## Flash Cards

to accompany

# A First Course in Linear Algebra

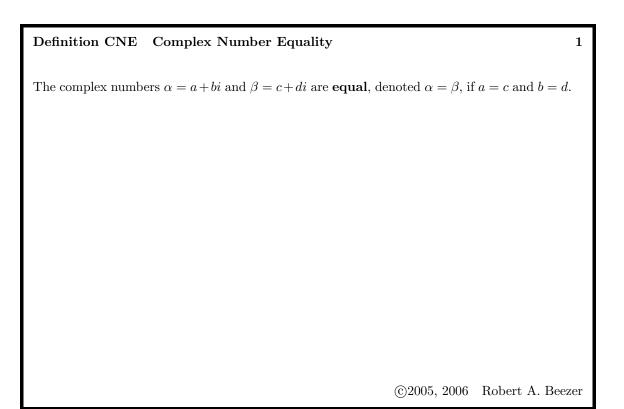
by Robert A. Beezer Department of Mathematics and Computer Science University of Puget Sound

Version 2.22

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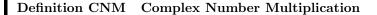
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## Definition CNA Complex Number Addition

 $\mathbf{2}$ 

The **sum** of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha + \beta$ , is (a + c) + (b + d)i.



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The **product** of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha\beta$ , is (ac - bd) + (ad + bc)i.

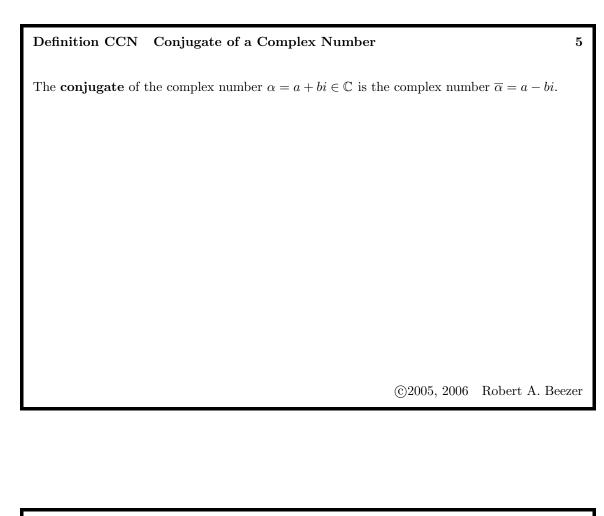
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#### Theorem PCNA Properties of Complex Number Arithmetic

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The operations of addition and multiplication of complex numbers have the following properties.

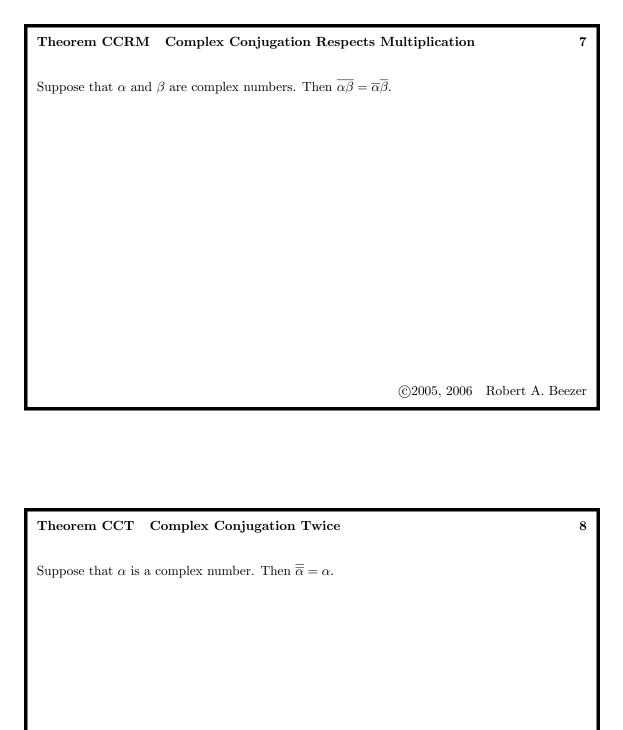
- ACCN Additive Closure, Complex Numbers If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha + \beta \in \mathbb{C}$ .
- MCCN Multiplicative Closure, Complex Numbers If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha\beta \in \mathbb{C}$ .
- CACN Commutativity of Addition, Complex Numbers For any  $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$ .
- CMCN Commutativity of Multiplication, Complex Numbers For any  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha\beta = \beta\alpha$ .
- AACN Additive Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .
- MACN Multiplicative Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ .
- DCN Distributivity, Complex Numbers For any  $\alpha$ ,  $\beta$ ,  $\gamma \in \mathbb{C}$ ,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ .
- **ZCN Zero, Complex Numbers** There is a complex number 0 = 0 + 0i so that for any  $\alpha \in \mathbb{C}$ ,  $0 + \alpha = \alpha$ .
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any  $\alpha \in \mathbb{C}$ ,  $1\alpha = \alpha$ .
- AICN Additive Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$  there exists  $-\alpha \in \mathbb{C}$  so that  $\alpha + (-\alpha) = 0$
- MICN Multiplicative Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$  there exists  $\frac{1}{\alpha} \in \mathbb{C}$  so that  $\alpha\left(\frac{1}{\alpha}\right) = 1$ .

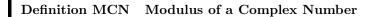


## Theorem CCRA Complex Conjugation Respects Addition

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Suppose that  $\alpha$  and  $\beta$  are complex numbers. Then  $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$ .





a

The **modulus** of the complex number  $\alpha = a + bi \in \mathbb{C}$ , is the nonnegative real number

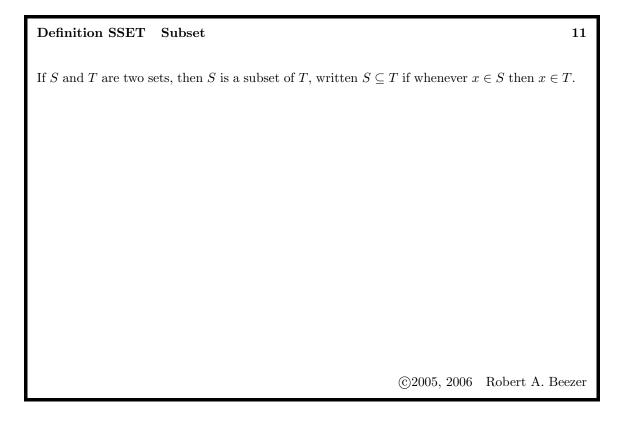
$$|\alpha| = \sqrt{\alpha \overline{\alpha}} = \sqrt{a^2 + b^2}.$$

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#### Definition SET Set

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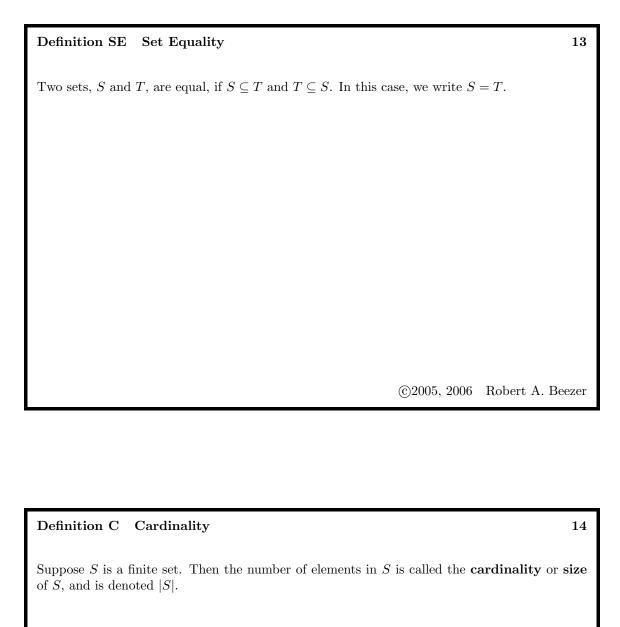
A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write  $x \in S$ . If x is not in S, then we write  $x \notin S$ . We refer to the objects in a set as its elements.

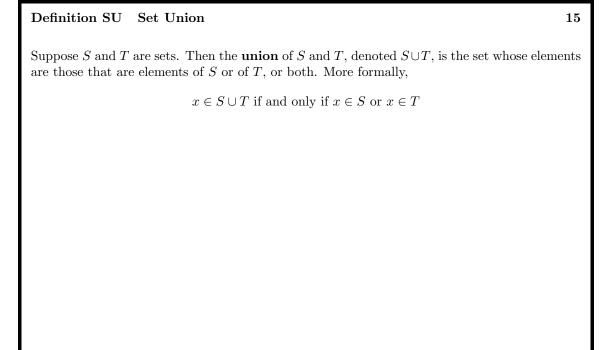


## Definition ES Empty Set

12

The empty set is the set with no elements. Its is denoted by  $\emptyset$ .





#### Definition SI Set Intersection

**16** 

Suppose S and T are sets. Then the **intersection** of S and T, denoted  $S \cap T$ , is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$  if and only if  $x \in S$  and  $x \in T$ 

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Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted  $\overline{S}$ , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$  if and only if  $x \in U$  and  $x \notin S$ 

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#### Definition SLE System of Linear Equations

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A system of linear equations is a collection of m equations in the variable quantities  $x_1, x_2, x_3, \ldots, x_n$  of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

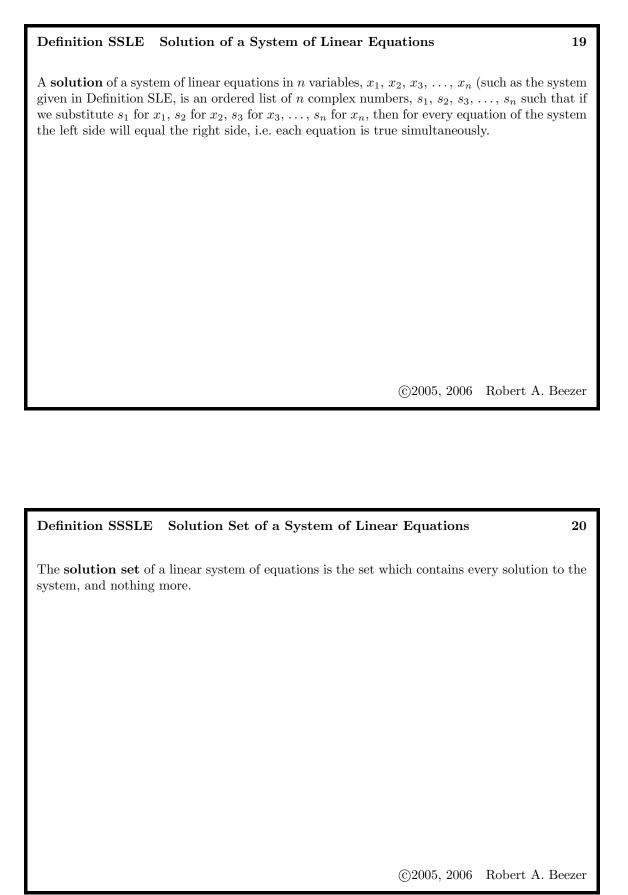
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

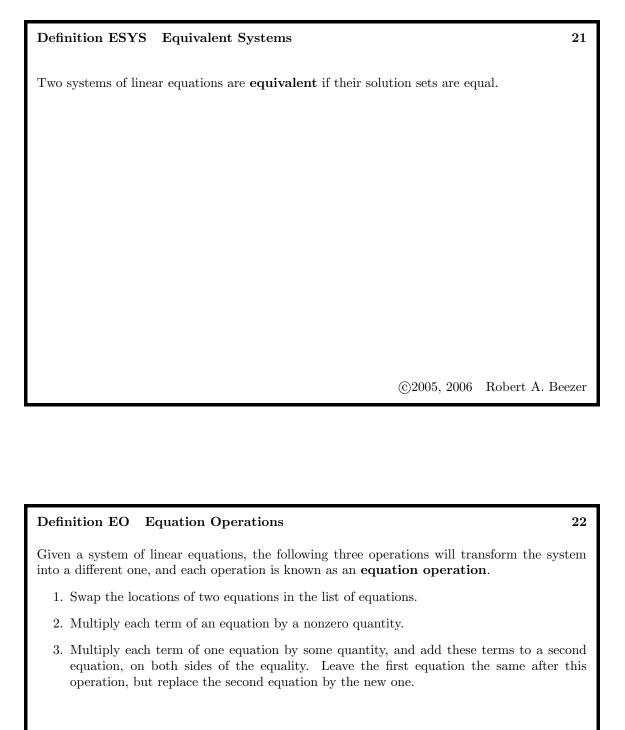
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

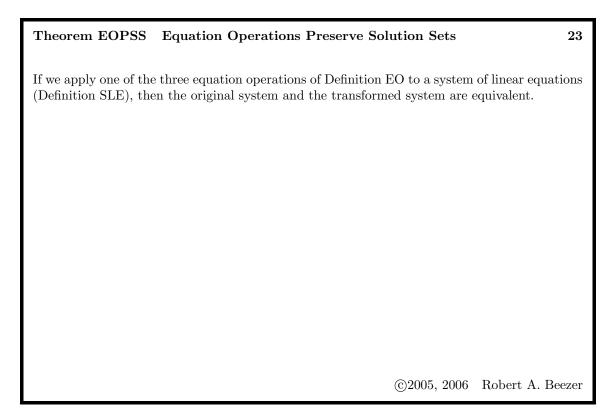
$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of  $a_{ij}$ ,  $b_i$  and  $x_j$  are from the set of complex numbers,  $\mathbb{C}$ .







#### Definition M Matrix 24

An  $m \times n$  matrix is a rectangular layout of numbers from  $\mathbb{C}$  having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet  $(A, B, C, \ldots)$  to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation  $[A]_{ij}$  will refer to the complex number in row i and column j of A.

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{z}$ . Some books like to write vectors with arrows, such as  $\vec{u}$ . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the **entry** or **component** that is number i in the list that is the vector  $\mathbf{v}$  we write  $[\mathbf{v}]_i$ .

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#### Definition ZCV Zero Column Vector

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The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly,  $[\mathbf{0}]_i = 0$  for  $1 \le i \le m$ .

#### Definition CM Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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#### Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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#### Definition SOLV Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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#### Definition MRLS Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and  $\mathbf{b}$  is the vector of constants, then we will write  $\mathcal{LS}(A, \mathbf{b})$  as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

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Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants  $\mathbf{b}$ . Then the **augmented matrix** of the system of equations is the  $m \times (n+1)$  matrix whose first n columns are the columns of A and whose last column (number n+1) is the column vector  $\mathbf{b}$ . This matrix will be written as  $[A \mid \mathbf{b}]$ .

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#### Definition RO Row Operations

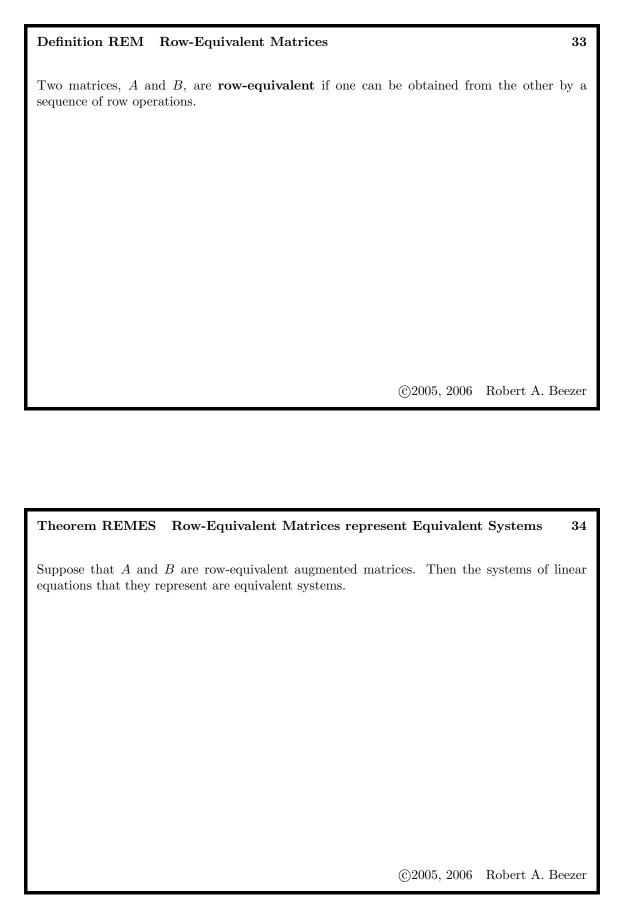
**32** 

The following three operations will transform an  $m \times n$  matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1.  $R_i \leftrightarrow R_j$ : Swap the location of rows i and j.
- 2.  $\alpha R_i$ : Multiply row i by the nonzero scalar  $\alpha$ .
- 3.  $\alpha R_i + R_j$ : Multiply row i by the scalar  $\alpha$  and add to row j.



#### Definition RREF Reduced Row-Echelon Form

35

A matrix is in reduced row-echelon form if it meets all of the following conditions:

- 1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  where  $d_1 < d_2 < d_3 < \cdots < d_r$ , while the columns that are not pivot columns will be denoted as  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  where  $f_1 < f_2 < f_3 < \cdots < f_{n-r}$ .

