

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Theorem DSLI Direct Sums and Linear Independence

214

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose that R is a linearly independent subset of U and S is a linearly independent subset of W. Then $R \cup S$ is a linearly independent subset of V.



Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then $\dim(V) = \dim(U) + \dim(W)$.

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Theorem RDS Repeated Direct Sums

216

Suppose V is a vector space with subspaces U and W with $V=U\oplus W$. Suppose that X and Y are subspaces of W with $W=X\oplus Y$. Then $V=U\oplus X\oplus Y$.

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

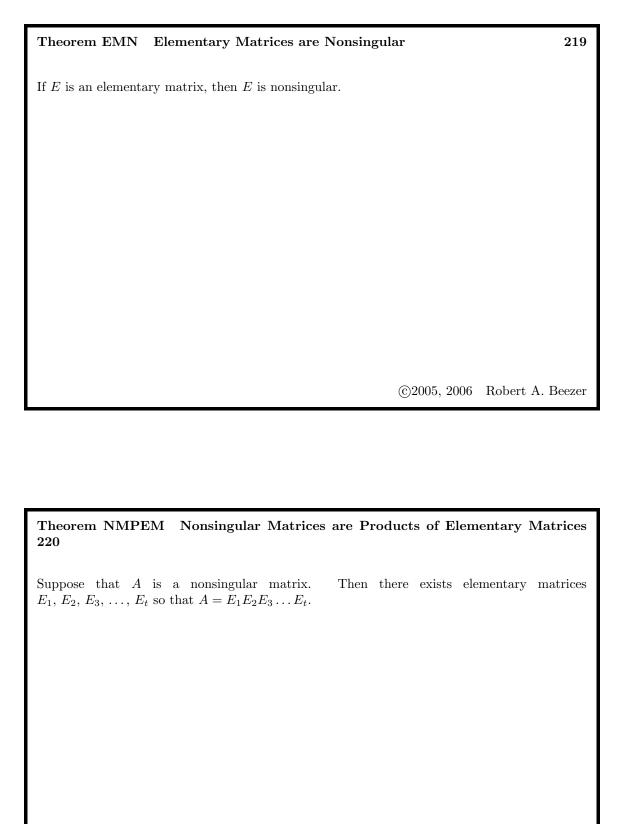
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Theorem EMDRO Elementary Matrices Do Row Operations

218

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.





Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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Definition DM Determinant of a Matrix

222

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

If A is a 1×1 matrix, then $det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\begin{split} \det{(A)} &= [A]_{11} \det{(A\,(1|1))} - [A]_{12} \det{(A\,(1|2))} + [A]_{13} \det{(A\,(1|3))} - \\ & [A]_{14} \det{(A\,(1|4))} + \dots + (-1)^{n+1} \, [A]_{1n} \det{(A\,(1|n))} \end{split}$$

Theorem DMST	Determinant	of Matricos	of Sizo	Two
i neorem Divis i	Determinant	or matrices	or Size	$1\mathbf{wo}$

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det{(A)} = ad - bc$

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Theorem DER Determinant Expansion about Rows

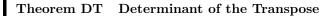
224

223

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A\left(i|1\right))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A\left(i|2\right))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A\left(i|3\right))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A\left(i|n\right))} \qquad 1 \leq i \leq n \end{split}$$

which is known as **expansion** about row i.



Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

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Theorem DEC Determinant Expansion about Columns

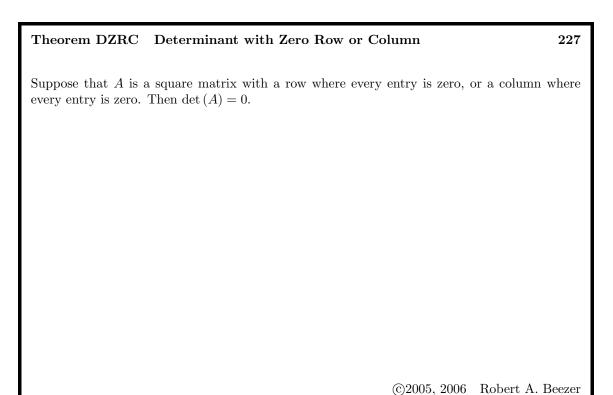
226

225

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

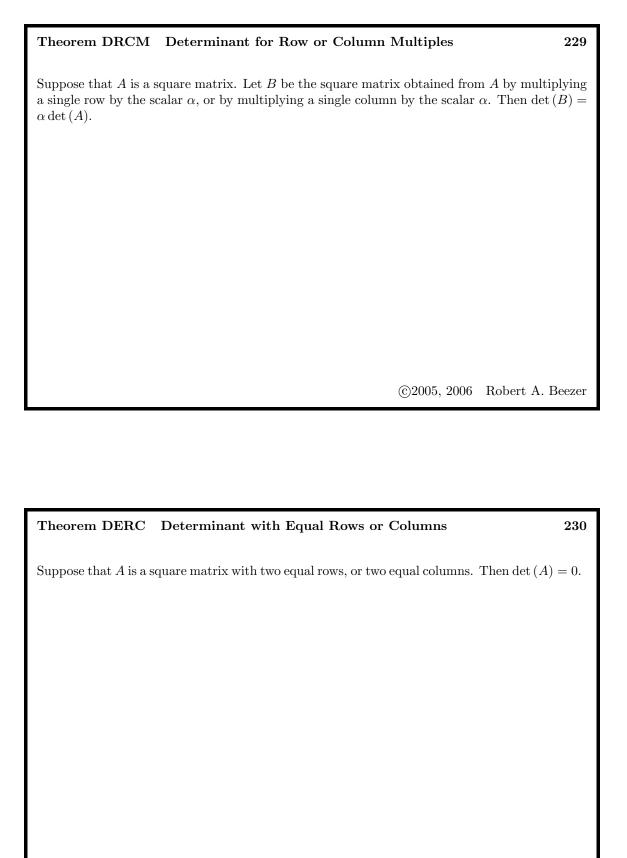
which is known as **expansion** about column j.