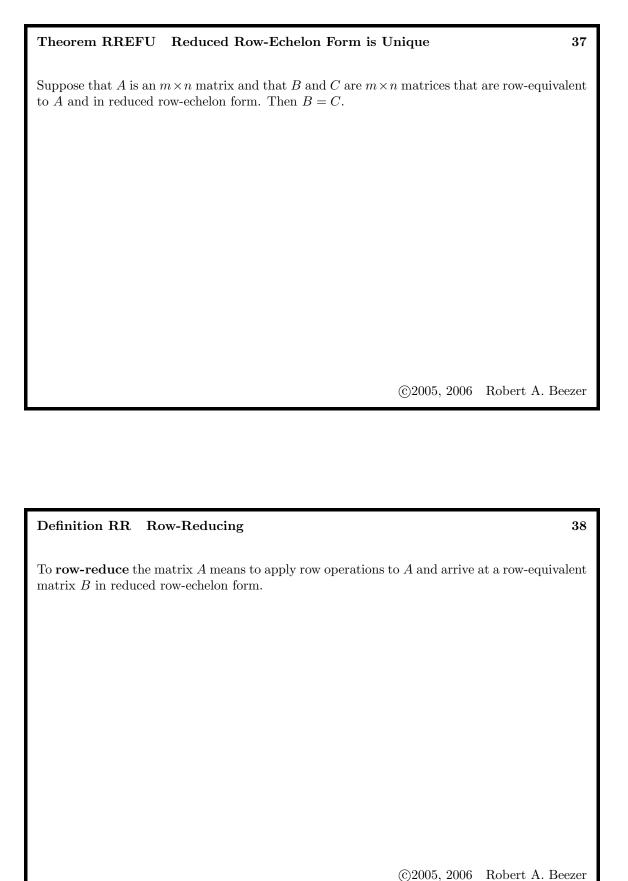
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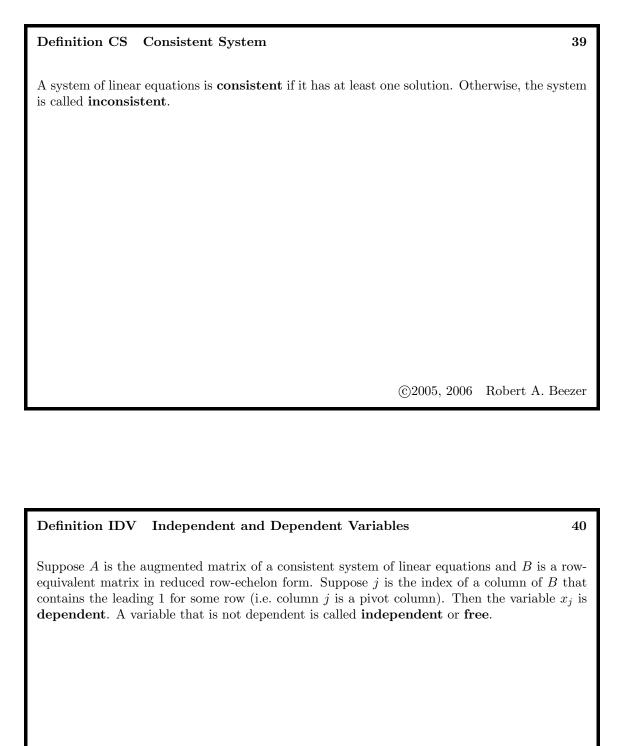
#### Theorem REMEF Row-Equivalent Matrix in Echelon Form

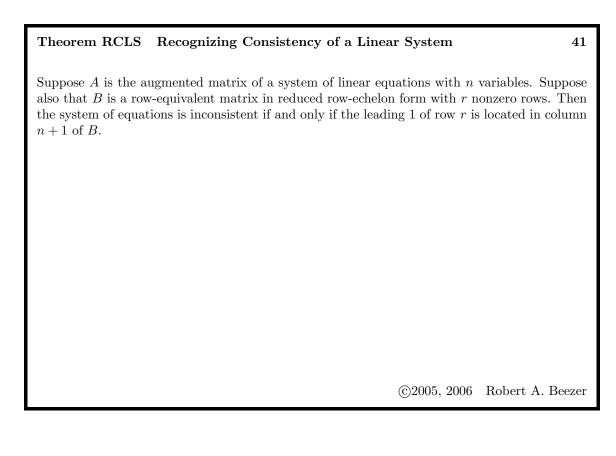
36

Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.



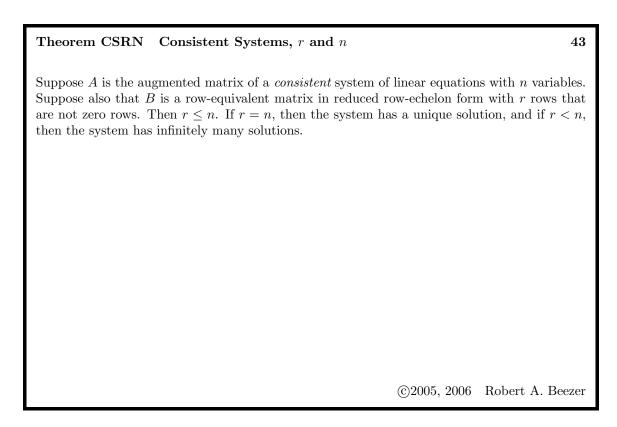




#### Theorem ISRN Inconsistent Systems, r and n

42

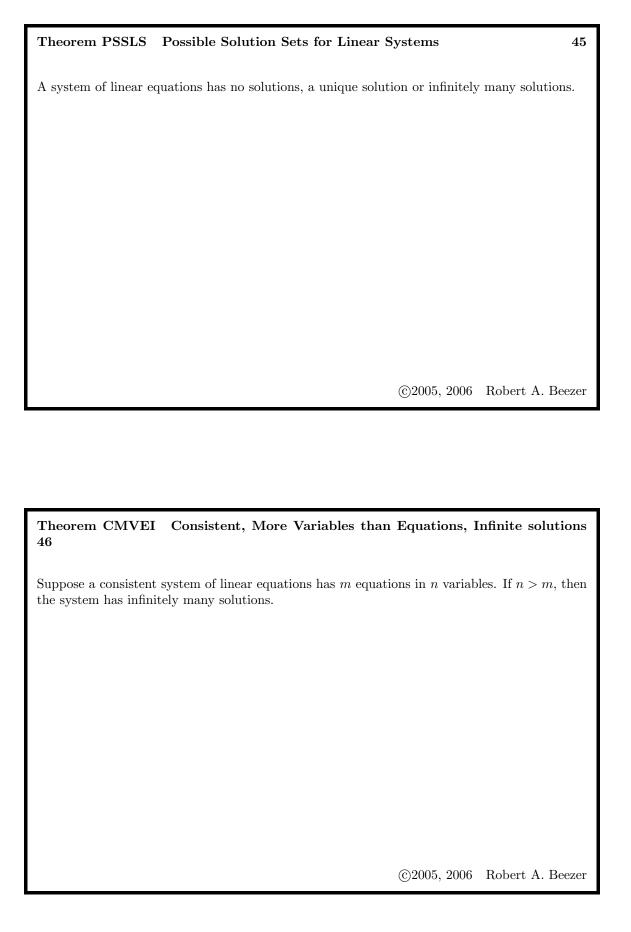
Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r=n+1, then the system of equations is inconsistent.

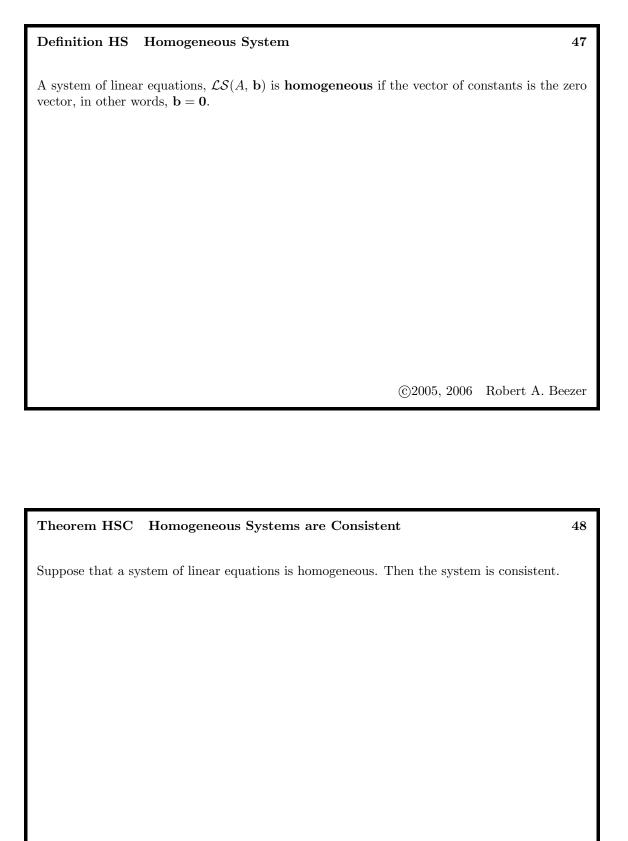


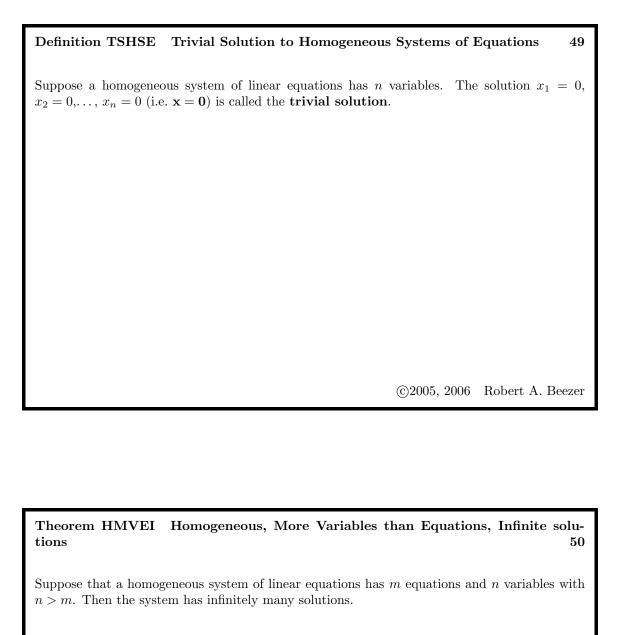
#### Theorem FVCS Free Variables for Consistent Systems

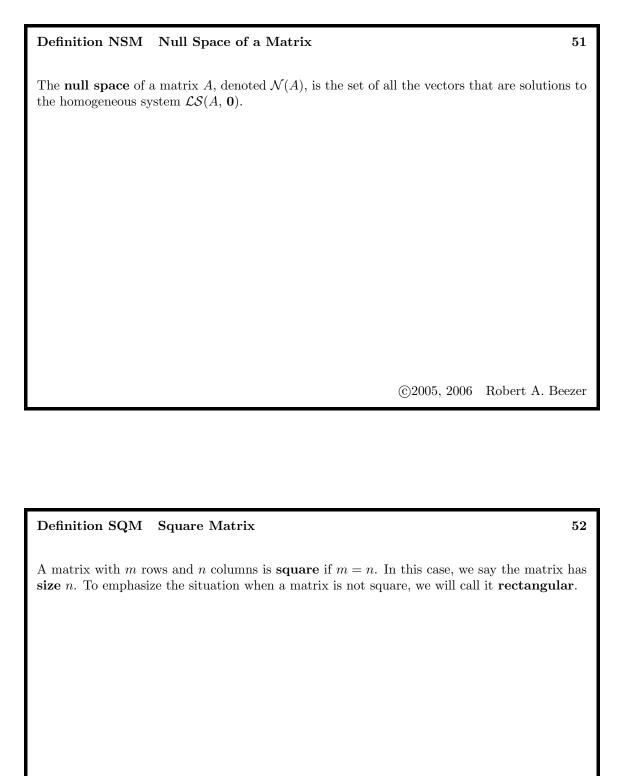
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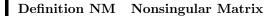
Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.











**53** 

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations  $\mathcal{LS}(A, \mathbf{0})$  is  $\{\mathbf{0}\}$ , i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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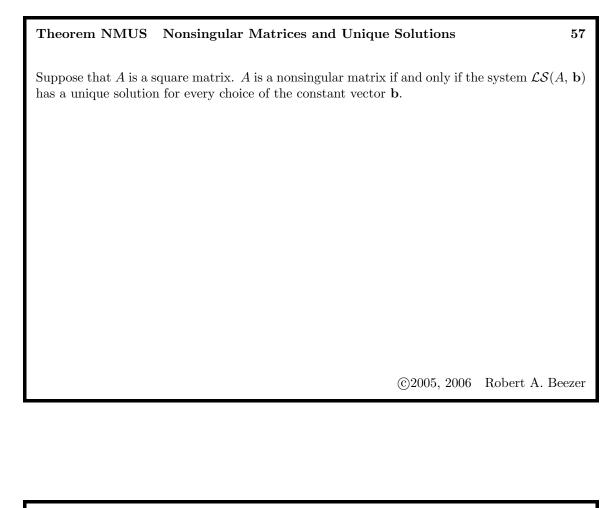
## Definition IM Identity Matrix

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The  $m \times m$  identity matrix,  $I_m$ , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \le i, j \le m$$

Theorem NMRRI	Nonsingular Matrices Row Reduce to the Identity matrix 5	5
	square matrix and $B$ is a row-equivalent matrix in reduced row-echelo	n
form. Then $A$ is nonst	ingular if and only if $B$ is the identity matrix.	
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Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 5	6
Suppose that $A$ is a s	square matrix. Then $A$ is nonsingular if and only if the null space of $A$	
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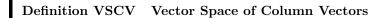


#### Theorem NME1 Nonsingular Matrix Equivalences, Round 1

**58** 

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .



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The vector space  $\mathbb{C}^m$  is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers,  $\mathbb{C}$ .

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## Definition CVE Column Vector Equality

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Suppose that  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ . Then  $\mathbf{u}$  and  $\mathbf{v}$  are **equal**, written  $\mathbf{u} = \mathbf{v}$  if

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

## Definition CVA Column Vector Addition

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Suppose that  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ . The **sum** of  $\mathbf{u}$  and  $\mathbf{v}$  is the vector  $\mathbf{u} + \mathbf{v}$  defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \le i \le m$$

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## Definition CVSM Column Vector Scalar Multiplication

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Suppose  $\mathbf{u} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ , then the **scalar multiple** of  $\mathbf{u}$  by  $\alpha$  is the vector  $\alpha \mathbf{u}$  defined by

$$[\alpha \mathbf{u}]_i = \alpha \left[ \mathbf{u} \right]_i$$

$$1 \leq i \leq m$$

#### Theorem VSPCV Vector Space Properties of Column Vectors

Suppose that  $\mathbb{C}^m$  is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ .
- SCC Scalar Closure, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha \mathbf{u} \in \mathbb{C}^m$ .
- CC Commutativity, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AAC Additive Associativity, Column Vectors If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- **ZC Zero Vector, Column Vectors** There is a vector, **0**, called the **zero vector**, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in \mathbb{C}^m$ .
- AIC Additive Inverses, Column Vectors If  $\mathbf{u} \in \mathbb{C}^m$ , then there exists a vector  $-\mathbf{u} \in \mathbb{C}^m$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMAC Scalar Multiplication Associativity, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVAC Distributivity across Vector Addition, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSAC Distributivity across Scalar Addition, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ .
- OC One Column Vectors If  $u \in \mathbb{C}^m$  then 1u u

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#### Definition LCCV Linear Combination of Column Vectors

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Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  from  $\mathbb{C}^m$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their **linear** combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$$

Denote the columns of the  $m \times n$  matrix A as the vectors  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ . Then  $\mathbf{x}$  is a solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$  if and only if  $\mathbf{b}$  equals the linear combination of the columns of A formed with the entries of  $\mathbf{x}$ ,

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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#### Theorem VFSLS Vector Form of Solutions to Linear Systems

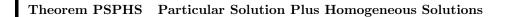
66

Suppose that  $[A \mid \mathbf{b}]$  is the augmented matrix for a consistent linear system  $\mathcal{LS}(A, \mathbf{b})$  of m equations in n variables. Let B be a row-equivalent  $m \times (n+1)$  matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$ , and columns with leading 1's (pivot columns) having indices  $D = \{d_1, d_2, d_3, \ldots, d_r\}$ . Define vectors  $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$  of size n by

$$\begin{aligned} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} \end{aligned}$$

Then the set of solutions to the system of equations  $\mathcal{LS}(A, \mathbf{b})$  is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} | \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$



67

Suppose that **w** is one solution to the linear system of equations  $\mathcal{LS}(A, b)$ . Then **y** is a solution to  $\mathcal{LS}(A, b)$  if and only if  $\mathbf{y} = \mathbf{w} + \mathbf{z}$  for some vector  $\mathbf{z} \in \mathcal{N}(A)$ .

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#### Definition SSCV Span of a Set of Column Vectors

68

Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

#### Theorem SSNS Spanning Sets for Null Spaces

69

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the column indices where B has leading 1's (pivot columns) and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the set of column indices where B does not have leading 1's. Construct the n-r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n-r$  of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle$$

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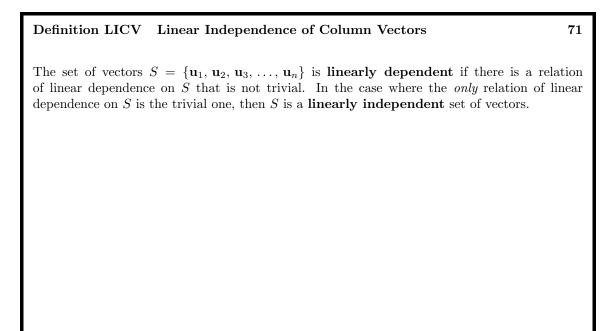
#### Definition RLDCV Relation of Linear Dependence for Column Vectors

70

Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ , a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

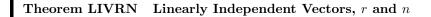
is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is the **trivial relation of linear dependence** on S.



# Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems

Suppose that A is an  $m \times n$  matrix and  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  has a unique solution.

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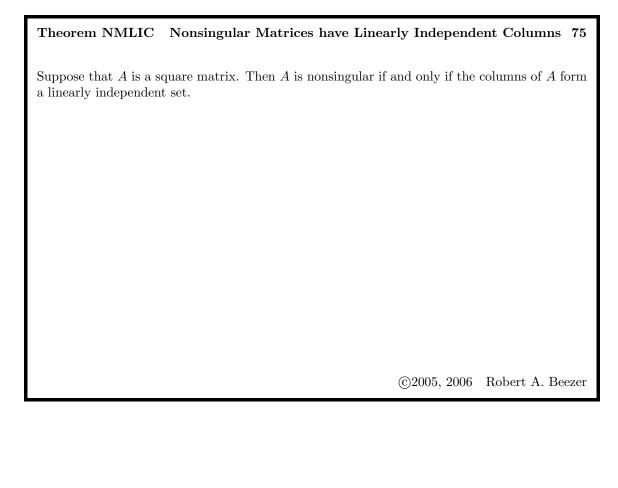
74

Suppose that A is an  $m \times n$  matrix and  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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### ${\bf Theorem~MVSLD~~More~Vectors~than~Size~implies~Linear~Dependence}$

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$  is the set of vectors in  $\mathbb{C}^m$ , and that n > m. Then S is a linearly dependent set.



#### Theorem NME2 Nonsingular Matrix Equivalences, Round 2

**76** 

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{0\}$ .
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A form a linearly independent set.

### Theorem BNS Basis for Null Spaces

77

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n-r$  of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set  $S = \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\}$ . Then

- 1.  $\mathcal{N}(A) = \langle S \rangle$ .
- 2. S is a linearly independent set.

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#### Theorem DLDS Dependency in Linearly Dependent Sets

**78** 

Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors. Then S is a linearly dependent set if and only if there is an index  $t, 1 \le t \le n$  such that  $\mathbf{u}_t$  is a linear combination of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$ .

## Theorem BS Basis of a Span

**79** 

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  is a set of column vectors. Define  $W = \langle S \rangle$  and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with  $D = \{d_1, d_2, d_3, \dots, d_r\}$  the set of column indices corresponding to the pivot columns of B. Then

- 1.  $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$  is a linearly independent set.
- 2.  $W = \langle T \rangle$ .

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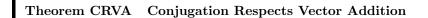
## Definition CCCV Complex Conjugate of a Column Vector

80

Suppose that **u** is a vector from  $\mathbb{C}^m$ . Then the conjugate of the vector,  $\overline{\mathbf{u}}$ , is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$



Suppose  ${\bf x}$  and  ${\bf y}$  are two vectors from  $\mathbb{C}^m.$  Then

$$\overline{\mathbf{x}+\mathbf{y}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$$

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# Theorem CRSM Conjugation Respects Vector Scalar Multiplication

82

Suppose **x** is a vector from  $\mathbb{C}^m$ , and  $\alpha \in \mathbb{C}$  is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

# Definition IP Inner Product

83

Given the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  the **inner product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity in  $\mathbb{C}$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[ \mathbf{u} \right]_1 \overline{\left[ \mathbf{v} \right]_1} + \left[ \mathbf{u} \right]_2 \overline{\left[ \mathbf{v} \right]_2} + \left[ \mathbf{u} \right]_3 \overline{\left[ \mathbf{v} \right]_3} + \dots + \left[ \mathbf{u} \right]_m \overline{\left[ \mathbf{v} \right]_m} = \sum_{i=1}^m \left[ \mathbf{u} \right]_i \overline{\left[ \mathbf{v} \right]_i}$$

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#### Theorem IPVA Inner Product and Vector Addition

84

Suppose  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ . Then

1. 
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

2. 
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

	Theorem IPSM	Inner Product	and Scalar	Multiplication
--	--------------	---------------	------------	----------------

Suppose  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ . Then

- 1.  $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- 2.  $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$

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# Theorem IPAC Inner Product is Anti-Commutative

86

Suppose that **u** and **v** are vectors in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ .

### Definition NV Norm of a Vector

87

The  $\bf{norm}$  of the vector  $\bf{u}$  is the scalar quantity in  $\mathbb C$ 

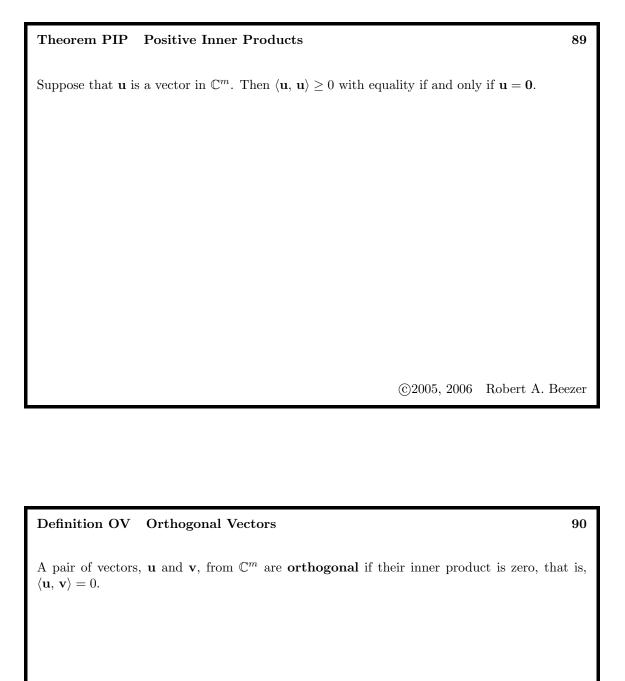
$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

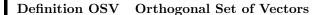
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#### 

88

Suppose that **u** is a vector in  $\mathbb{C}^m$ . Then  $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ .





Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors from  $\mathbb{C}^m$ . Then S is an **orthogonal** set if every pair of different vectors from S is orthogonal, that is  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  whenever  $i \neq j$ .

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#### Definition SUV Standard Unit Vectors

92

Let  $\mathbf{e}_j \in \mathbb{C}^m$ ,  $1 \leq j \leq m$  denote the column vectors defined by

$$\left[\mathbf{e}_{j}\right]_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j | \, 1 \le j \le m\}$$

is the set of standard unit vectors in  $\mathbb{C}^m$ .



Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

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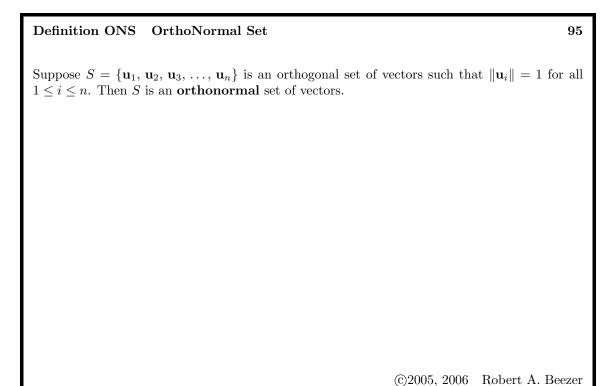
#### Theorem GSP Gram-Schmidt Procedure

94

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$  is a linearly independent set of vectors in  $\mathbb{C}^m$ . Define the vectors  $\mathbf{u}_i$ ,  $1 \le i \le p$  by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

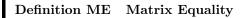
Then if  $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$ , then T is an orthogonal set of non-zero vectors, and  $\langle T \rangle = \langle S \rangle$ .



### Definition VSM Vector Space of $m \times n$ Matrices

96

The vector space  $M_{mn}$  is the set of all  $m \times n$  matrices with entries from the set of complex numbers.



The  $m \times n$  matrices A and B are **equal**, written A = B provided  $[A]_{ij} = [B]_{ij}$  for all  $1 \le i \le m$ , 1 < j < n.

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### Definition MA Matrix Addition

98

Given the  $m \times n$  matrices A and B, define the **sum** of A and B as an  $m \times n$  matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} 1 \le i \le m, \ 1 \le j \le n$$

Given the  $m \times n$  matrix A and the scalar  $\alpha \in \mathbb{C}$ , the **scalar multiple** of A is an  $m \times n$  matrix, written  $\alpha A$  and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

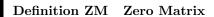
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#### Theorem VSPM Vector Space Properties of Matrices

100

Suppose that  $M_{mn}$  is the set of all  $m \times n$  matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If  $A, B \in M_{mn}$ , then  $A + B \in M_{mn}$ .
- SCM Scalar Closure, Matrices If  $\alpha \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha A \in M_{mn}$ .
- CM Commutativity, Matrices If  $A, B \in M_{mn}$ , then A + B = B + A.
- AAM Additive Associativity, Matrices If  $A, B, C \in M_{mn}$ , then A + (B + C) = (A + B) + C.
- **ZM Zero Vector, Matrices** There is a matrix,  $\mathcal{O}$ , called the **zero matrix**, such that  $A + \mathcal{O} = A$  for all  $A \in M_{mn}$ .
- AIM Additive Inverses, Matrices If  $A \in M_{mn}$ , then there exists a matrix  $-A \in M_{mn}$  so that  $A + (-A) = \mathcal{O}$ .
- SMAM Scalar Multiplication Associativity, Matrices If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha(\beta A) = (\alpha \beta)A$ .
- DMAM Distributivity across Matrix Addition, Matrices If  $\alpha \in \mathbb{C}$  and  $A, B \in M_{mn}$ , then  $\alpha(A+B) = \alpha A + \alpha B$ .
- DSAM Distributivity across Scalar Addition, Matrices If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $(\alpha + \beta)A = \alpha A + \beta A$ .
- OM One Matrices If  $A \in M_{min}$  then 1A A



The  $m \times n$  zero matrix is written as  $\mathcal{O} = \mathcal{O}_{m \times n}$  and defined by  $[\mathcal{O}]_{ij} = 0$ , for all  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ .

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# Definition TM Transpose of a Matrix

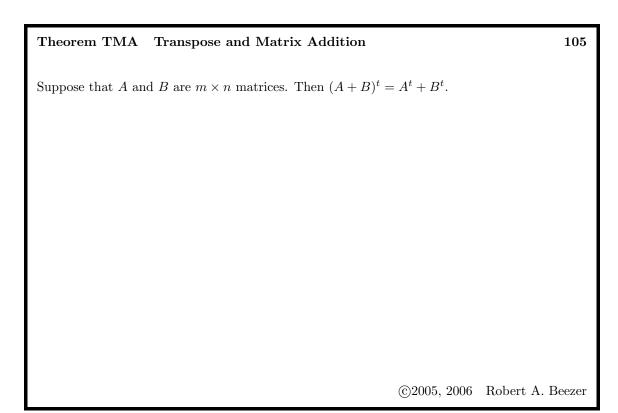
102

Given an  $m \times n$  matrix A, its **transpose** is the  $n \times m$  matrix  $A^t$  given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix		103
The matrix A is <b>symmetric</b> if $A = A^t$ .		
The matrix A is symmetric if $A = A$ .		
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	6)2000, 2000	Robert A. Beezer
Theorem SMS Symmetric Matrices are Square		104
Theorem Sivis Symmetric Matrices are Square		104

Suppose that A is a symmetric matrix. Then A is square.



# Theorem TMSM Transpose and Matrix Scalar Multiplication

106

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $(\alpha A)^t = \alpha A^t$ .



Suppose that A is an  $m \times n$  matrix. Then  $\left(A^{t}\right)^{t} = A$ .

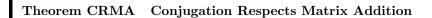
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# Definition CCM Complex Conjugate of a Matrix

108

Suppose A is an  $m \times n$  matrix. Then the **conjugate** of A, written  $\overline{A}$  is an  $m \times n$  matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$



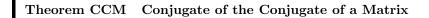
110

Suppose that A and B are  $m \times n$  matrices. Then  $\overline{A+B} = \overline{A} + \overline{B}$ .

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# Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $\overline{\alpha A} = \overline{\alpha} \overline{A}$ .



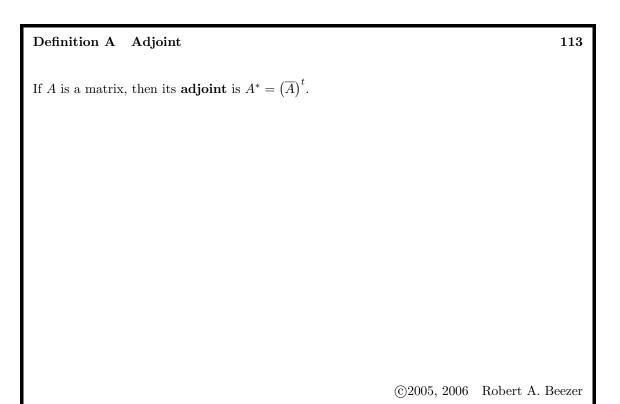
Suppose that A is an  $m \times n$  matrix. Then  $\overline{\overline{(A)}} = A$ .

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# Theorem MCT Matrix Conjugation and Transposes

112

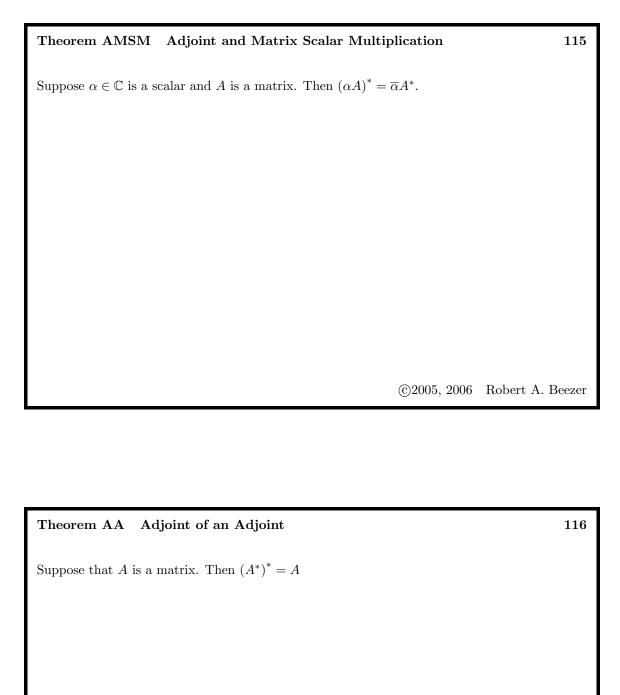
Suppose that A is an  $m \times n$  matrix. Then  $\overline{(A^t)} = \left(\overline{A}\right)^t$ .



# Theorem AMA Adjoint and Matrix Addition

114

Suppose A and B are matrices of the same size. Then  $(A + B)^* = A^* + B^*$ .





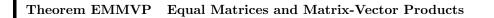
Suppose A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$  and  $\mathbf{u}$  is a vector of size n. Then the **matrix-vector product** of A with  $\mathbf{u}$  is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

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## Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 118

The set of solutions to the linear system  $\mathcal{LS}(A, \mathbf{b})$  equals the set of solutions for  $\mathbf{x}$  in the vector equation  $A\mathbf{x} = \mathbf{b}$ .



Suppose that A and B are  $m \times n$  matrices such that  $A\mathbf{x} = B\mathbf{x}$  for every  $\mathbf{x} \in \mathbb{C}^n$ . Then A = B.

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## Definition MM Matrix Multiplication

**120** 

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix with columns  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ . Then the **matrix product** of A with B is the  $m \times p$  matrix where column i is the matrix-vector product  $A\mathbf{B}_i$ . Symbolically,

$$AB = A \left[ \mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[ A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

#### Theorem EMP **Entries of Matrix Products**

121

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then for  $1 \le i \le m$ ,  $1 \le j \le p$ , the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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## Theorem MMZM Matrix Multiplication and the Zero Matrix

122

Suppose A is an  $m \times n$  matrix. Then

- 1.  $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$ 2.  $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

# Theorem MMIM Matrix Multiplication and Identity Matrix

123

124

Suppose A is an  $m \times n$  matrix. Then

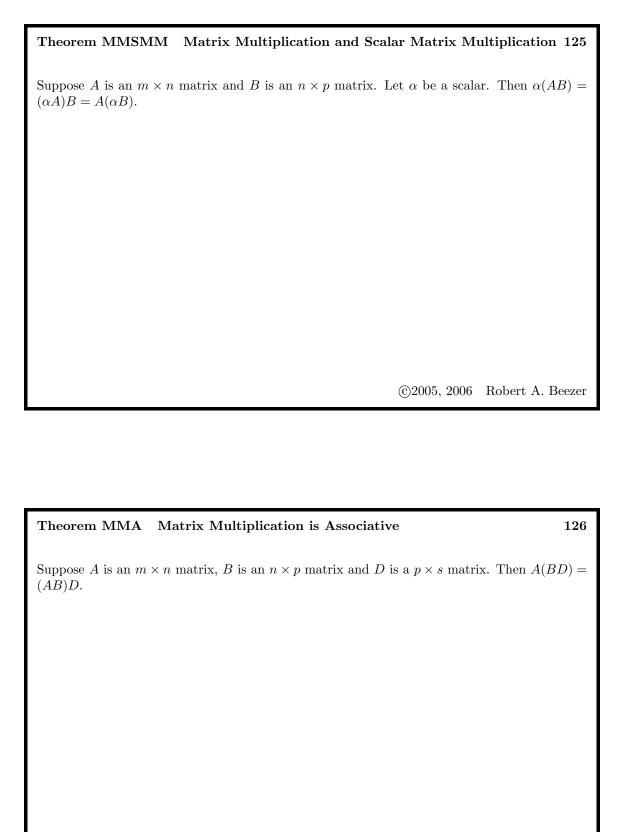
- 1.  $AI_n = A$
- $2. \quad I_m A = A$

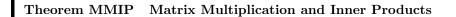
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## Theorem MMDAA Matrix Multiplication Distributes Across Addition

Suppose A is an  $m \times n$  matrix and B and C are  $n \times p$  matrices and D is a  $p \times s$  matrix. Then

- 1. A(B+C) = AB + AC
- $2. \quad (B+C)D = BD + CD$





**128** 

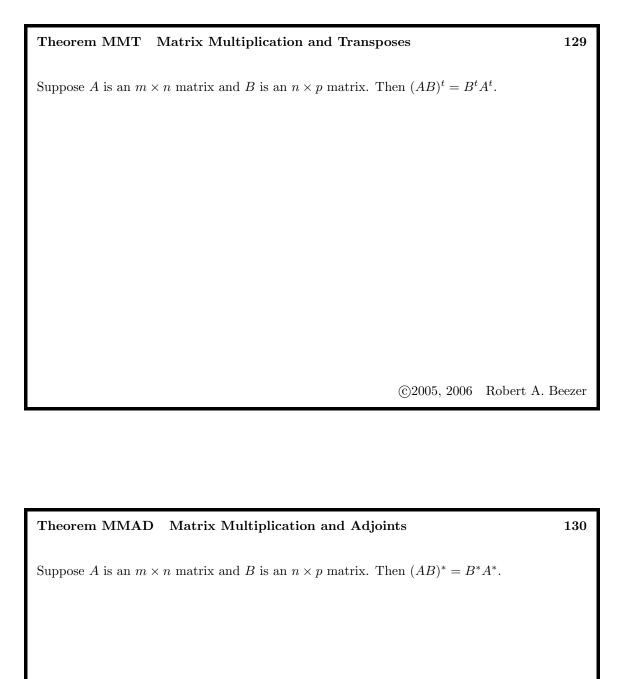
If we consider the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  as  $m \times 1$  matrices then

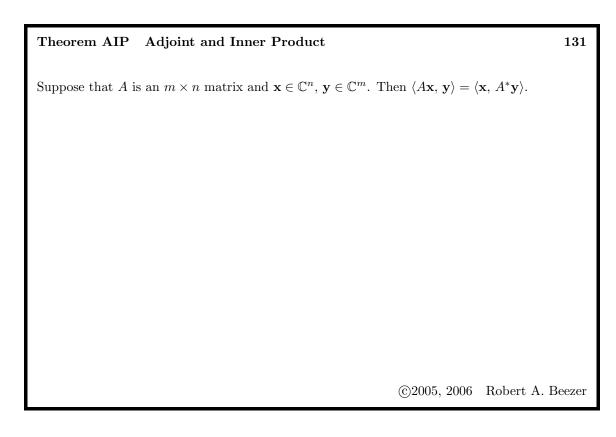
$$\langle \mathbf{u}, \, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

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# Theorem MMCC Matrix Multiplication and Complex Conjugation

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then  $\overline{AB} = \overline{A} \overline{B}$ .