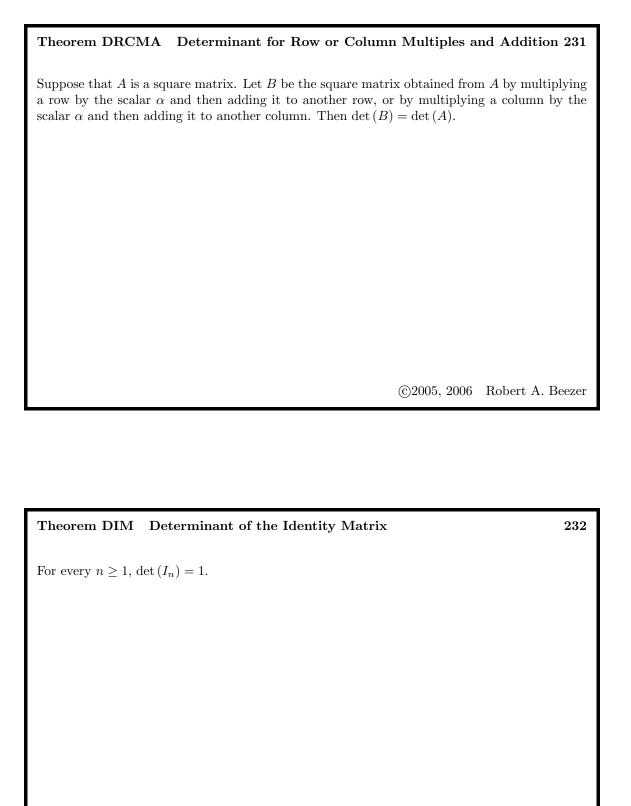


Theorem DRCS Determinant for Row or Column Swap

228

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.





For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

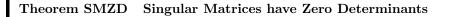
- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$

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Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 234

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det\left(EA\right) = \det\left(E\right)\det\left(A\right)$$



Let A be a square matrix. Then A is singular if and only if det(A) = 0.

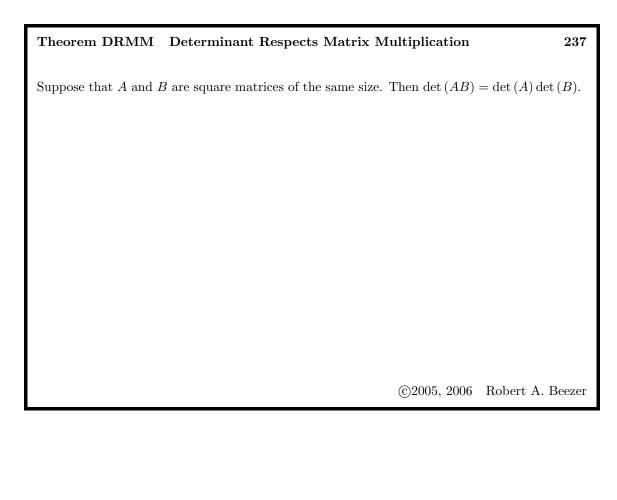
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Theorem NME7 Nonsingular Matrix Equivalences, Round 7

236

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.

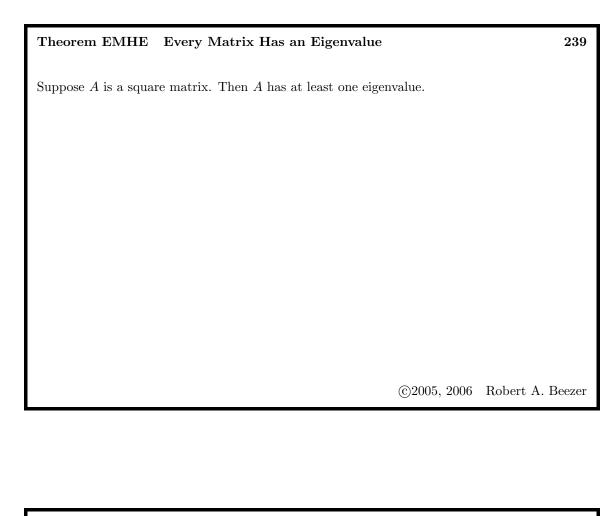


Definition EEM Eigenvalues and Eigenvectors of a Matrix

238

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda \mathbf{x}$$

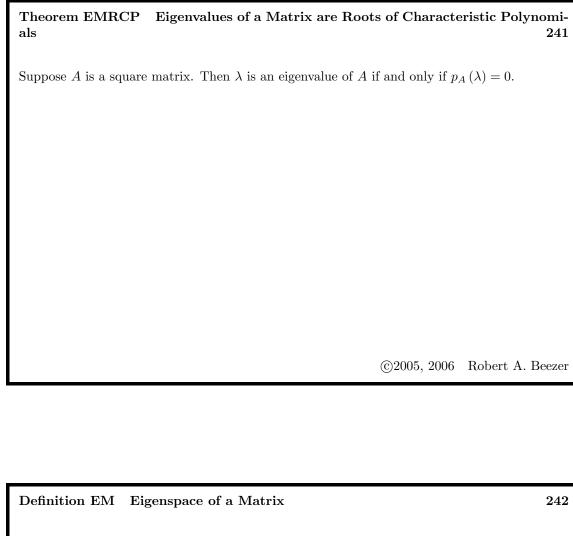


Definition CP Characteristic Polynomial

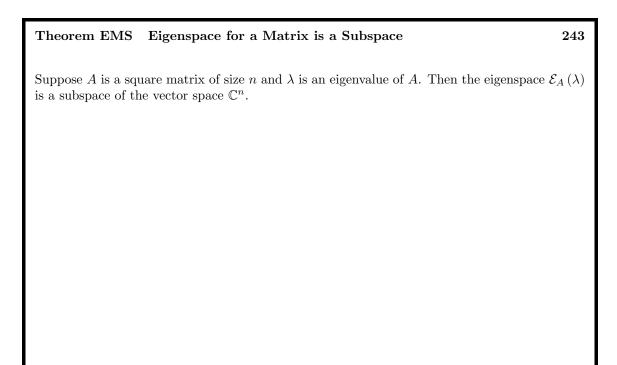
240

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_{A}(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$



Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.