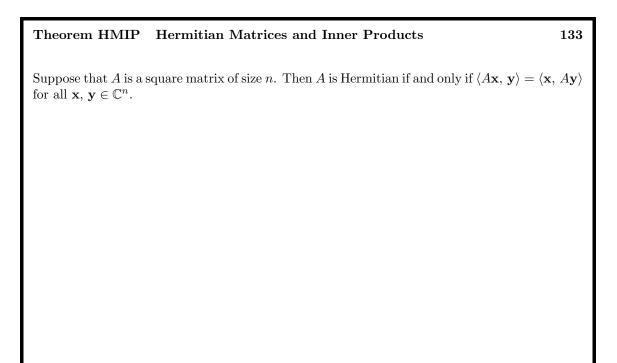




# Definition HM Hermitian Matrix

132

The square matrix A is **Hermitian** (or **self-adjoint**) if  $A = A^*$ .



### Definition MI Matrix Inverse

134

Suppose A and B are square matrices of size n such that  $AB = I_n$  and  $BA = I_n$ . Then A is **invertible** and B is the **inverse** of A. In this situation, we write  $B = A^{-1}$ .

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#### Theorem TTMI Two-by-Two Matrix Inverse

135

136

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

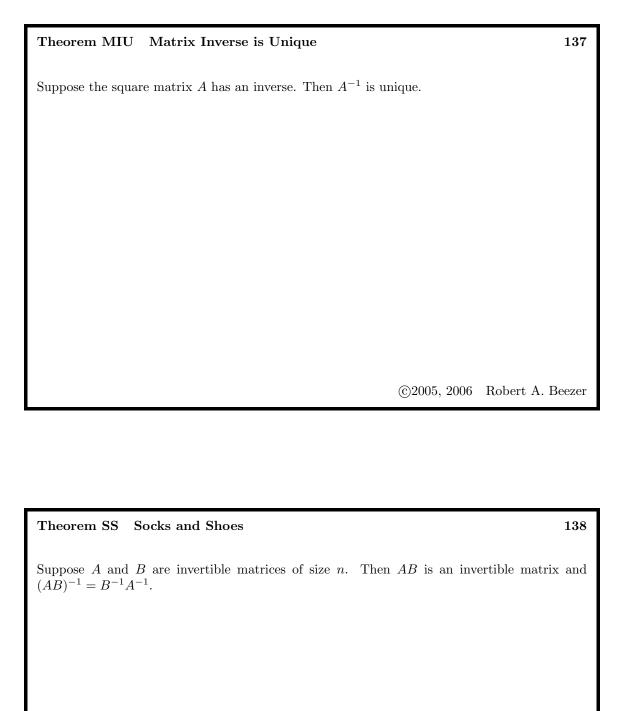
Then A is invertible if and only if  $ad - bc \neq 0$ . When A is invertible, then

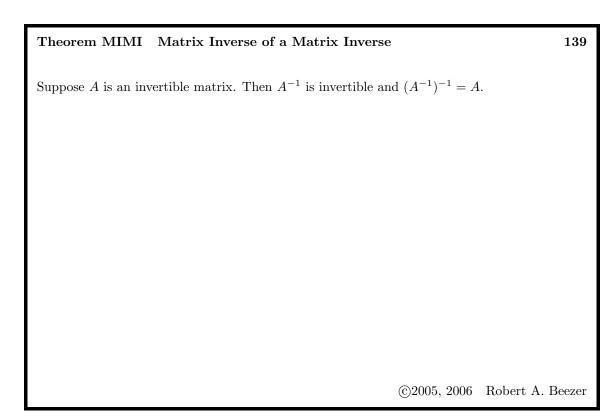
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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### Theorem CINM Computing the Inverse of a Nonsingular Matrix

Suppose A is a nonsingular square matrix of size n. Create the  $n \times 2n$  matrix M by placing the  $n \times n$  identity matrix  $I_n$  to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then  $AJ = I_n$ .





# Theorem MIT Matrix Inverse of a Transpose

140

Suppose A is an invertible matrix. Then  $A^t$  is invertible and  $(A^t)^{-1} = (A^{-1})^t$ .



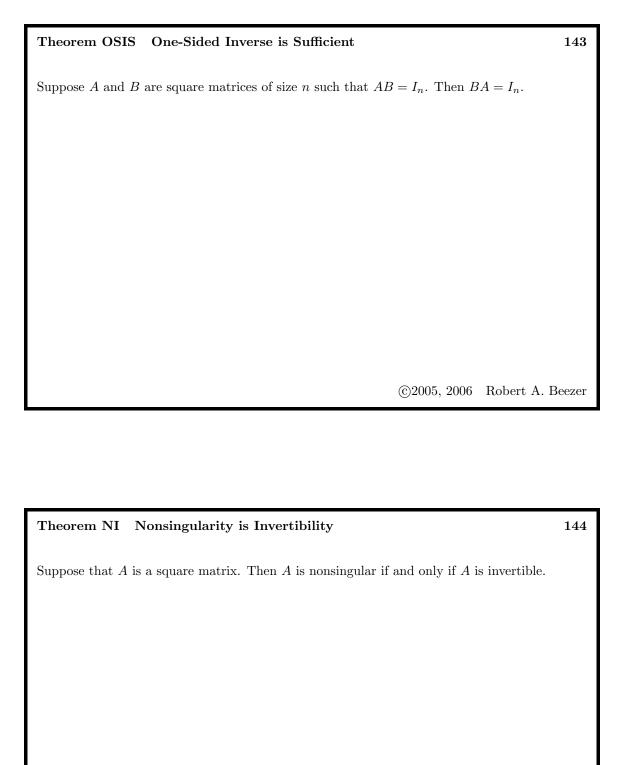
Suppose A is an invertible matrix and  $\alpha$  is a nonzero scalar. Then  $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$  and  $\alpha A$  is invertible.

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# Theorem NPNT Nonsingular Product has Nonsingular Terms

142

Suppose that A and B are square matrices of size n. The product AB is nonsingular if and only if A and B are both nonsingular.



#### Theorem NME3 Nonsingular Matrix Equivalences, Round 3

145

Suppose that A is a square matrix of size n. The following are equivalent.

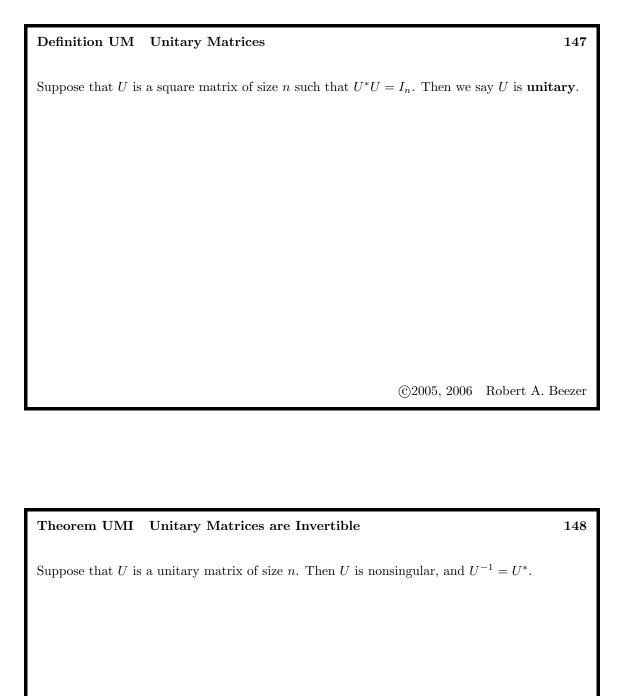
- 1. A is nonsingular.
- $2.\ A$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

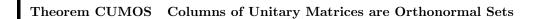
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#### Theorem SNCM Solution with Nonsingular Coefficient Matrix

146

Suppose that A is nonsingular. Then the unique solution to  $\mathcal{LS}(A, \mathbf{b})$  is  $A^{-1}\mathbf{b}$ .





Suppose that A is a square matrix of size n with columns  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ . Then A is a unitary matrix if and only if S is an orthonormal set.

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# Theorem UMPIP Unitary Matrices Preserve Inner Products

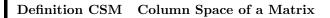
150

Suppose that U is a unitary matrix of size n and  $\mathbf{u}$  and  $\mathbf{v}$  are two vectors from  $\mathbb{C}^n$ . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

and

$$||U\mathbf{v}|| = ||\mathbf{v}||$$



Suppose that A is an  $m \times n$  matrix with columns  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ . Then the **column space** of A, written  $\mathcal{C}(A)$ , is the subset of  $\mathbb{C}^m$  containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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## Theorem CSCS Column Spaces and Consistent Systems

152

Suppose A is an  $m \times n$  matrix and **b** is a vector of size m. Then  $\mathbf{b} \in \mathcal{C}(A)$  if and only if  $\mathcal{LS}(A, \mathbf{b})$  is consistent.

## Theorem BCS Basis of the Column Space

153

Suppose that A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the set of column indices where B has leading 1's. Let  $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$ . Then

- 1. T is a linearly independent set.
- 2.  $C(A) = \langle T \rangle$ .

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## Theorem CSNM Column Space of a Nonsingular Matrix

**154** 

Suppose A is a square matrix of size n. Then A is nonsingular if and only if  $\mathcal{C}(A) = \mathbb{C}^n$ .

#### Theorem NME4 Nonsingular Matrix Equivalences, Round 4

155

Suppose that A is a square matrix of size n. The following are equivalent.

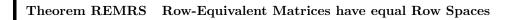
- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .

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## Definition RSM Row Space of a Matrix

156

Suppose A is an  $m \times n$  matrix. Then the **row space** of A,  $\mathcal{R}(A)$ , is the column space of  $A^t$ , i.e.  $\mathcal{R}(A) = \mathcal{C}(A^t)$ .



Suppose A and B are row-equivalent matrices. Then  $\mathcal{R}(A) = \mathcal{R}(B)$ .

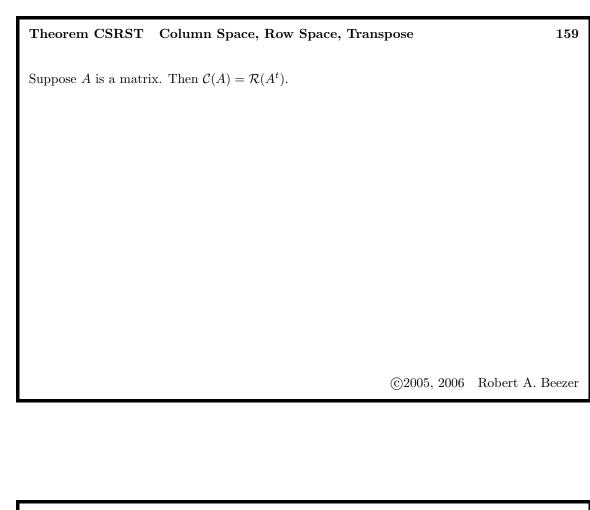
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# Theorem BRS Basis for the Row Space

158

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of  $B^t$ . Then

- 1.  $\mathcal{R}(A) = \langle S \rangle$ .
- 2. S is a linearly independent set.



# Definition LNS Left Null Space

160

Suppose A is an  $m \times n$  matrix. Then the **left null space** is defined as  $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$ .

Suppose A is an  $m \times n$  matrix. Add m new columns to A that together equal an  $m \times m$  identity matrix to form an  $m \times (n+m)$  matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the  $m \times n$  matrix formed from the first n columns of N and let J denote the  $m \times m$  matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the  $r \times n$  matrix formed from all of the non-zero rows of B. Let K be the  $r \times m$  matrix formed from the first r rows of J, while L will be the  $(m-r) \times m$  matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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#### Theorem PEEF Properties of Extended Echelon Form

162

Suppose that A is an  $m \times n$  matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^m$ , then  $A\mathbf{x} = \mathbf{y}$  if and only if  $B\mathbf{x} = J\mathbf{y}$ .
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

Suppose A is an  $m \times n$  matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C,  $\mathcal{N}(A) = \mathcal{N}(C)$ .
- 2. The row space of A is the row space of C,  $\mathcal{R}(A) = \mathcal{R}(C)$ .
- 3. The column space of A is the null space of L,  $C(A) = \mathcal{N}(L)$ .
- 4. The left null space of A is the row space of L,  $\mathcal{L}(A) = \mathcal{R}(L)$ .

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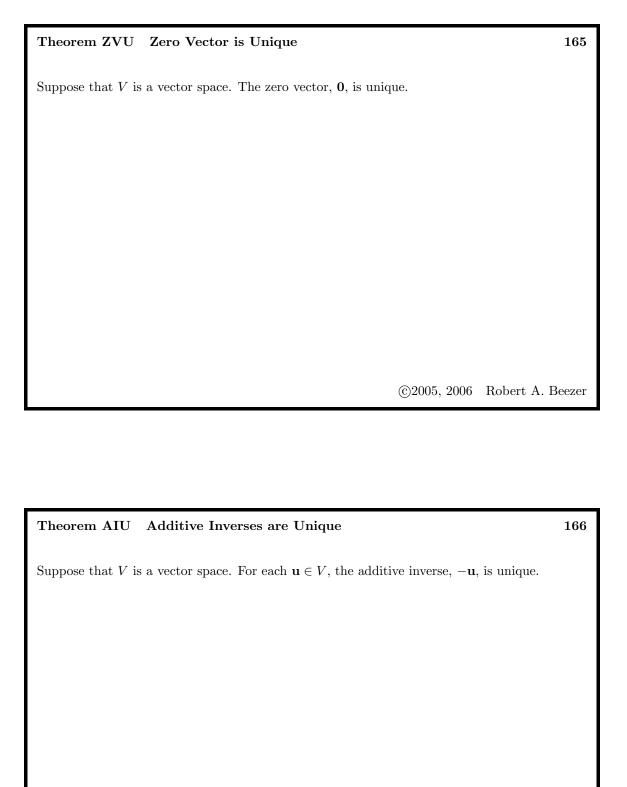
#### Definition VS Vector Space

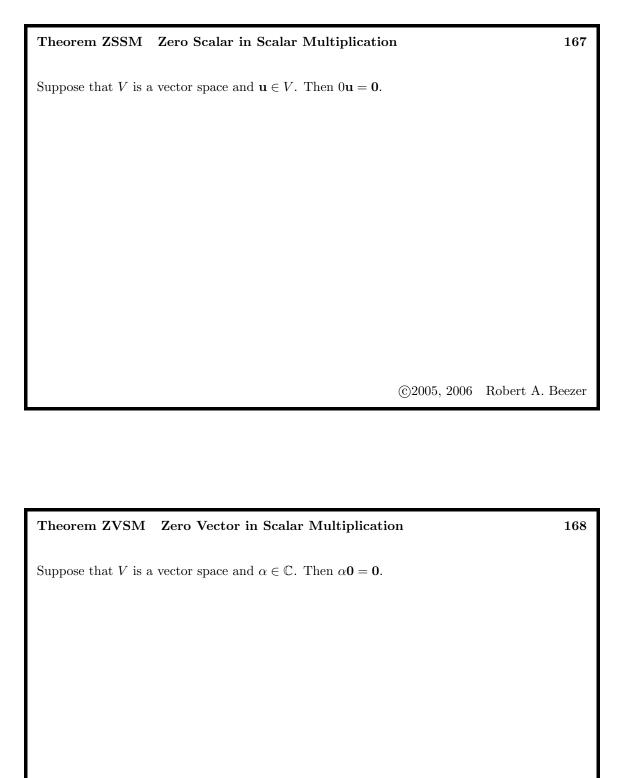
164

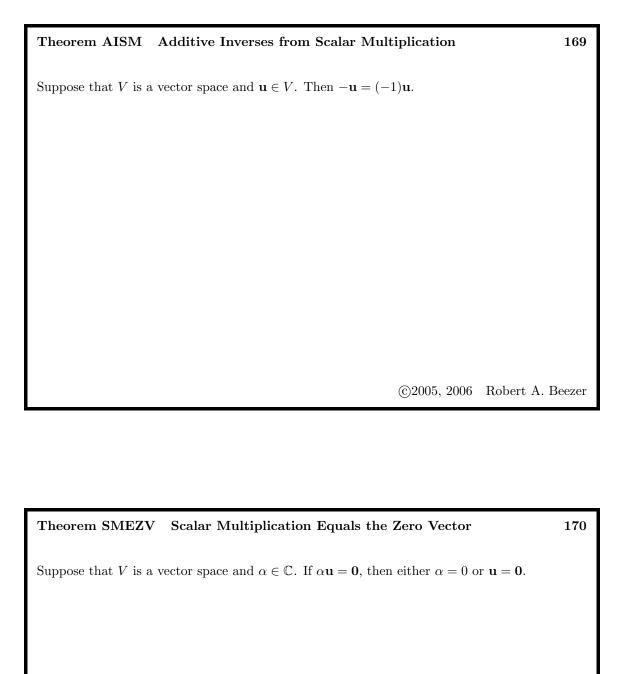
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** over  $\mathbb{C}$  if the following ten properties hold.