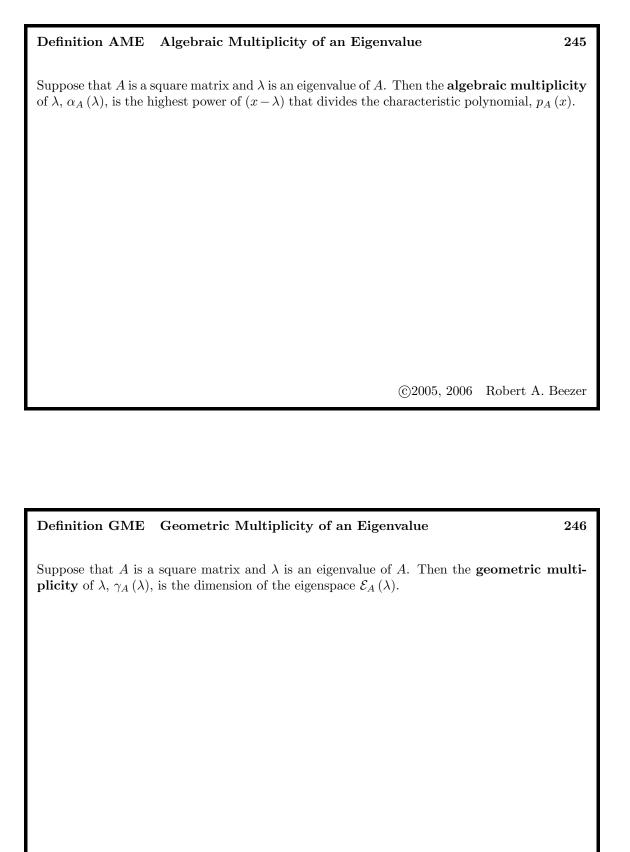
Theorem EMNS Eigenspace of a Matrix is a Null Space

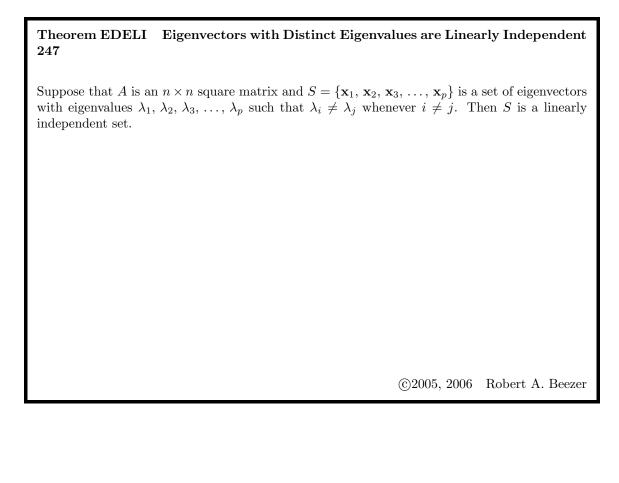
244

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

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Theorem SMZE Singular Matrices have Zero Eigenvalues

248

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NME8 Nonsingular Matrix Equivalences, Round 8

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

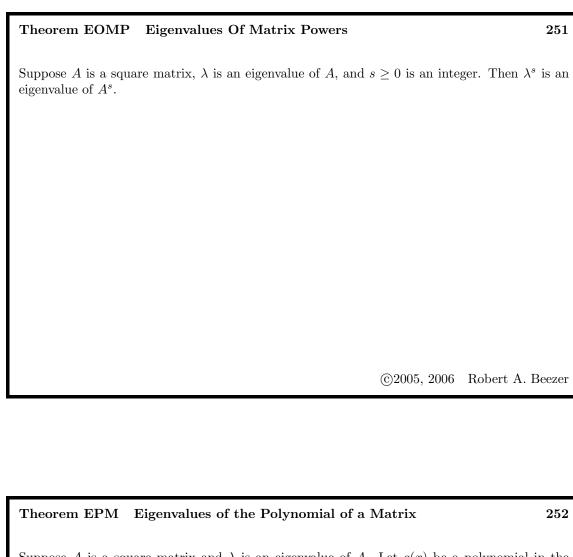
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Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix

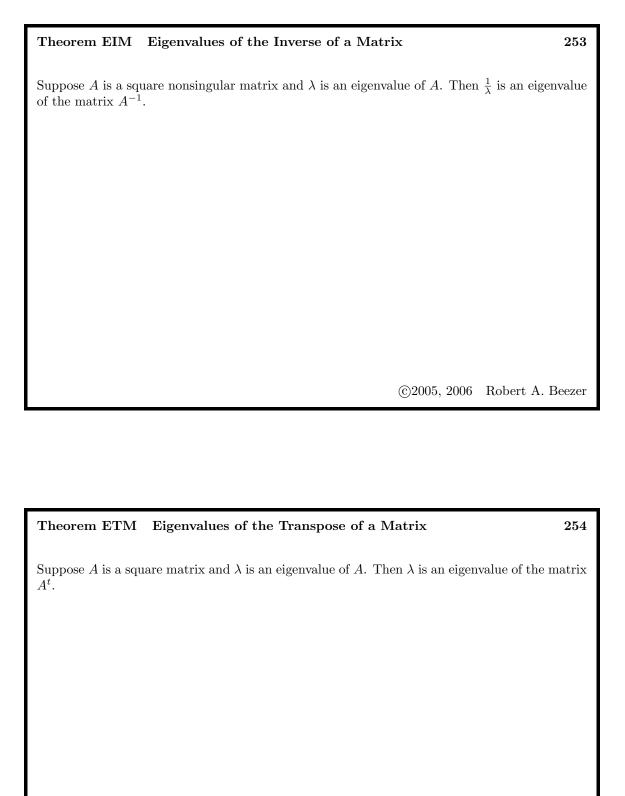
250

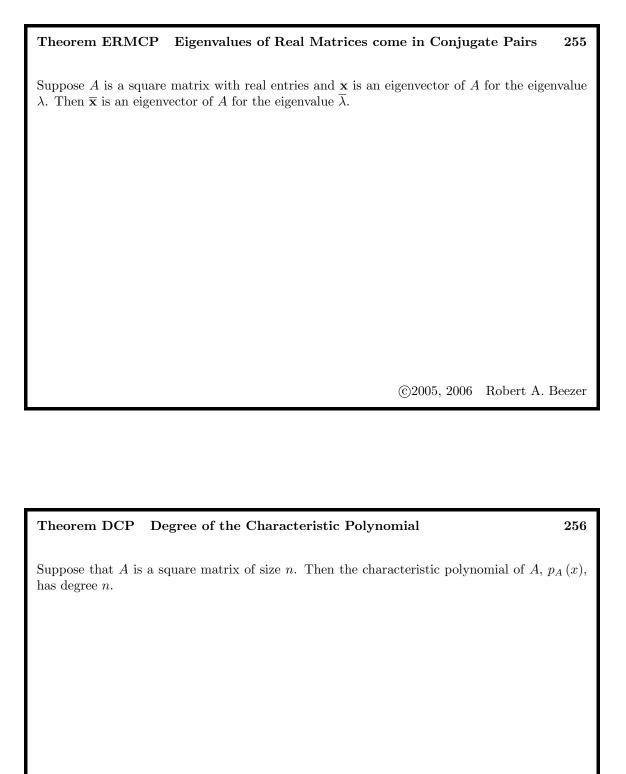
249

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA .



Suppose A is a square matrix and λ is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix q(A).





Theorem NEM Number of Eigenvalues of a Matrix

257

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

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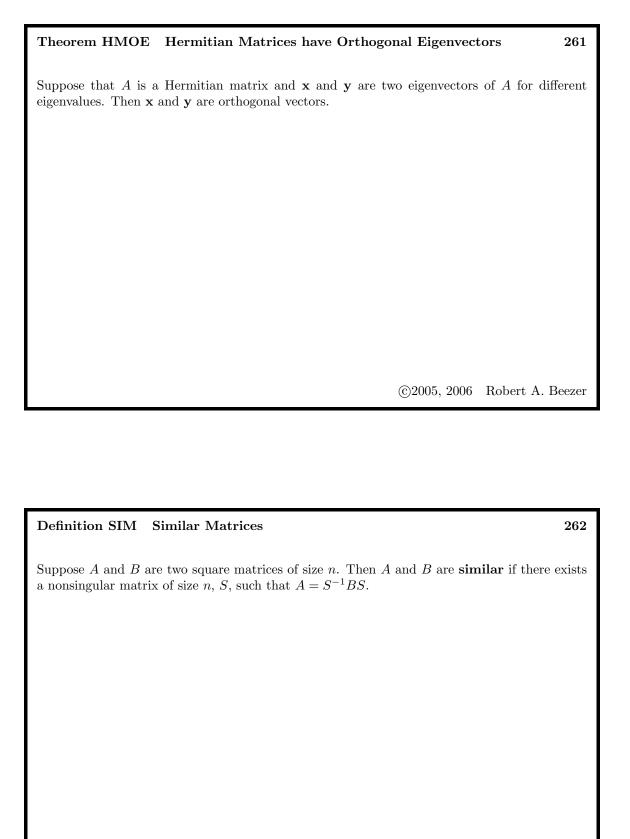
Theorem ME Multiplicities of an Eigenvalue

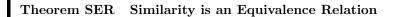
258

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix	259
Suppose that A is a square matrix of size n . Then A cannot have more than	n distinct eigen-
values.	
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Theorem HMRE Hermitian Matrices have Real Eigenvalues	260
Theorem HMRE Hermitian Matrices have Real Eigenvalues Suppose that A is a Hermitian matrix and λ is an eigenvalue of A . Then $\lambda \in \mathbb{R}$	





Suppose A, B and C are square matrices of size n. Then

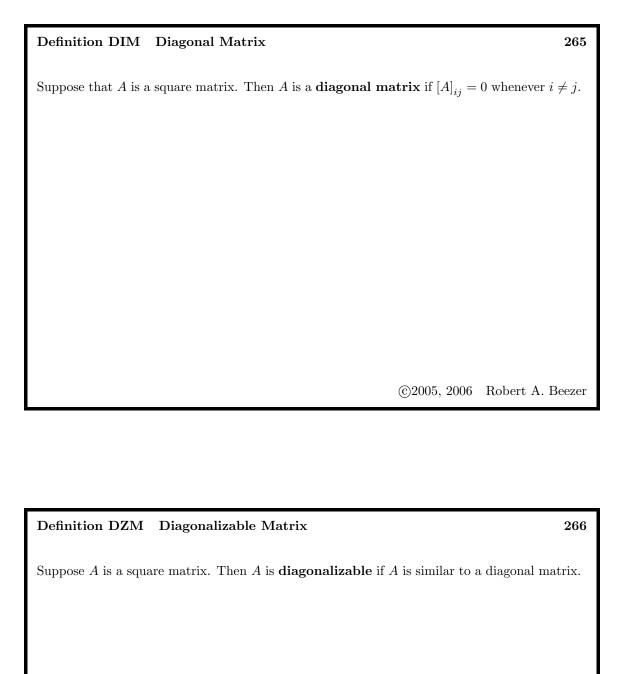
- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

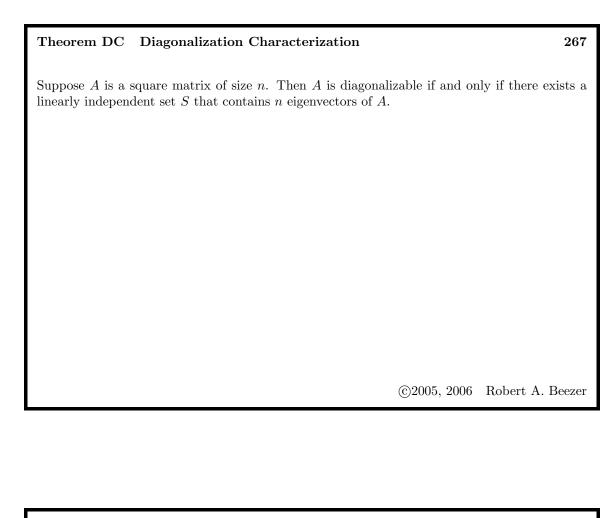
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Theorem SMEE Similar Matrices have Equal Eigenvalues

264

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $p_{A}(x) = p_{B}(x)$.

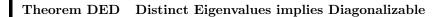




Theorem DMFE Diagonalizable Matrices have Full Eigenspaces

268

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.



Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

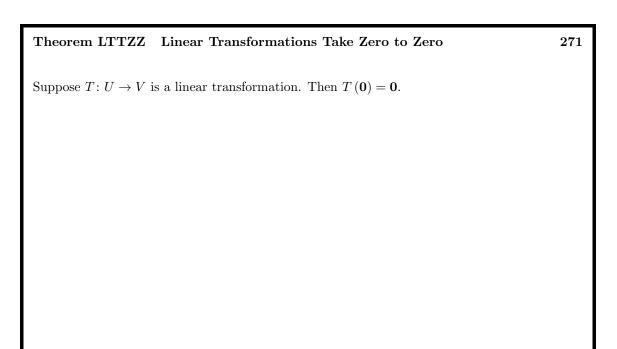
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Definition LT Linear Transformation

270

A linear transformation, $T \colon U \to V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

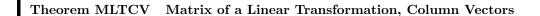
- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$



Theorem MBLT Matrices Build Linear Transformations

272

Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \to \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.



Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

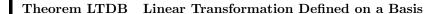
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Theorem LTLC Linear Transformations and Linear Combinations

274

Suppose that $T: U \to V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$



Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for the vector space U and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation, $T: U \to V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i$, $1 \le i \le n$.

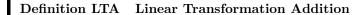
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Definition PI Pre-Image

276

Suppose that $T:U\to V$ is a linear transformation. For each ${\bf v}$, define the **pre-image** of ${\bf v}$ to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{v} \}$$



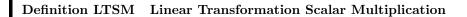
Suppose that $T: U \to V$ and $S: U \to V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T+S: U \to V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 278

Suppose that $T\colon U\to V$ and $S\colon U\to V$ are two linear transformations with the same domain and codomain. Then $T+S\colon U\to V$ is a linear transformation.



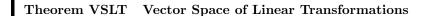
Suppose that $T: U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T: U \to V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 280

Suppose that $T:U\to V$ is a linear transformation and $\alpha\in\mathbb{C}$. Then $(\alpha T)\colon U\to V$ is a linear transformation.



Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{L}T(U,V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

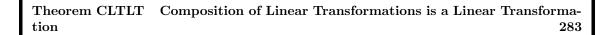
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Definition LTC Linear Transformation Composition

282

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are linear transformations. Then the **composition** of S and T is the function $(S\circ T)\colon U\to W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$



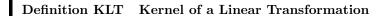
Suppose that $T\colon U\to V$ and $S\colon V\to W$ are linear transformations. Then $(S\circ T)\colon U\to W$ is a linear transformation.

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Definition ILT Injective Linear Transformation

 $\mathbf{284}$

Suppose $T: U \to V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.



Suppose $T\colon U\to V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0} \}$$

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Theorem KLTS Kernel of a Linear Transformation is a Subspace

286

Suppose that $T:U\to V$ is a linear transformation. Then the kernel of $T,\,\mathcal{K}(T),$ is a subspace of U.

Theorem KPI Kernel and Pre-Image

287

288

Suppose $T: U \to V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

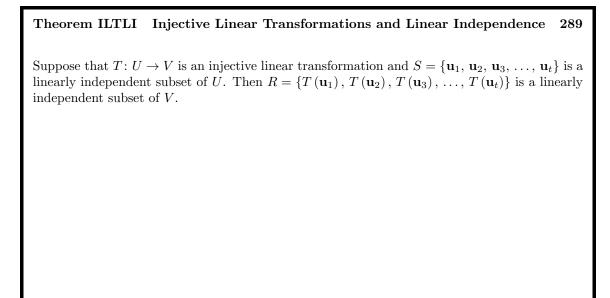
$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$

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Theorem KILT Kernel of an Injective Linear Transformation

of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}.$

Suppose that $T: U \to V$ is a linear transformation. Then T is injective if and only if the kernel

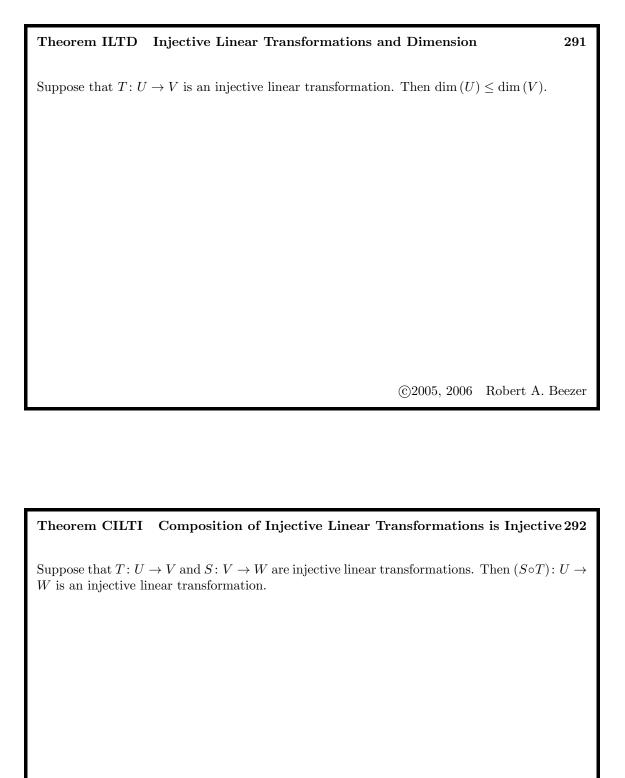


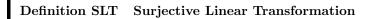
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Theorem ILTB Injective Linear Transformations and Bases

290

Suppose that $T: U \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.





Suppose $T: U \to V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

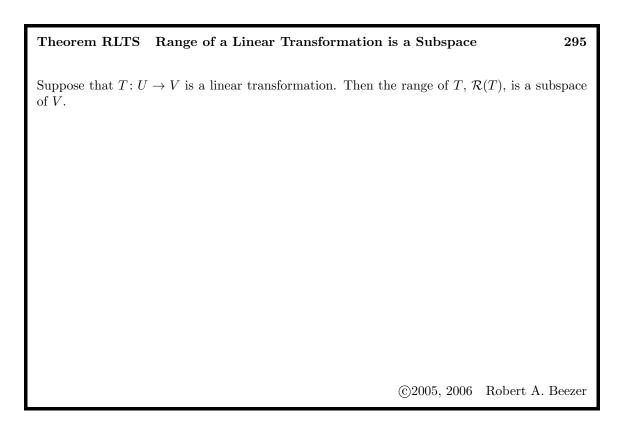
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Definition RLT Range of a Linear Transformation

294

Suppose $T\colon U\to V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) | \mathbf{u} \in U \}$$



Theorem RSLT Range of a Surjective Linear Transformation

296

Suppose that $T: U \to V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT Spanning Set for Range of a Linear Transformation

297

Suppose that $T: U \to V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), ..., T(\mathbf{u}_t)\}$$

spans $\mathcal{R}(T)$.

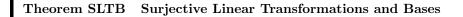
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Theorem RPI Range and Pre-Image

298

Suppose that $T \colon U \to V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T)$$
 if and only if $T^{-1}(\mathbf{v}) \neq \emptyset$



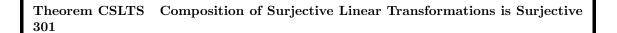
Suppose that $T: U \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.

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Theorem SLTD Surjective Linear Transformations and Dimension

300

Suppose that $T \colon U \to V$ is a surjective linear transformation. Then $\dim(U) \ge \dim(V)$.



Suppose that $T\colon U\to V$ and $S\colon V\to W$ are surjective linear transformations. Then $(S\circ T)\colon U\to W$ is a surjective linear transformation.

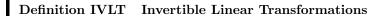
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Definition IDLT Identity Linear Transformation

302

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \to W, \qquad I_W (\mathbf{w}) = \mathbf{w}$$



Suppose that $T\colon U\to V$ is a linear transformation. If there is a function $S\colon V\to U$ such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is invertible. In this case, we call S the inverse of T and write $S = T^{-1}$.

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Theorem ILTLT $\,$ Inverse of a Linear Transformation is a Linear Transformation $\,$ 304

Suppose that $T\colon U\to V$ is an invertible linear transformation. Then the function $T^{-1}\colon V\to U$ is a linear transformation.

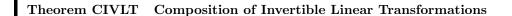


Suppose that $T: U \to V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$.

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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective 306

Suppose $T\colon U\to V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.



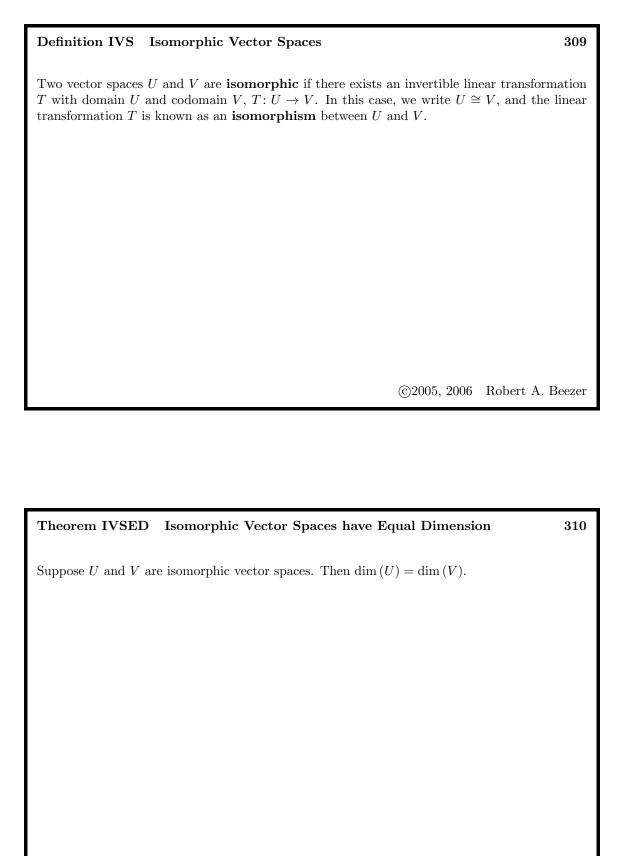
Suppose that $T:U\to V$ and $S:V\to W$ are invertible linear transformations. Then the composition, $(S\circ T):U\to W$ is an invertible linear transformation.

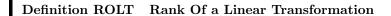
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Theorem ICLT Inverse of a Composition of Linear Transformations

308

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are invertible linear transformations. Then $S\circ T$ is invertible and $(S\circ T)^{-1}=T^{-1}\circ S^{-1}$.





Suppose that $T: U \to V$ is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

$$r(T) = \dim (\mathcal{R}(T))$$

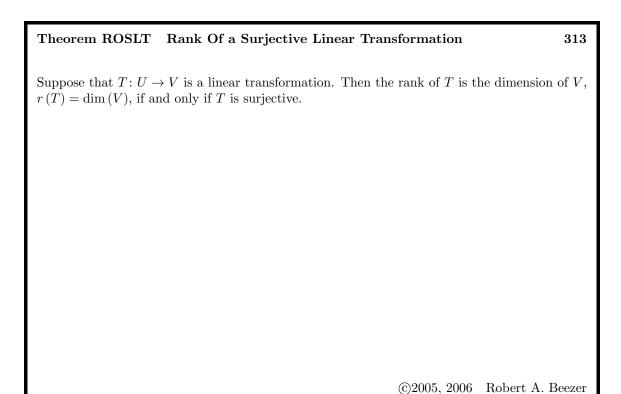
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Definition NOLT Nullity Of a Linear Transformation

312

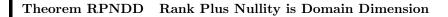
Suppose that $T:U\to V$ is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

$$n(T) = \dim (\mathcal{K}(T))$$



314

Suppose that $T:U\to V$ is a linear transformation. Then the nullity of T is zero, $n\left(T\right)=0$, if and only if T is injective.