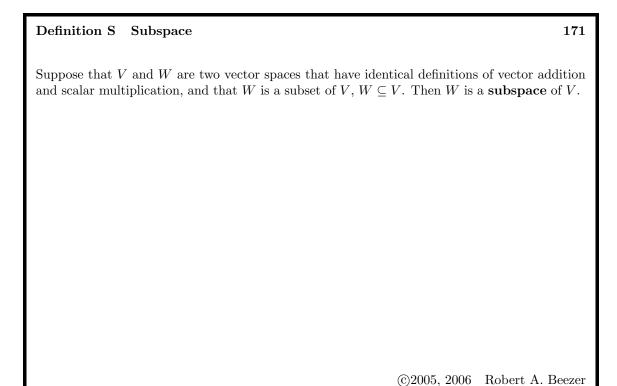
- AC Additive Closure If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} \in V$ .
- SC Scalar Closure If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha \mathbf{u} \in V$ .
- C Commutativity If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AA Additive Associativity If  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w} \in V$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in V$ .
- AI Additive Inverses If  $\mathbf{u} \in V$ , then there exists a vector  $-\mathbf{u} \in V$  so that  $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ .
- SMA Scalar Multiplication Associativity If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVA Distributivity across Vector Addition If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSA Distributivity across Scalar Addition If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .
- One If  $u \in V$  then 1u = u

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.







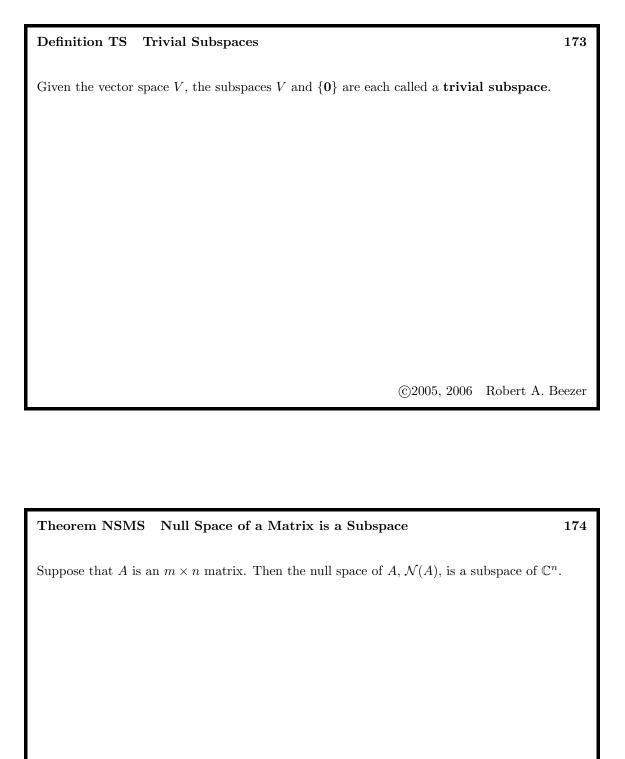


# Theorem TSS Testing Subsets for Subspaces

172

Suppose that V is a vector space and W is a subset of V,  $W \subseteq V$ . Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty,  $W \neq \emptyset$ .
- 2. If  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ , then  $\mathbf{x} + \mathbf{y} \in W$ .
- 3. If  $\alpha \in \mathbb{C}$  and  $\mathbf{x} \in W$ , then  $\alpha \mathbf{x} \in W$ .



### Definition LC Linear Combination

175

Suppose that V is a vector space. Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

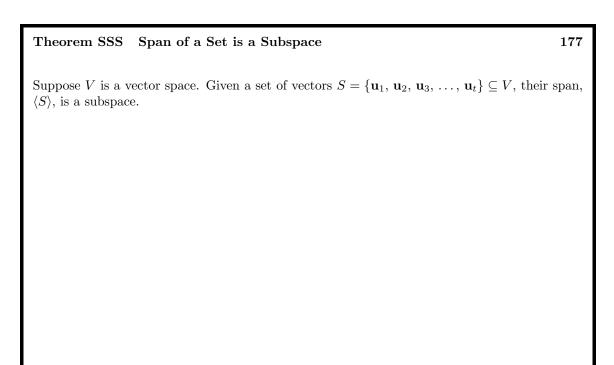
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#### Definition SS Span of a Set

176

Suppose that V is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$



# Theorem CSMS Column Space of a Matrix is a Subspace

178

Suppose that A is an  $m \times n$  matrix. Then  $\mathcal{C}(A)$  is a subspace of  $\mathbb{C}^m$ .

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Theorem RSMS Row Space of a Matrix is a Subspace	179
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$ .	
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Theorem LNSMS Left Null Space of a Matrix is a Subspace	180
Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$ .	

# Definition RLD Relation of Linear Dependence

181

Suppose that V is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_n\}$ , an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

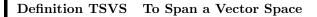
is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is a **trivial relation of linear dependence** on S.

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### Definition LI Linear Independence

182

Suppose that V is a vector space. The set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.



Suppose V is a vector space. A subset S of V is a **spanning set** for V if  $\langle S \rangle = V$ . In this case, we also say S **spans** V.

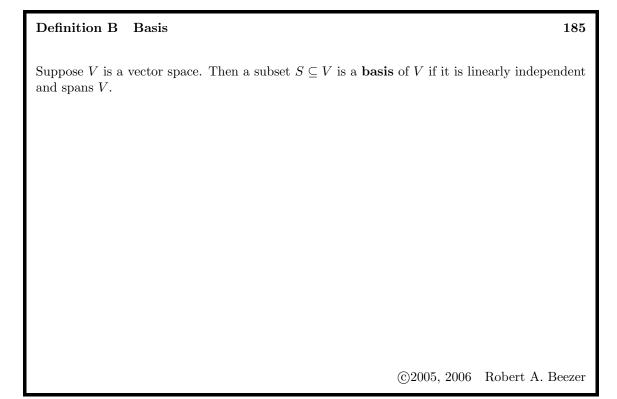
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## Theorem VRRB Vector Representation Relative to a Basis

184

Suppose that V is a vector space and  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  is a linearly independent set that spans V. Let  $\mathbf{w}$  be any vector in V. Then there exist *unique* scalars  $a_1, a_2, a_3, \dots, a_m$  such that

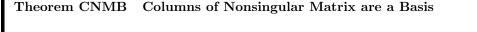
$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$



## Theorem SUVB Standard Unit Vectors are a Basis

186

The set of standard unit vectors for  $\mathbb{C}^m$  (Definition SUV),  $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i | 1 \leq i \leq m\}$  is a basis for the vector space  $\mathbb{C}^m$ .



Suppose that A is a square matrix of size m. Then the columns of A are a basis of  $\mathbb{C}^m$  if and only if A is nonsingular.

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# Theorem NME5 Nonsingular Matrix Equivalences, Round 5

188

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .

#### Theorem COB Coordinates and Orthonormal Bases

189

190

Suppose that  $B = \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_p\}$  is an orthonormal basis of the subspace W of  $\mathbb{C}^m$ . For any  $\mathbf{w} \in W$ ,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$

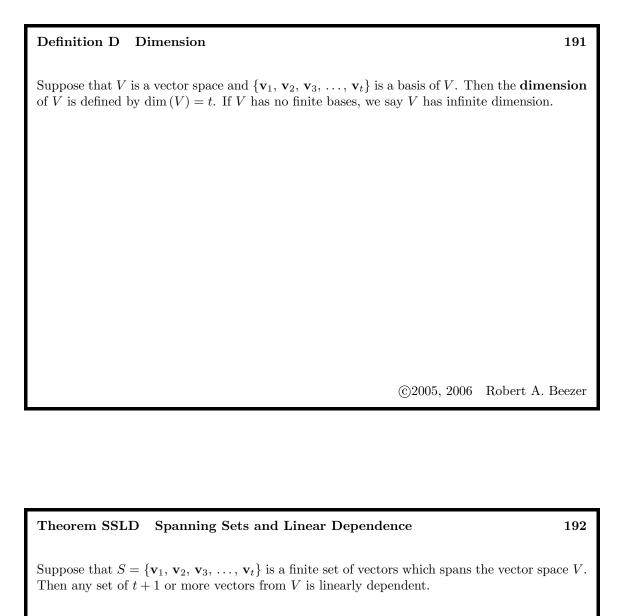
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## 

Let A be an  $n \times n$  matrix and  $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$  be an orthonormal basis of  $\mathbb{C}^n$ . Define

$$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$$

Then A is a unitary matrix if and only if C is an orthonormal basis of  $\mathbb{C}^n$ .



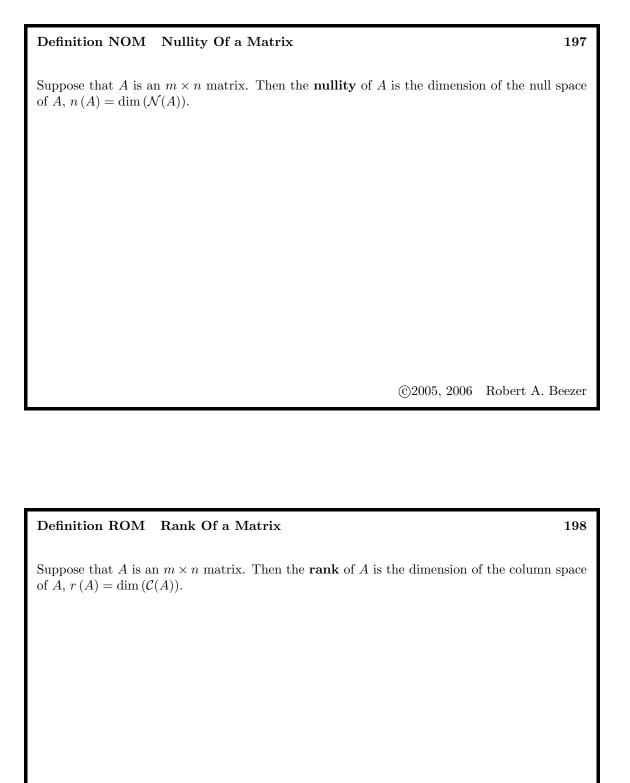
Theorem bis bases have identical Sizes	195
Suppose that $V$ is a vector space with a finite basis $B$ and a second basis	C. Then $B$ and $C$
have the same size.	
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Theorem DCM Dimension of $\mathbb{C}^m$	194
The dimension of $\mathbb{C}^m$ (Example VSCV) is $m$ .	

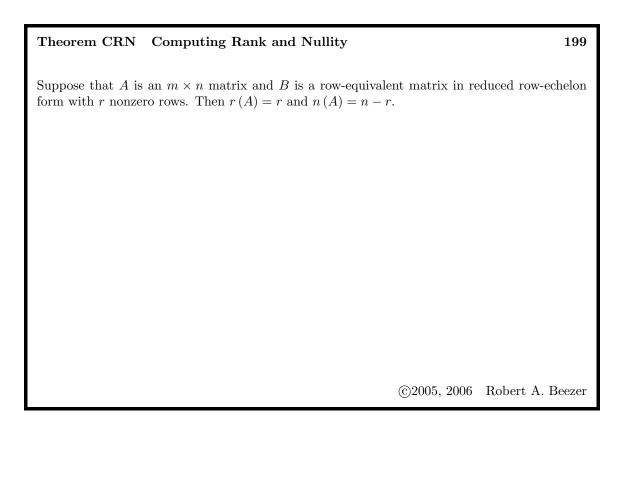
	195
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# Theorem DM Dimension of $M_{mn}$

196

The dimension of  $M_{mn}$  (Example VSM) is mn.





# Theorem RPNC Rank Plus Nullity is Columns

200

Suppose that A is an  $m \times n$  matrix. Then r(A) + n(A) = n.

#### Theorem RNNM Rank and Nullity of a Nonsingular Matrix

201

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

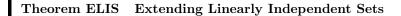
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#### Theorem NME6 Nonsingular Matrix Equivalences, Round 6

202

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.



Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose  $\mathbf{w}$  is a vector such that  $\mathbf{w} \notin \langle S \rangle$ . Then the set  $S' = S \cup \{\mathbf{w}\}$  is linearly independent.

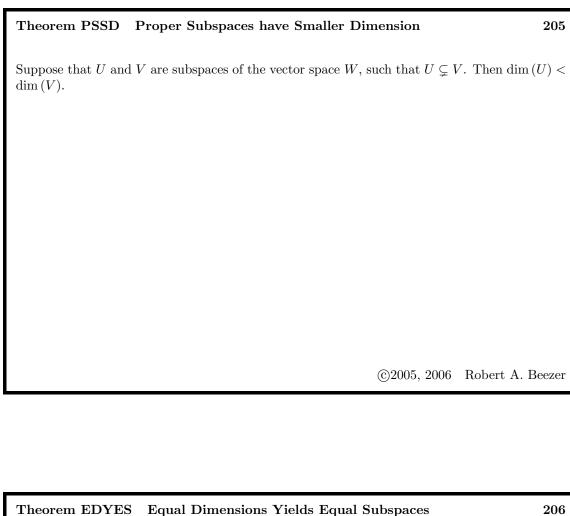
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#### Theorem G Goldilocks

204

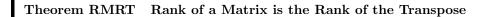
Suppose that V is a vector space of dimension t. Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.



# Theorem EDYES Equal Dimensions Yields Equal Subspaces

Suppose that U and V are subspaces of the vector space W, such that  $U \subseteq V$  and dim (U) = $\dim(V)$ . Then U = V.



Suppose A is an  $m \times n$  matrix. Then  $r(A) = r(A^t)$ .

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# Theorem DFS Dimensions of Four Subspaces

208

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim  $(\mathcal{N}(A)) = n r$
- 2. dim  $(\mathcal{C}(A)) = r$
- 3. dim  $(\mathcal{R}(A)) = r$
- 4. dim  $(\mathcal{L}(A)) = m r$

#### Definition DS Direct Sum

209

Suppose that V is a vector space with two subspaces U and W such that for every  $\mathbf{v} \in V$ ,

- 1. There exists vectors  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  such that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If  $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$  and  $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$  where  $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$  then  $\mathbf{u}_1 = \mathbf{u}_2$  and  $\mathbf{w}_1 = \mathbf{w}_2$ .

Then V is the **direct sum** of U and W and we write  $V = U \oplus W$ .

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#### Theorem DSFB Direct Sum From a Basis

210

Suppose that V is a vector space with a basis  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  and  $m \leq n$ . Define

$$U = \langle \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_m\} \rangle \qquad W = \langle \{\mathbf{v}_{m+1}, \, \mathbf{v}_{m+2}, \, \mathbf{v}_{m+3}, \, \dots, \, \mathbf{v}_n\} \rangle$$

Then  $V = U \oplus W$ .



Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that  $V = U \oplus W$ .

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#### Theorem DSZV Direct Sums and Zero Vectors

212

Suppose U and W are subspaces of the vector space V. Then  $V = U \oplus W$  if and only if

- 1. For every  $\mathbf{v} \in V$ , there exists vectors  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  such that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$ .
- 2. Whenever  $\mathbf{0} = \mathbf{u} + \mathbf{w}$  with  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  then  $\mathbf{u} = \mathbf{w} = \mathbf{0}$ .

#### Theorem DSZI Direct Sums and Zero Intersection

213

Suppose U and W are subspaces of the vector space V. Then  $V=U\oplus W$  if and only if

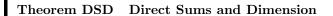
- 1. For every  $\mathbf{v} \in V$ , there exists vectors  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  such that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$ .
- 2.  $U \cap W = \{0\}.$

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#### Theorem DSLI Direct Sums and Linear Independence

214

Suppose U and W are subspaces of the vector space V with  $V = U \oplus W$ . Suppose that R is a linearly independent subset of U and S is a linearly independent subset of W. Then  $R \cup S$  is a linearly independent subset of V.



Suppose U and W are subspaces of the vector space V with  $V=U\oplus W$ . Then  $\dim(V)=\dim(U)+\dim(W)$ .

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## Theorem RDS Repeated Direct Sums

216

Suppose V is a vector space with subspaces U and W with  $V=U\oplus W$ . Suppose that X and Y are subspaces of W with  $W=X\oplus Y$ . Then  $V=U\oplus X\oplus Y$ .

1. For  $i \neq j$ ,  $E_{i,j}$  is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For  $\alpha \neq 0$ ,  $E_i(\alpha)$  is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For  $i \neq j$ ,  $E_{i,j}(\alpha)$  is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

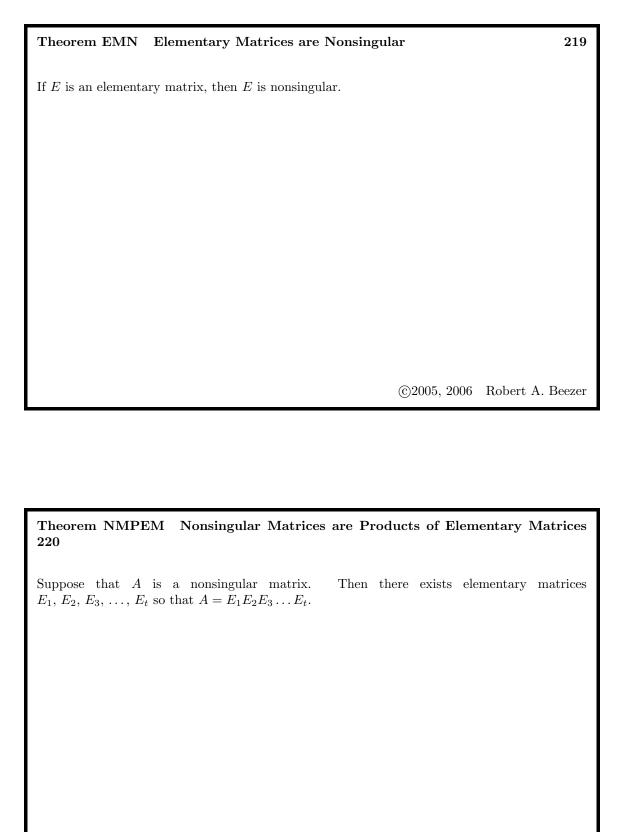
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#### Theorem EMDRO Elementary Matrices Do Row Operations

218

Suppose that A is an  $m \times n$  matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

- 1. If the row operation swaps rows i and j, then  $B = E_{i,j}A$ .
- 2. If the row operation multiplies row i by  $\alpha$ , then  $B = E_i(\alpha) A$ .
- 3. If the row operation multiplies row i by  $\alpha$  and adds the result to row j, then  $B = E_{i,j}(\alpha) A$ .