Suppose that $T\colon U\to V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

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Definition VR Vector Representation

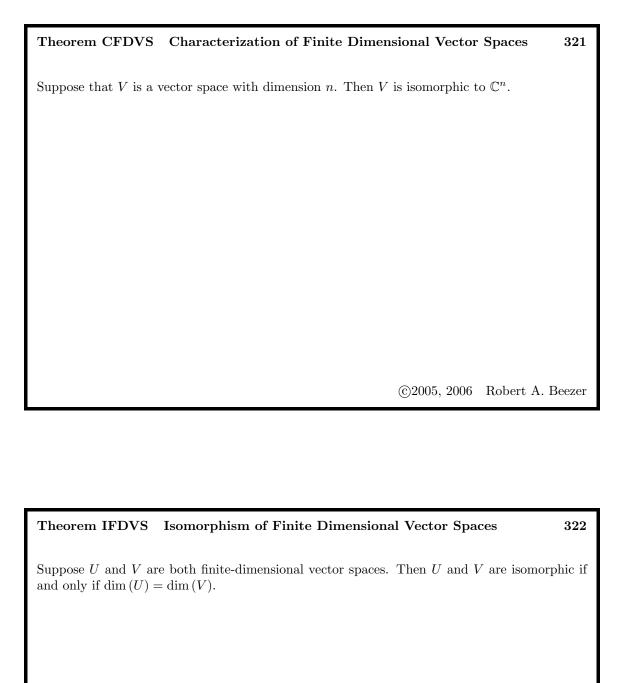
316

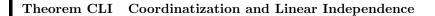
Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \to \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by

$$\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \dots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$$

Theorem VRLT	Vector Representation is a Linear	Transformation	on 317
The function ρ_B (D	Definition VR) is a linear transformation.		
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Theorem VRI V	Vector Representation is Injective		318
	definition VP) is an injective linear trans		310

Theorem VRS Vector Representation is Surjective	319
The function ρ_B (Definition VR) is a surjective linear transformation.	
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Theorem VRILT Vector Representation is an Invertible Linear 320	Transformation
The function ρ_B (Definition VR) is an invertible linear transformation.	
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Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n .

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Theorem CSS Coordinatization and Spanning Sets

324

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$.

Definition MR Matrix Representation

325

Suppose that $T: U \to V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\rho_{C} \left(T \left(\mathbf{u}_{1} \right) \right) \middle| \rho_{C} \left(T \left(\mathbf{u}_{2} \right) \right) \middle| \rho_{C} \left(T \left(\mathbf{u}_{3} \right) \right) \middle| \dots \middle| \rho_{C} \left(T \left(\mathbf{u}_{n} \right) \right) \right]$$

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Theorem FTMR Fundamental Theorem of Matrix Representation

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Suppose that $T: U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 327

Suppose that $T\colon U\to V$ and $S\colon U\to V$ are linear transformations, B is a basis of U and C is a basis of V. Then

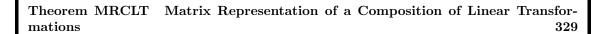
$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 328

Suppose that $T:U\to V$ is a linear transformation, $\alpha\in\mathbb{C},\,B$ is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$



Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

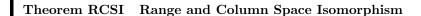
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Theorem KNSI Kernel and Null Space Isomorphism

330

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$



Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

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Theorem IMR Invertible Matrix Representations

332

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix. When T is invertible,

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

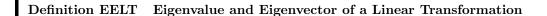
Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

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Theorem NME9 Nonsingular Matrix Equivalences, Round 9

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.



Suppose that $T: V \to V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

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Definition CBM Change-of-Basis Matrix

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Suppose that V is a vector space, and $I_V \colon V \to V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$



Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

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Theorem ICBM Inverse of Change-of-Basis Matrix

338

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB Matrix Representation and Change of Basis

339

Suppose that $T\colon U\to V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

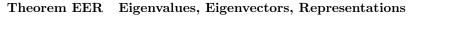
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Theorem SCB Similarity and Change of Basis

340

Suppose that $T: V \to V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$



Suppose that $T\colon V\to V$ is a linear transformation and B is a basis of V. Then $\mathbf{v}\in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

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Definition NLT Nilpotent Linear Transformation

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Suppose that $T \colon V \to V$ is a linear transformation such that there is an integer p > 0 such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.

Definition JB Jordan Block

343

Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

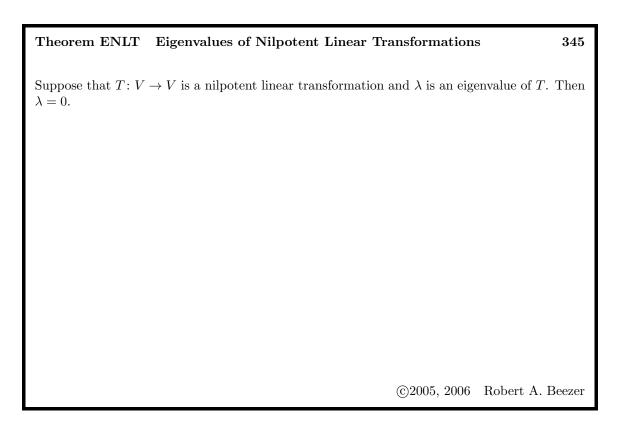
$$\left[J_{n}\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$

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Theorem NJB Nilpotent Jordan Blocks

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The Jordan block $J_{n}\left(0\right)$ is nilpotent of index n.



Theorem DNLT Diagonalizable Nilpotent Linear Transformations

346

Suppose the linear transformation $T\colon V\to V$ is nilpotent. Then T is diagonalizable if and only T is the zero linear transformation.

Theorem KPLT Kernels of Powers of Linear Transformations

347

Suppose $T: V \to V$ is a linear transformation, where dim (V) = n. Then there is an integer m, $0 \le m \le n$, such that

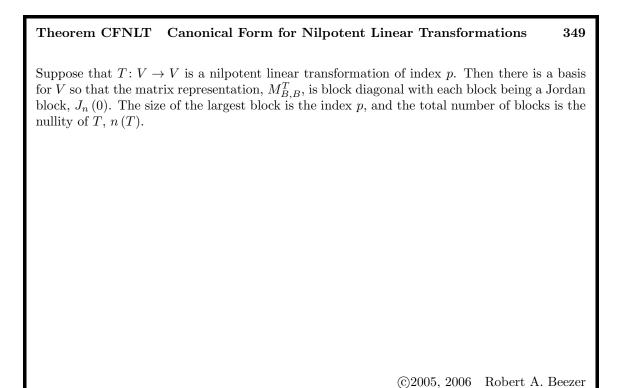
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

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Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 348

Suppose $T:V\to V$ is a nilpotent linear transformation with index p and dim (V)=n. Then $0\leq p\leq n$ and

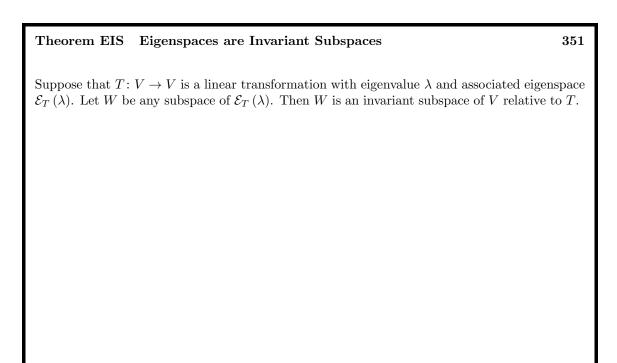
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$



Definition IS Invariant Subspace

350

Suppose that $T: V \to V$ is a linear transformation and W is a subspace of V. Suppose further that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$. Then W is an **invariant subspace** of V relative to T.

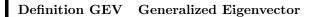


Theorem KPIS Kernels of Powers are Invariant Subspaces

352

Suppose that $T\colon V\to V$ is a linear transformation. Then $\mathcal{K}\big(T^k\big)$ is an invariant subspace of V

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Suppose that $T: V \to V$ is a linear transformation. Suppose further that for $\mathbf{x} \neq \mathbf{0}$, $(T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0}$ for some k > 0. Then \mathbf{x} is a **generalized eigenvector** of T with eigenvalue λ .

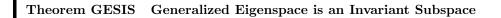
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Definition GES Generalized Eigenspace

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Suppose that $T\colon V\to V$ is a linear transformation. Define the **generalized eigenspace** of T for λ as

$$\mathcal{G}_{T}(\lambda) = \left\{ \mathbf{x} | \left(T - \lambda I_{V}\right)^{k}(\mathbf{x}) = \mathbf{0} \text{ for some } k \geq 0 \right\}$$



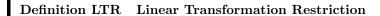
Suppose that $T: V \to V$ is a linear transformation. Then the generalized eigenspace $\mathcal{G}_T(\lambda)$ is an invariant subspace of V relative to T.

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Theorem GEK Generalized Eigenspace as a Kernel

356

Suppose that $T: V \to V$ is a linear transformation, $\dim(V) = n$, and λ is an eigenvalue of T. Then $\mathcal{G}_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n)$.



Suppose that $T \colon V \to V$ is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by

$$T|_U \colon U \to U$$

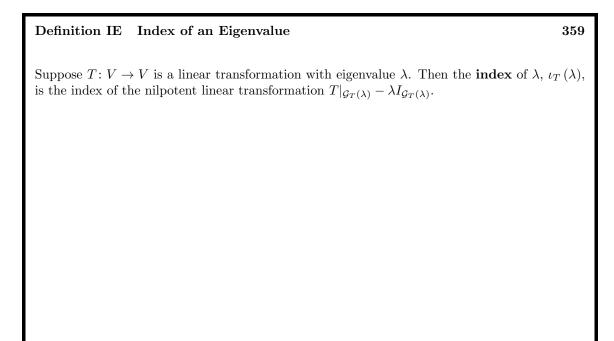
$$T|_{U}(\mathbf{u}) = T(\mathbf{u})$$

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Theorem RGEN Restriction to Generalized Eigenspace is Nilpotent

358

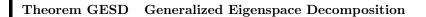
Suppose $T: V \to V$ is a linear transformation with eigenvalue λ . Then the linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is nilpotent.



Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace 360

Suppose that $T: V \to V$ is a linear transformation with eigenvalue λ . Then there is a basis of the the generalized eigenspace $\mathcal{G}_T(\lambda)$ such that the restriction $T|_{\mathcal{G}_T(\lambda)}: \mathcal{G}_T(\lambda) \to \mathcal{G}_T(\lambda)$ has a matrix representation that is block diagonal where each block is a Jordan block of the form $J_n(\lambda)$.

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Suppose that $T: V \to V$ is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$. Then

$$V = \mathcal{G}_{T}(\lambda_{1}) \oplus \mathcal{G}_{T}(\lambda_{2}) \oplus \mathcal{G}_{T}(\lambda_{3}) \oplus \cdots \oplus \mathcal{G}_{T}(\lambda_{m})$$

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Theorem DGES Dimension of Generalized Eigenspaces

362

Suppose $T: V \to V$ is a linear transformation with eigenvalue λ . Then the dimension of the generalized eigenspace for λ is the algebraic multiplicity of λ , dim $(\mathcal{G}_T(\lambda_i)) = \alpha_T(\lambda_i)$.

Definition JCF Jordan Canonical Form

363

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A square matrix is in Jordan canonical form if it meets the following requirements:

- 1. The matrix is block diagonal.
- 2. Each block is a Jordan block.
- 3. If $\rho < \lambda$ then the block $J_k(\rho)$ occupies rows with indices greater than the indices of the rows occupied by $J_{\ell}(\lambda)$.
- 4. If $\rho = \lambda$ and $\ell < k$, then the block $J_{\ell}(\lambda)$ occupies rows with indices greater than the indices of the rows occupied by $J_{k}(\lambda)$.

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Theorem JCFLT Jordan Canonical Form for a Linear Transformation

Suppose $T: V \to V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

- 1. The matrix representation is in Jordan canonical form.
- 2. If $J_k(\lambda)$ is one of the Jordan blocks, then λ is an eigenvalue of T.
- 3. For a fixed value of λ , the largest block of the form $J_k(\lambda)$ has size equal to the index of λ , $\iota_T(\lambda)$.
- 4. For a fixed value of λ , the number of blocks of the form $J_k(\lambda)$ is the geometric multiplicity of λ , $\gamma_T(\lambda)$.
- 5. For a fixed value of λ , the number of rows occupied by blocks of the form $J_k(\lambda)$ is the algebraic multiplicity of λ , $\alpha_T(\lambda)$.

Theorem CHT	Cayley-Hamilton Theorem	365	
Suppose A is a square matrix with characteristic polynomial $p_{A}(x)$. Then $p_{A}(A) = \mathcal{O}$.			
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