Suppose that A is an  $m \times n$  matrix. Then the **submatrix** A(i|j) is the  $(m-1) \times (n-1)$  matrix obtained from A by removing row i and column j.

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#### Definition DM Determinant of a Matrix

222

Suppose A is a square matrix. Then its **determinant**,  $\det(A) = |A|$ , is an element of  $\mathbb{C}$  defined recursively by:

If A is a  $1 \times 1$  matrix, then  $\det(A) = [A]_{11}$ .

If A is a matrix of size n with  $n \geq 2$ , then

$$\begin{split} \det{(A)} &= [A]_{11} \det{(A\,(1|1))} - [A]_{12} \det{(A\,(1|2))} + [A]_{13} \det{(A\,(1|3))} - \\ & [A]_{14} \det{(A\,(1|4))} + \dots + (-1)^{n+1} \, [A]_{1n} \det{(A\,(1|n))} \end{split}$$

Theorem DMST	Determinant	of Matricos	of Sizo	Two
i neorem i divis i	Determinant	or matrices	or Size	$1\mathbf{wo}$

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $\det{(A)} = ad - bc$ 

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# Theorem DER Determinant Expansion about Rows

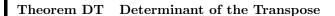
224

223

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[ A \right]_{i1} \det{(A\left(i|1\right))} + (-1)^{i+2} \left[ A \right]_{i2} \det{(A\left(i|2\right))} \\ &+ (-1)^{i+3} \left[ A \right]_{i3} \det{(A\left(i|3\right))} + \dots + (-1)^{i+n} \left[ A \right]_{in} \det{(A\left(i|n\right))} \qquad 1 \leq i \leq n \end{split}$$

which is known as **expansion** about row i.



Suppose that A is a square matrix. Then  $\det(A^t) = \det(A)$ .

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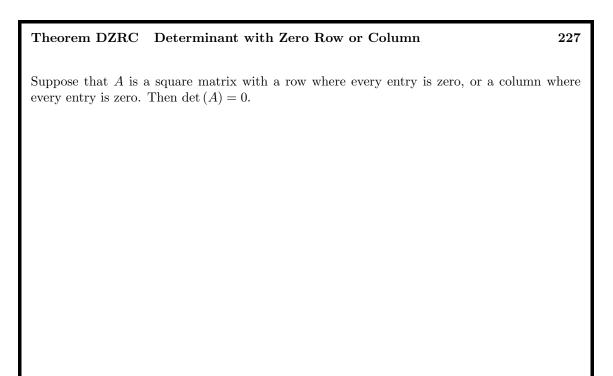
## Theorem DEC Determinant Expansion about Columns

226

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[ A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[ A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[ A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[ A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

which is known as **expansion** about column j.

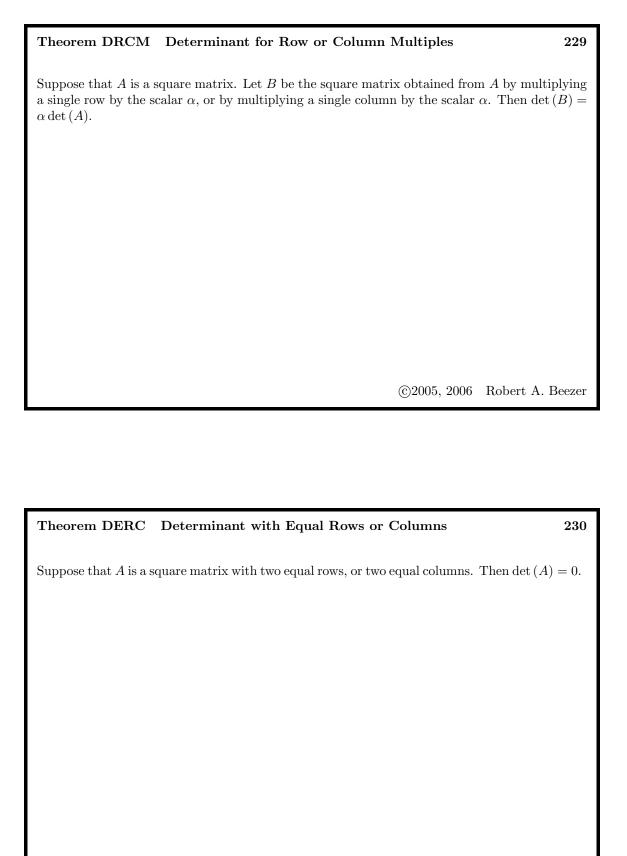


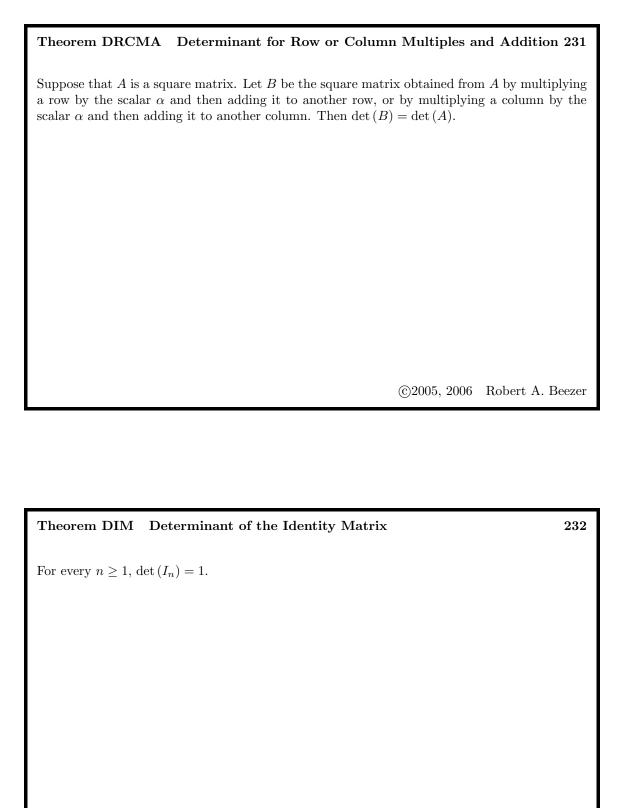
# Theorem DRCS Determinant for Row or Column Swap

228

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then  $\det(B) = -\det(A)$ .

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For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

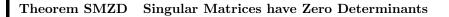
- 1.  $\det(E_{i,j}) = -1$
- 2.  $\det (E_i(\alpha)) = \alpha$
- 3.  $\det (E_{i,j}(\alpha)) = 1$

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# Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 234

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det(EA) = \det(E)\det(A)$$



Let A be a square matrix. Then A is singular if and only if det(A) = 0.

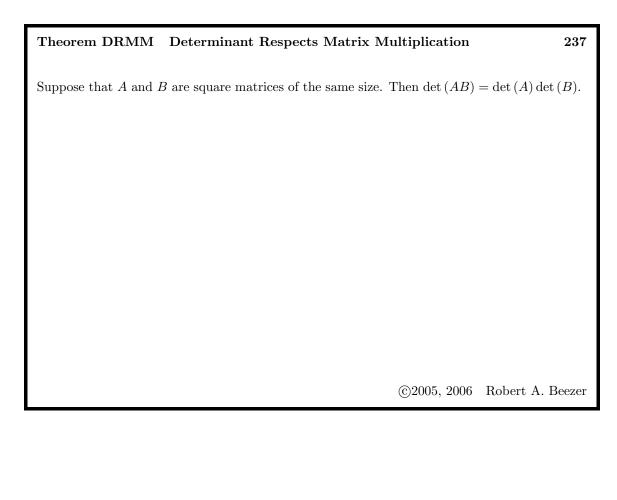
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# Theorem NME7 Nonsingular Matrix Equivalences, Round 7

236

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $det(A) \neq 0$ .

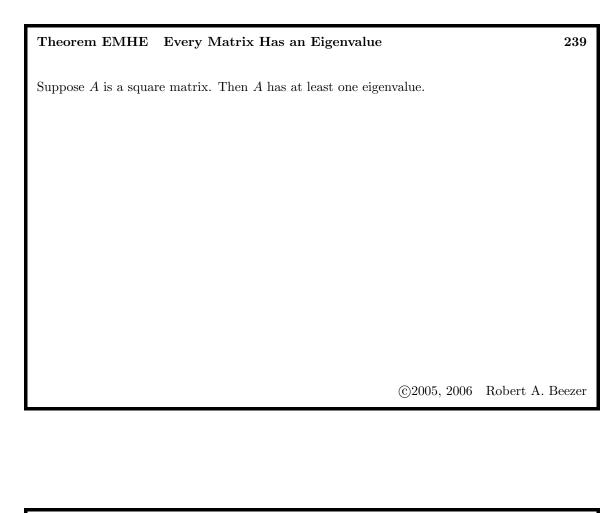


# Definition EEM Eigenvalues and Eigenvectors of a Matrix

238

Suppose that A is a square matrix of size n,  $\mathbf{x} \neq \mathbf{0}$  is a vector in  $\mathbb{C}^n$ , and  $\lambda$  is a scalar in  $\mathbb{C}$ . Then we say  $\mathbf{x}$  is an **eigenvector** of A with **eigenvalue**  $\lambda$  if

$$A\mathbf{x} = \lambda \mathbf{x}$$

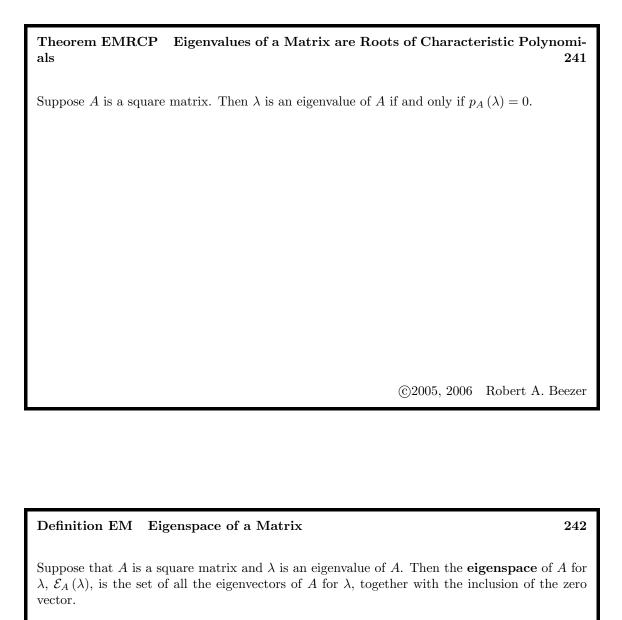


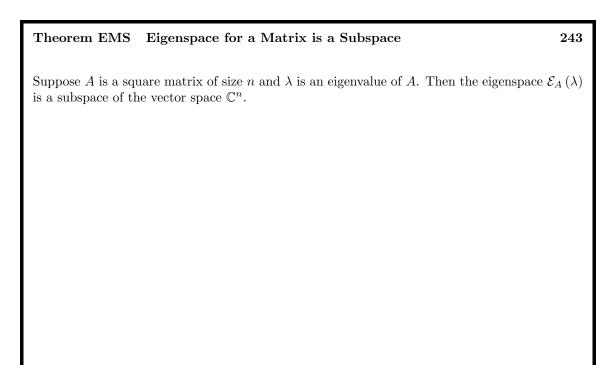
## Definition CP Characteristic Polynomial

240

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial  $p_{A}(x)$  defined by

$$p_A(x) = \det(A - xI_n)$$





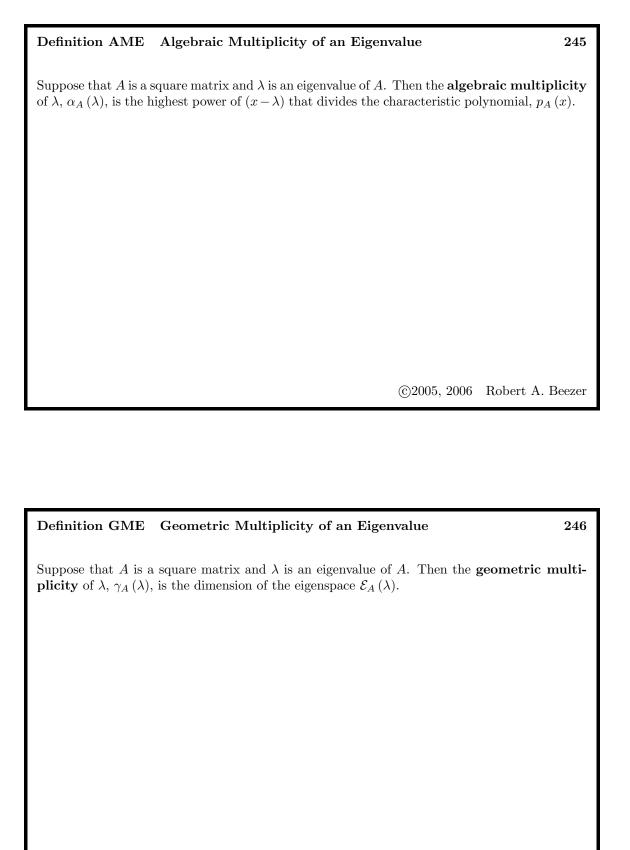
# Theorem EMNS Eigenspace of a Matrix is a Null Space

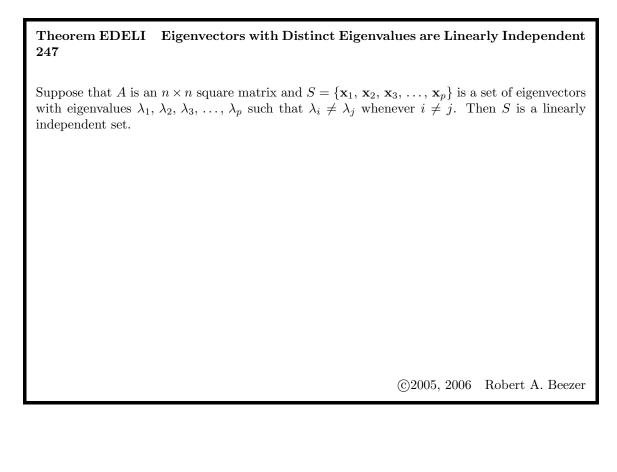
244

Suppose A is a square matrix of size n and  $\lambda$  is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

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# Theorem SMZE Singular Matrices have Zero Eigenvalues

248

Suppose A is a square matrix. Then A is singular if and only if  $\lambda = 0$  is an eigenvalue of A.

#### Theorem NME8 Nonsingular Matrix Equivalences, Round 8

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .
- 12.  $\lambda = 0$  is not an eigenvalue of A.

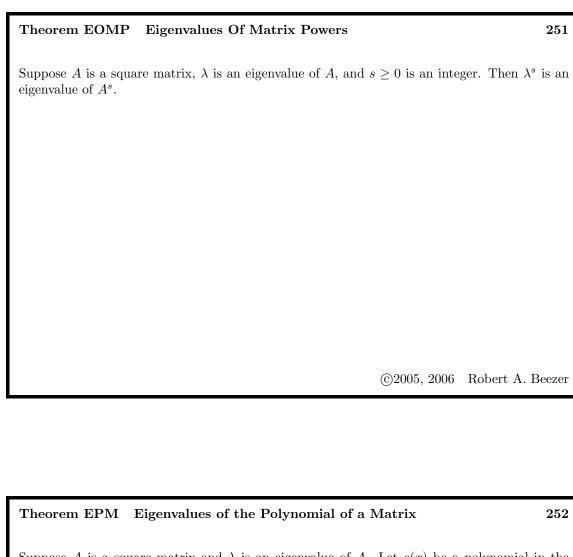
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## Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix

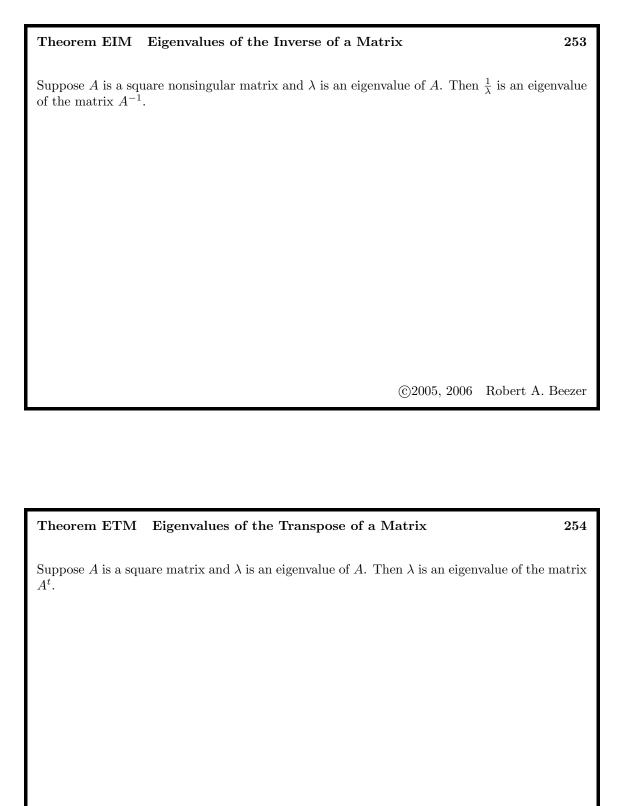
**250** 

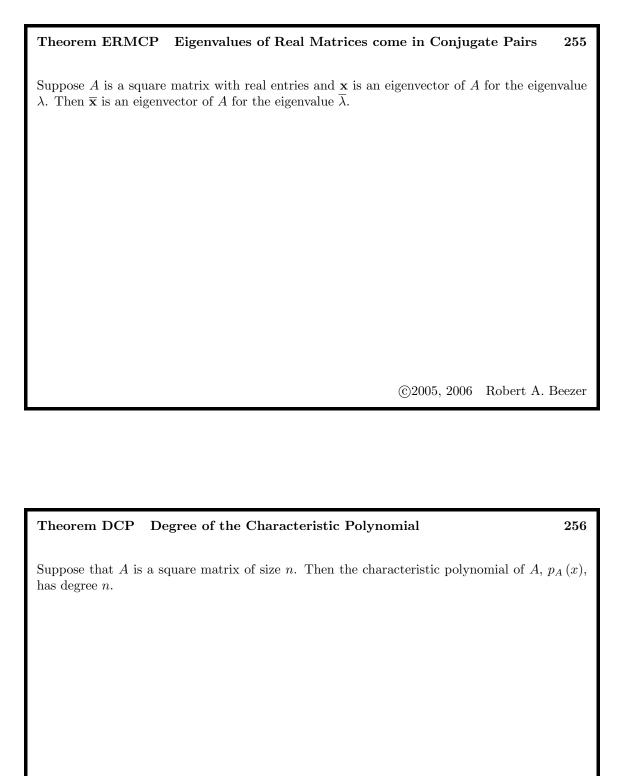
249

Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Then  $\alpha\lambda$  is an eigenvalue of  $\alpha A$ .



Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then  $q(\lambda)$  is an eigenvalue of the matrix q(A).





# Theorem NEM Number of Eigenvalues of a Matrix

257

Suppose that A is a square matrix of size n with distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ . Then

$$\sum_{i=1}^{k} \alpha_A \left( \lambda_i \right) = n$$

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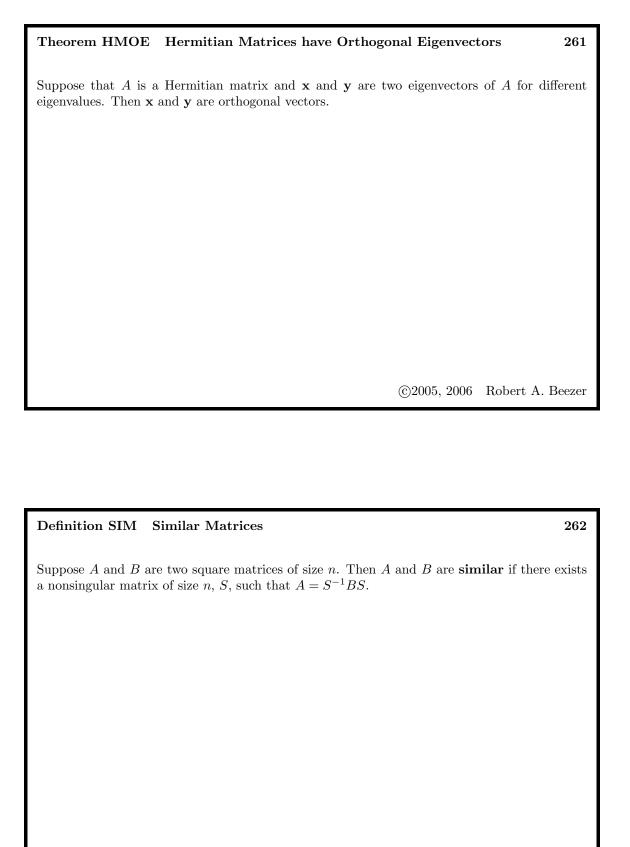
## Theorem ME Multiplicities of an Eigenvalue

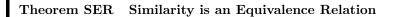
258

Suppose that A is a square matrix of size n and  $\lambda$  is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix	259
Suppose that $A$ is a square matrix of size $n$ . Then $A$ cannot have more than	n distinct eigen-
values.	
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Theorem HMRE Hermitian Matrices have Real Eigenvalues	260
Theorem HMRE Hermitian Matrices have Real Eigenvalues Suppose that $A$ is a Hermitian matrix and $\lambda$ is an eigenvalue of $A$ . Then $\lambda \in \mathbb{R}$	





Suppose A, B and C are square matrices of size n. Then

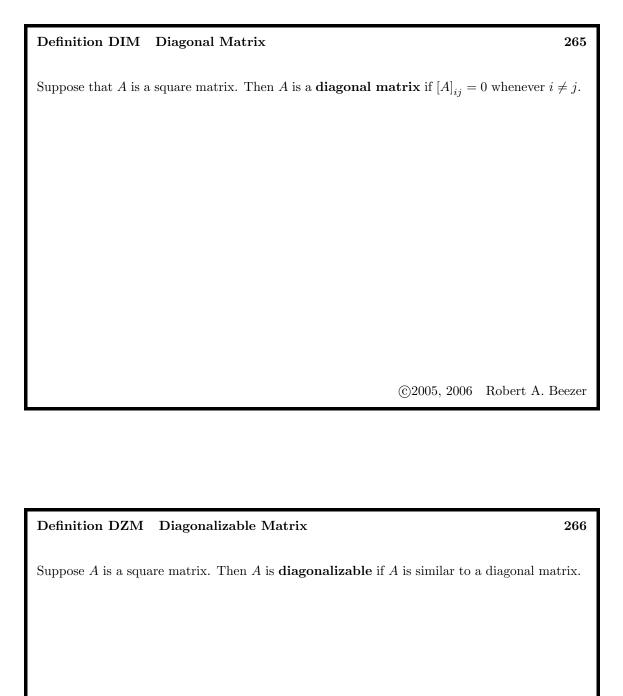
- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

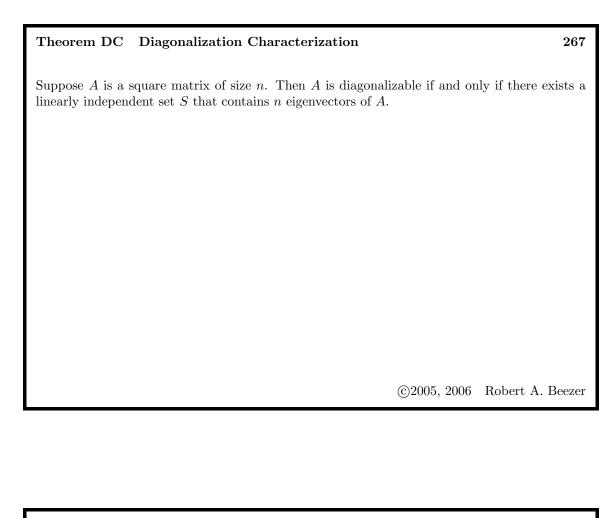
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## Theorem SMEE Similar Matrices have Equal Eigenvalues

264

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is,  $p_{A}(x) = p_{B}(x)$ .

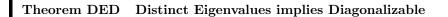




# Theorem DMFE Diagonalizable Matrices have Full Eigenspaces

**268** 

Suppose A is a square matrix. Then A is diagonalizable if and only if  $\gamma_A(\lambda) = \alpha_A(\lambda)$  for every eigenvalue  $\lambda$  of A.



Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

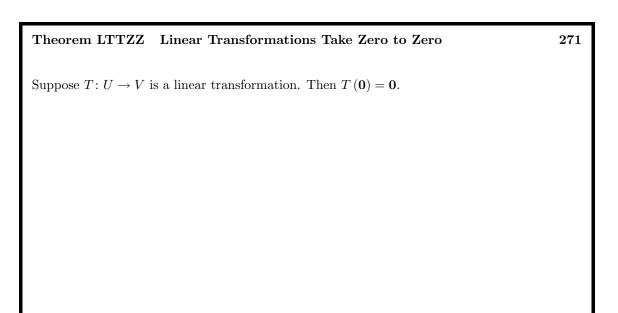
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#### Definition LT Linear Transformation

270

A linear transformation,  $T \colon U \to V$ , is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1.  $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$  for all  $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2.  $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$  for all  $\mathbf{u} \in U$  and all  $\alpha \in \mathbb{C}$

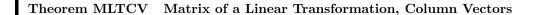


#### Theorem MBLT Matrices Build Linear Transformations

272

Suppose that A is an  $m \times n$  matrix. Define a function  $T: \mathbb{C}^n \to \mathbb{C}^m$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Then T is a linear transformation.

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Suppose that  $T: \mathbb{C}^n \to \mathbb{C}^m$  is a linear transformation. Then there is an  $m \times n$  matrix A such that  $T(\mathbf{x}) = A\mathbf{x}$ .

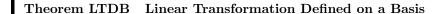
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#### Theorem LTLC Linear Transformations and Linear Combinations

274

Suppose that  $T: U \to V$  is a linear transformation,  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$  are vectors from U and  $a_1, a_2, a_3, \ldots, a_t$  are scalars from  $\mathbb{C}$ . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$



Suppose  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a basis for the vector space U and  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$  is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation,  $T: U \to V$ , such that  $T(\mathbf{u}_i) = \mathbf{v}_i$ ,  $1 \le i \le n$ .

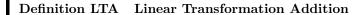
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#### Definition PI Pre-Image

276

Suppose that  $T\colon U\to V$  is a linear transformation. For each  ${\bf v}$ , define the **pre-image** of  ${\bf v}$  to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{v} \}$$



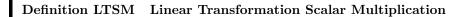
Suppose that  $T: U \to V$  and  $S: U \to V$  are two linear transformations with the same domain and codomain. Then their **sum** is the function  $T+S: U \to V$  whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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#### Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 278

Suppose that  $T\colon U\to V$  and  $S\colon U\to V$  are two linear transformations with the same domain and codomain. Then  $T+S\colon U\to V$  is a linear transformation.



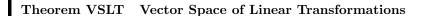
Suppose that  $T: U \to V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then the **scalar multiple** is the function  $\alpha T: U \to V$  whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

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# Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 280

Suppose that  $T:U\to V$  is a linear transformation and  $\alpha\in\mathbb{C}$ . Then  $(\alpha T)\colon U\to V$  is a linear transformation.



Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V,  $\mathcal{L}T(U,V)$  is a vector space when the operations are those given in Definition LTA and Definition LTSM.

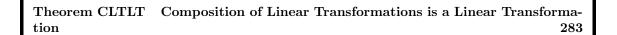
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## Definition LTC Linear Transformation Composition

282

Suppose that  $T\colon U\to V$  and  $S\colon V\to W$  are linear transformations. Then the **composition** of S and T is the function  $(S\circ T)\colon U\to W$  whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$



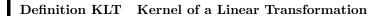
Suppose that  $T\colon U\to V$  and  $S\colon V\to W$  are linear transformations. Then  $(S\circ T)\colon U\to W$  is a linear transformation.

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## Definition ILT Injective Linear Transformation

 $\mathbf{284}$ 

Suppose  $T: U \to V$  is a linear transformation. Then T is **injective** if whenever  $T(\mathbf{x}) = T(\mathbf{y})$ , then  $\mathbf{x} = \mathbf{y}$ .



Suppose  $T: U \to V$  is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0} \}$$

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## Theorem KLTS Kernel of a Linear Transformation is a Subspace

286

Suppose that  $T:U\to V$  is a linear transformation. Then the kernel of  $T,\,\mathcal{K}(T),$  is a subspace of U.

# Theorem KPI Kernel and Pre-Image

287

288

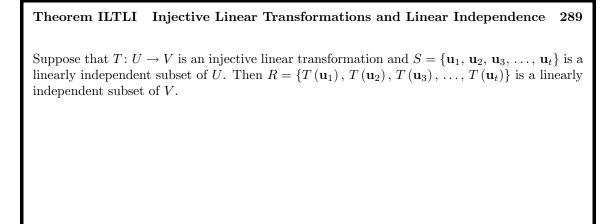
Suppose  $T: U \to V$  is a linear transformation and  $\mathbf{v} \in V$ . If the preimage  $T^{-1}(\mathbf{v})$  is non-empty, and  $\mathbf{u} \in T^{-1}(\mathbf{v})$  then

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$

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# Theorem KILT Kernel of an Injective Linear Transformation

Suppose that  $T: U \to V$  is a linear transformation. Then T is injective if and only if the kernel of T is trivial,  $\mathcal{K}(T) = \{\mathbf{0}\}.$ 

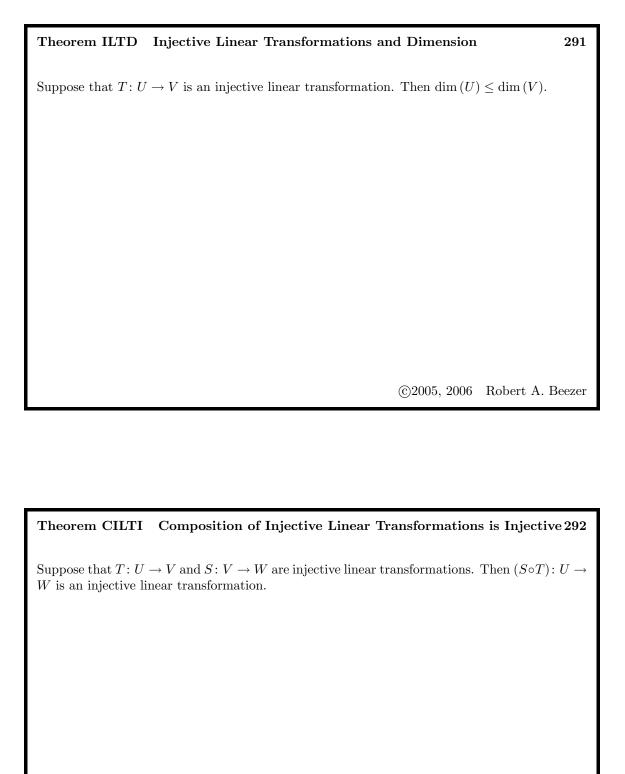


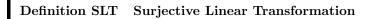
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#### Theorem ILTB Injective Linear Transformations and Bases

290

Suppose that  $T: U \to V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is injective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a linearly independent subset of V.





Suppose  $T: U \to V$  is a linear transformation. Then T is **surjective** if for every  $\mathbf{v} \in V$  there exists a  $\mathbf{u} \in U$  so that  $T(\mathbf{u}) = \mathbf{v}$ .

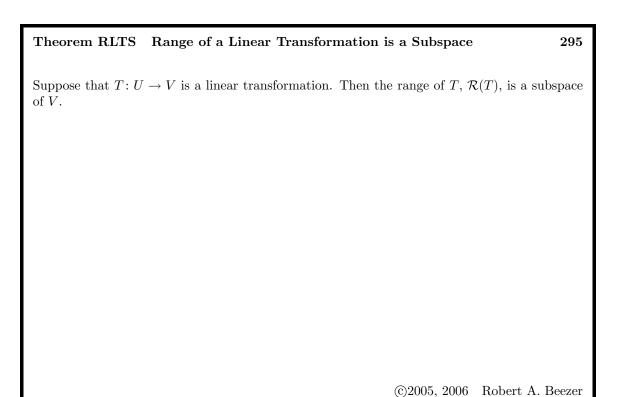
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# Definition RLT Range of a Linear Transformation

294

Suppose  $T \colon U \to V$  is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) | \mathbf{u} \in U \}$$



# Theorem RSLT Range of a Surjective Linear Transformation

**296** 

Suppose that  $T \colon U \to V$  is a linear transformation. Then T is surjective if and only if the range of T equals the codomain,  $\mathcal{R}(T) = V$ .

# Theorem SSRLT Spanning Set for Range of a Linear Transformation

297

Suppose that  $T: U \to V$  is a linear transformation and  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$  spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

spans  $\mathcal{R}(T)$ .

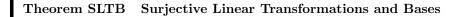
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## Theorem RPI Range and Pre-Image

298

Suppose that  $T\colon U\to V$  is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T)$$
 if and only if  $T^{-1}(\mathbf{v}) \neq \emptyset$ 



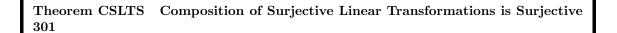
Suppose that  $T: U \to V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is surjective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a spanning set for V.

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## Theorem SLTD Surjective Linear Transformations and Dimension

300

Suppose that  $T \colon U \to V$  is a surjective linear transformation. Then  $\dim(U) \ge \dim(V)$ .



Suppose that  $T\colon U\to V$  and  $S\colon V\to W$  are surjective linear transformations. Then  $(S\circ T)\colon U\to W$  is a surjective linear transformation.

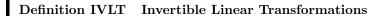
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# Definition IDLT Identity Linear Transformation

302

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \to W, \qquad I_W (\mathbf{w}) = \mathbf{w}$$



Suppose that  $T\colon U\to V$  is a linear transformation. If there is a function  $S\colon V\to U$  such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write  $S = T^{-1}$ .

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# Theorem ILTLT $\,$ Inverse of a Linear Transformation is a Linear Transformation $\,$ 304

Suppose that  $T\colon U\to V$  is an invertible linear transformation. Then the function  $T^{-1}\colon V\to U$  is a linear transformation.

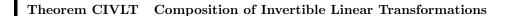


Suppose that  $T: U \to V$  is an invertible linear transformation. Then  $T^{-1}$  is an invertible linear transformation and  $(T^{-1})^{-1} = T$ .

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# Theorem ILTIS Invertible Linear Transformations are Injective and Surjective 306

Suppose  $T\colon U\to V$  is a linear transformation. Then T is invertible if and only if T is injective and surjective.



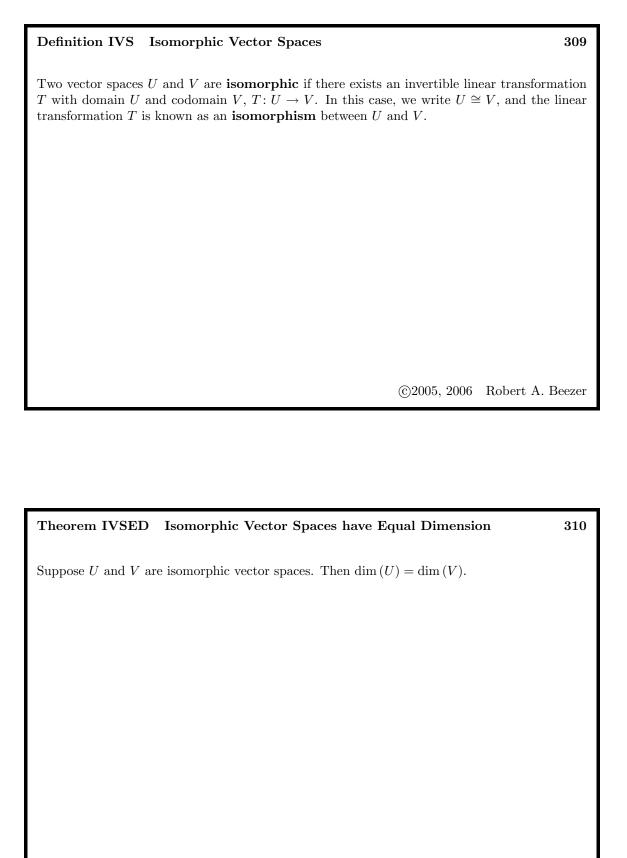
Suppose that  $T\colon U\to V$  and  $S\colon V\to W$  are invertible linear transformations. Then the composition,  $(S\circ T)\colon U\to W$  is an invertible linear transformation.

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## Theorem ICLT Inverse of a Composition of Linear Transformations

308

Suppose that  $T\colon U\to V$  and  $S\colon V\to W$  are invertible linear transformations. Then  $S\circ T$  is invertible and  $\left(S\circ T\right)^{-1}=T^{-1}\circ S^{-1}$ .





Suppose that  $T:U\to V$  is a linear transformation. Then the **rank** of T,r(T), is the dimension of the range of T,

$$r(T) = \dim \left( \mathcal{R}(T) \right)$$

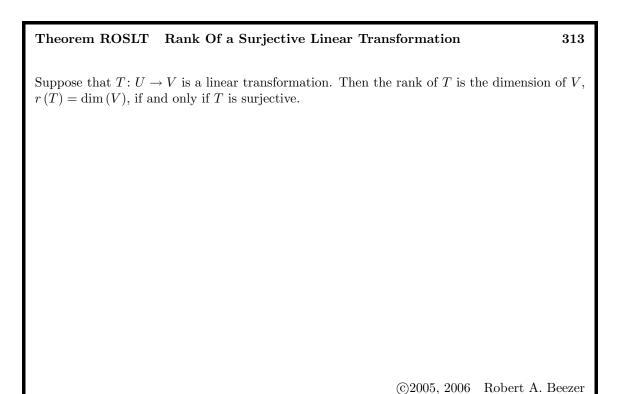
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# Definition NOLT Nullity Of a Linear Transformation

**312** 

Suppose that  $T:U\to V$  is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

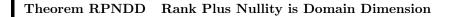
$$n(T) = \dim (\mathcal{K}(T))$$



# Theorem NOILT Nullity Of an Injective Linear Transformation

**314** 

Suppose that  $T: U \to V$  is a linear transformation. Then the nullity of T is zero, n(T) = 0, if and only if T is injective.



Suppose that  $T\colon U\to V$  is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

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## Definition VR Vector Representation

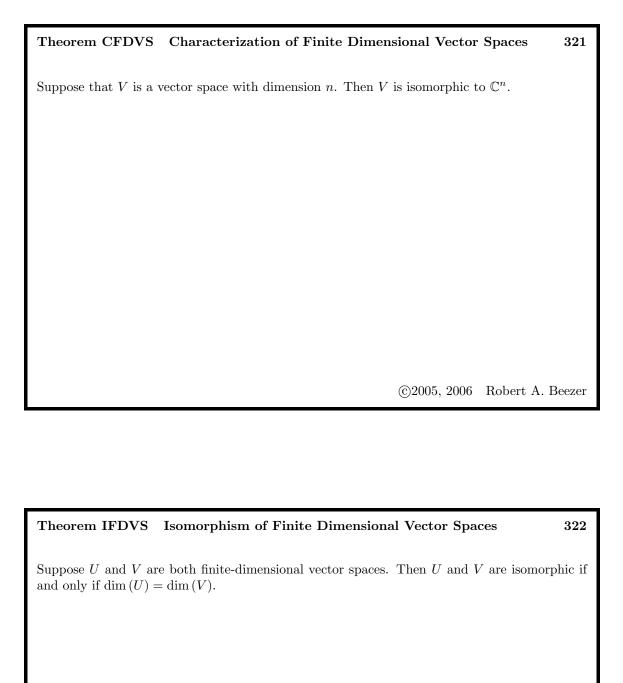
316

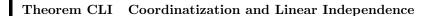
Suppose that V is a vector space with a basis  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ . Define a function  $\rho_B \colon V \to \mathbb{C}^n$  as follows. For  $\mathbf{w} \in V$  define the column vector  $\rho_B(\mathbf{w}) \in \mathbb{C}^n$  by

$$\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \dots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$$

Theorem VRLT	Vector Representation is a Linear	Transformation	on 317
The function $\rho_B$ (D	Definition VR) is a linear transformation.		
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		©2005, 2006	Robert A. Beezer
Theorem VRI	Vector Representation is Injective		318
The function a (D	Antipition VP) is an injective linear trans	farmation	

Theorem VRS Vector Representation is Surjective	319			
The function $\rho_B$ (Definition VR) is a surjective linear transformation.				
The function $\rho_B$ (Definition vit) is a surjective linear transformation.				
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Theorem VRILT Vector Representation is an Invertible Linear 320	Transformation			
The function $\rho_B$ (Definition VR) is an invertible linear transformation.				
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Suppose that U is a vector space with a basis B of size n. Then  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$  is a linearly independent subset of U if and only if  $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$  is a linearly independent subset of  $\mathbb{C}^n$ .

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# Theorem CSS Coordinatization and Spanning Sets

324

Suppose that U is a vector space with a basis B of size n. Then  $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$  if and only if  $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$ .

## Definition MR Matrix Representation

325

Suppose that  $T: U \to V$  is a linear transformation,  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the  $m \times n$  matrix,

$$M_{B,C}^{T} = \left[ \rho_{C} \left( T \left( \mathbf{u}_{1} \right) \right) \middle| \rho_{C} \left( T \left( \mathbf{u}_{2} \right) \right) \middle| \rho_{C} \left( T \left( \mathbf{u}_{3} \right) \right) \middle| \dots \middle| \rho_{C} \left( T \left( \mathbf{u}_{n} \right) \right) \right]$$

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### Theorem FTMR Fundamental Theorem of Matrix Representation

326

Suppose that  $T: U \to V$  is a linear transformation, B is a basis for U, C is a basis for V and  $M_{B,C}^T$  is the matrix representation of T relative to B and C. Then, for any  $\mathbf{u} \in U$ ,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

# Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 327

Suppose that  $T\colon U\to V$  and  $S\colon U\to V$  are linear transformations, B is a basis of U and C is a basis of V. Then

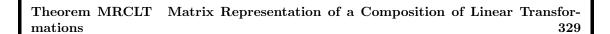
$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

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# Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 328

Suppose that  $T\colon U\to V$  is a linear transformation,  $\alpha\in\mathbb{C},\,B$  is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$



Suppose that  $T: U \to V$  and  $S: V \to W$  are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

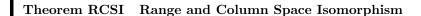
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## Theorem KNSI Kernel and Null Space Isomorphism

**330** 

Suppose that  $T: U \to V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of  $M_{B,C}^T$ ,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$



Suppose that  $T: U \to V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of  $M_{B,C}^T$ ,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

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#### Theorem IMR Invertible Matrix Representations

**332** 

Suppose that  $T: U \to V$  is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C,  $M_{B,C}^T$  is an invertible matrix. When T is invertible,

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

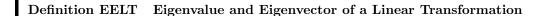
Suppose that A is a square matrix of size n and  $T: \mathbb{C}^n \to \mathbb{C}^n$  is the linear transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ . Then A is invertible matrix if and only if T is an invertible linear transformation.

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#### Theorem NME9 Nonsingular Matrix Equivalences, Round 9

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .
- 12.  $\lambda = 0$  is not an eigenvalue of A.
- 13. The linear transformation  $T: \mathbb{C}^n \to \mathbb{C}^n$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is invertible.



Suppose that  $T: V \to V$  is a linear transformation. Then a nonzero vector  $\mathbf{v} \in V$  is an **eigenvector** of T for the **eigenvalue**  $\lambda$  if  $T(\mathbf{v}) = \lambda \mathbf{v}$ .

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#### Definition CBM Change-of-Basis Matrix

336

335

Suppose that V is a vector space, and  $I_V: V \to V$  is the identity linear transformation on V. Let  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of  $I_V$  relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[ \rho_C \left( I_V \left( \mathbf{v}_1 \right) \right) \middle| \rho_C \left( I_V \left( \mathbf{v}_2 \right) \right) \middle| \rho_C \left( I_V \left( \mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left( I_V \left( \mathbf{v}_n \right) \right) \right]$$

$$= \left[ \rho_C \left( \mathbf{v}_1 \right) \middle| \rho_C \left( \mathbf{v}_2 \right) \middle| \rho_C \left( \mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left( \mathbf{v}_n \right) \right]$$



Suppose that  $\mathbf{v}$  is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

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# Theorem ICBM Inverse of Change-of-Basis Matrix

**338** 

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix  $C_{B,C}$  is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

# Theorem MRCB Matrix Representation and Change of Basis

339

Suppose that  $T\colon U\to V$  is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

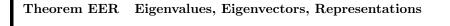
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# Theorem SCB Similarity and Change of Basis

**340** 

Suppose that  $T: V \to V$  is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$



Suppose that  $T\colon V\to V$  is a linear transformation and B is a basis of V. Then  $\mathbf{v}\in V$  is an eigenvector of T for the eigenvalue  $\lambda$  if and only if  $\rho_B(\mathbf{v})$  is an eigenvector of  $M_{B,B}^T$  for the eigenvalue  $\lambda$ .

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## Definition NLT Nilpotent Linear Transformation

342

Suppose that  $T: V \to V$  is a linear transformation such that there is an integer p > 0 such that  $T^p(\mathbf{v}) = \mathbf{0}$  for every  $\mathbf{v} \in V$ . The smallest p for which this condition is met is called the **index** of T.

# Definition JB Jordan Block

343

Given the scalar  $\lambda \in \mathbb{C}$ , the Jordan block  $J_n(\lambda)$  is the  $n \times n$  matrix defined by

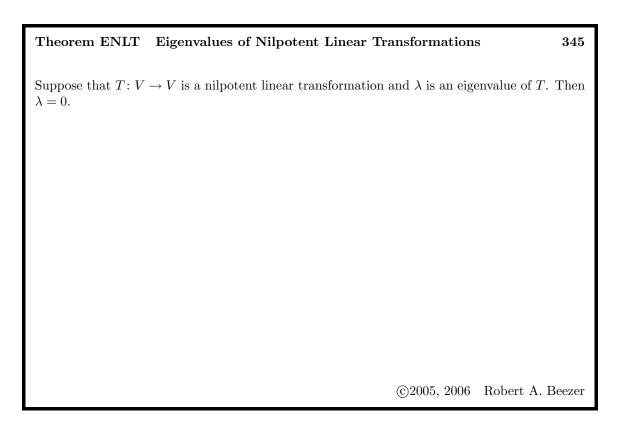
$$\left[J_{n}\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$

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## Theorem NJB Nilpotent Jordan Blocks

**344** 

The Jordan block  $J_{n}\left(0\right)$  is nilpotent of index n.



# Theorem DNLT Diagonalizable Nilpotent Linear Transformations

**346** 

Suppose the linear transformation  $T\colon V\to V$  is nilpotent. Then T is diagonalizable if and only T is the zero linear transformation.

### Theorem KPLT Kernels of Powers of Linear Transformations

347

Suppose  $T: V \to V$  is a linear transformation, where dim (V) = n. Then there is an integer m,  $0 \le m \le n$ , such that

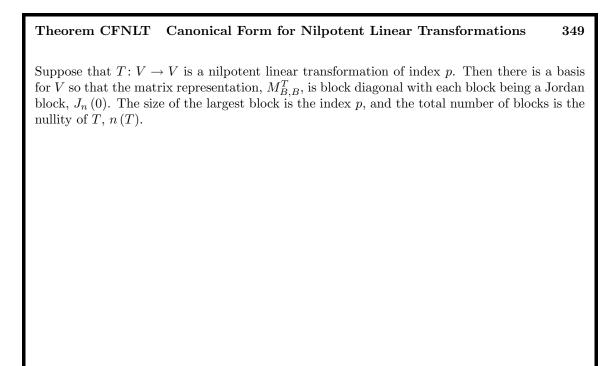
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

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### Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 348

Suppose  $T: V \to V$  is a nilpotent linear transformation with index p and dim (V) = n. Then  $0 \le p \le n$  and

$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$

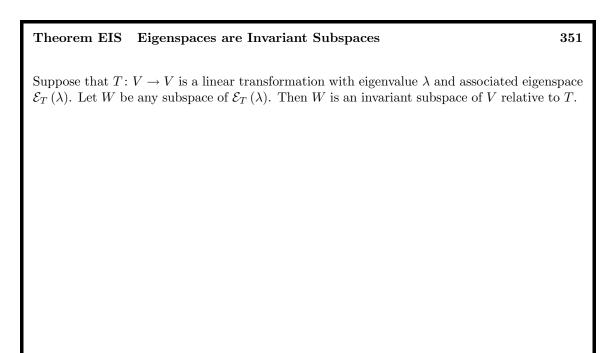


# Definition IS Invariant Subspace

350

Suppose that  $T: V \to V$  is a linear transformation and W is a subspace of V. Suppose further that  $T(\mathbf{w}) \in W$  for every  $\mathbf{w} \in W$ . Then W is an **invariant subspace** of V relative to T.

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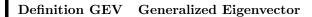


# Theorem KPIS Kernels of Powers are Invariant Subspaces

352

Suppose that  $T\colon V\to V$  is a linear transformation. Then  $\mathcal{K}\big(T^k\big)$  is an invariant subspace of V

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Suppose that  $T: V \to V$  is a linear transformation. Suppose further that for  $\mathbf{x} \neq \mathbf{0}$ ,  $(T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0}$  for some k > 0. Then  $\mathbf{x}$  is a **generalized eigenvector** of T with eigenvalue  $\lambda$ .

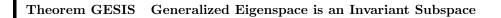
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# Definition GES Generalized Eigenspace

354

Suppose that  $T\colon V\to V$  is a linear transformation. Define the **generalized eigenspace** of T for  $\lambda$  as

$$\mathcal{G}_{T}(\lambda) = \left\{ \mathbf{x} | \left(T - \lambda I_{V}\right)^{k}(\mathbf{x}) = \mathbf{0} \text{ for some } k \geq 0 \right\}$$



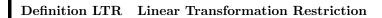
Suppose that  $T: V \to V$  is a linear transformation. Then the generalized eigenspace  $\mathcal{G}_T(\lambda)$  is an invariant subspace of V relative to T.

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# Theorem GEK Generalized Eigenspace as a Kernel

356

Suppose that  $T: V \to V$  is a linear transformation,  $\dim(V) = n$ , and  $\lambda$  is an eigenvalue of T. Then  $\mathcal{G}_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n)$ .



Suppose that  $T \colon V \to V$  is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by

$$T|_U \colon U \to U$$

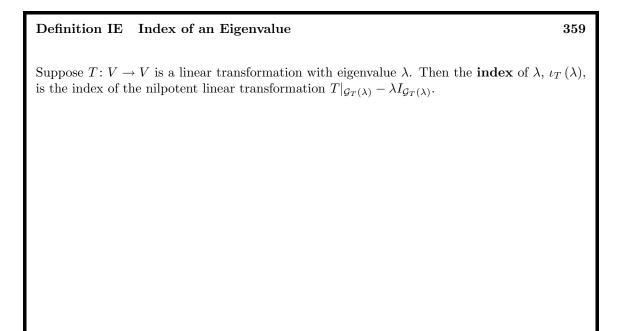
$$T|_{U}(\mathbf{u}) = T(\mathbf{u})$$

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# Theorem RGEN Restriction to Generalized Eigenspace is Nilpotent

358

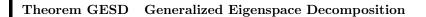
Suppose  $T: V \to V$  is a linear transformation with eigenvalue  $\lambda$ . Then the linear transformation  $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$  is nilpotent.



# Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace 360

Suppose that  $T: V \to V$  is a linear transformation with eigenvalue  $\lambda$ . Then there is a basis of the the generalized eigenspace  $\mathcal{G}_T(\lambda)$  such that the restriction  $T|_{\mathcal{G}_T(\lambda)}: \mathcal{G}_T(\lambda) \to \mathcal{G}_T(\lambda)$  has a matrix representation that is block diagonal where each block is a Jordan block of the form  $J_n(\lambda)$ .

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Suppose that  $T: V \to V$  is a linear transformation with distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$ . Then

$$V = \mathcal{G}_{T}(\lambda_{1}) \oplus \mathcal{G}_{T}(\lambda_{2}) \oplus \mathcal{G}_{T}(\lambda_{3}) \oplus \cdots \oplus \mathcal{G}_{T}(\lambda_{m})$$

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## Theorem DGES Dimension of Generalized Eigenspaces

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Suppose  $T: V \to V$  is a linear transformation with eigenvalue  $\lambda$ . Then the dimension of the generalized eigenspace for  $\lambda$  is the algebraic multiplicity of  $\lambda$ , dim  $(\mathcal{G}_T(\lambda_i)) = \alpha_T(\lambda_i)$ .

#### Definition JCF Jordan Canonical Form

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A square matrix is in **Jordan canonical form** if it meets the following requirements:

- 1. The matrix is block diagonal.
- 2. Each block is a Jordan block.
- 3. If  $\rho < \lambda$  then the block  $J_k(\rho)$  occupies rows with indices greater than the indices of the rows occupied by  $J_{\ell}(\lambda)$ .
- 4. If  $\rho = \lambda$  and  $\ell < k$ , then the block  $J_{\ell}(\lambda)$  occupies rows with indices greater than the indices of the rows occupied by  $J_{k}(\lambda)$ .

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#### Theorem JCFLT Jordan Canonical Form for a Linear Transformation 364

Suppose  $T:V\to V$  is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

- 1. The matrix representation is in Jordan canonical form.
- 2. If  $J_k(\lambda)$  is one of the Jordan blocks, then  $\lambda$  is an eigenvalue of T.
- 3. For a fixed value of  $\lambda$ , the largest block of the form  $J_k(\lambda)$  has size equal to the index of  $\lambda$ ,  $\iota_T(\lambda)$ .
- 4. For a fixed value of  $\lambda$ , the number of blocks of the form  $J_k(\lambda)$  is the geometric multiplicity of  $\lambda$ ,  $\gamma_T(\lambda)$ .
- 5. For a fixed value of  $\lambda$ , the number of rows occupied by blocks of the form  $J_k(\lambda)$  is the algebraic multiplicity of  $\lambda$ ,  $\alpha_T(\lambda)$ .

Theorem CHT	Cayley-Hamilton Theorem	365
Suppose $A$ is a sq	uare matrix with characteristic polynomial $p_{A}(x)$ . Then $p_{A}(A) = \mathcal{O}$ .	
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