Flash Cards

to accompany

A First Course in Linear Algebra

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Version 2.12

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The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are equal , denoted $\alpha = \beta$, if $a = c$ and $b = d$.	Definition CNE Complex Number Equality 1	
	The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are equal , denoted $\alpha = \beta$, if $a = c$ and $b = d$.	

Definition CNA	Complex Number Addition	2
The sum of the con	applex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + d)i$	į.
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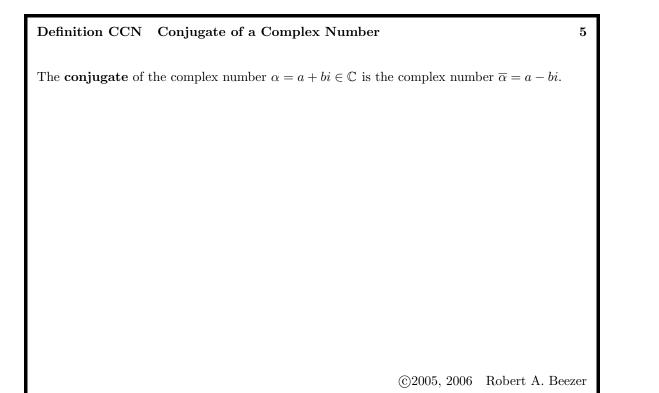
Definition CNM Complex Number Multiplication

The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is (ac - bd) + (ad + bc)i.

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Theorem PCNAProperties of Complex Number Arithmetic4The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers For any $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers For any $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}, \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers For any α , β , $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}, 0 + \alpha = \alpha$.
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$
- MICN Multiplicative Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha(\frac{1}{\alpha}) = 1$.



 Theorem CCRA
 Complex Conjugation Respects Addition
 6

 Suppose that α and β are complex numbers. Then $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
 6

 Output
 $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
 6

 Output
 $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
 6

 Couplex output
 $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
 6

 Output
 $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
 6

Theorem CCRM Complex Conjugation Respects Multiplication	7
Suppose that α and β are complex numbers. Then $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.	
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Theorem CCT Complex Conjugation Twice 8 Suppose that α is a complex number. Then $\overline{\overline{\alpha}} = \alpha$. ©2005, 2006 Robert A. Beezer

Definition MCN Modulus of a Complex Number

The **modulus** of the complex number $\alpha = a + bi \in \mathbb{C}$, is the nonnegative real number

 $|\alpha| = \sqrt{\alpha \overline{\alpha}} = \sqrt{a^2 + b^2}.$

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Definition SET Set

A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.

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Definition SSET Subset	11
If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever x	$x \in S$ then $x \in T$.
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Definition ES	Empty Set	12
The empty set is	the set with no elements. Its is denoted by \emptyset .	
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Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write S = T.

Definition SE Set Equality

Definition C Cardinality	14
Suppose S is a finite set. Then the number of elements in S is called the ca of S, and is denoted $ S $.	ardinality or size
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Definition SU Set Union

Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$

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Definition SI S	Set Intersection
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Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$ if and only if $x \in S$ and $x \in T$

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Definition SC Set Complement

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$

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Definition SLESystem of Linear Equations18A system of linear equations is a collection of m equations in the variable quantities
 $x_1, x_2, x_3, \dots, x_n$ of the form, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$
 $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$
 $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$
 \vdots
 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ where the values of a_{ij}, b_i and x_j are from the set of complex numbers, \mathbb{C} .©2005, 2006Robert A. Beezer

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Definition EO Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

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Definition M Matrix

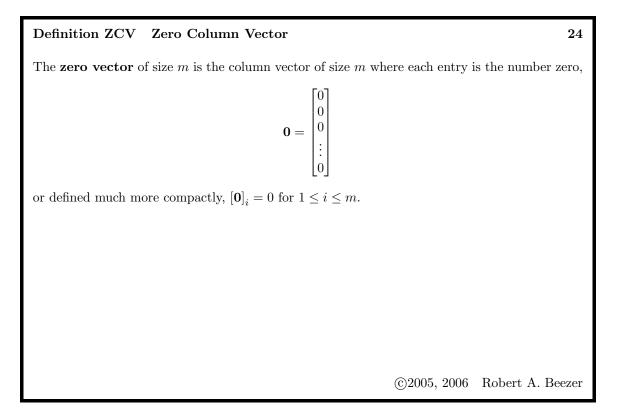
An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

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Definition CV Column Vector

A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$. Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$.



Definition CM Coefficient Matrix

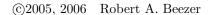
For a system of linear equations,

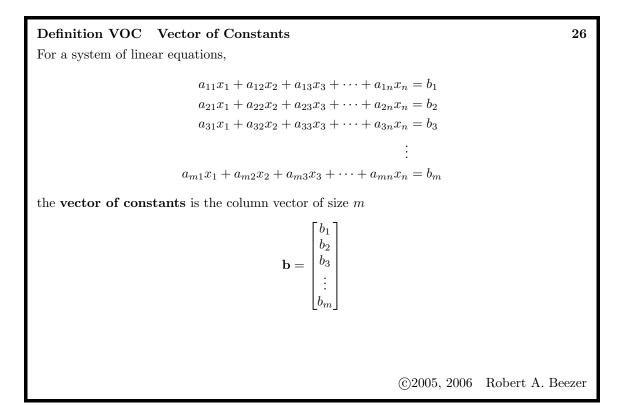
 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$:

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

the **coefficient matrix** is the $m \times n$ matrix

	a_{11}	a_{12}	$a_{13} \\ a_{23} \\ a_{33}$	• • •	a_{1n}
	a_{21}	a_{22}	a_{23}		a_{2n}
A =	a_{31}	a_{32}	a_{33}		a_{3n}
	:				
	•				
	a_{m1}	a_{m2}	a_{m3}	• • •	a_{mn}





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Definition SOLV Solution Vector

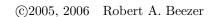
For a system of linear equations,

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



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Definition MRLS Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and **b** is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

Definition AM Augmented Matrix

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants **b**. Then the **augmented matrix** of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (number n + 1) is the column vector **b**. This matrix will be written as $[A \mid \mathbf{b}]$.

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Definition RO Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows *i* and *j*.
- 2. αR_i : Multiply row *i* by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row *i* by the scalar α and add to row *j*.

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Definition REM Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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Theorem REMES Row-Equivalent Matrices represent Equivalent Systems 32

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

Definition RREF Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r. A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

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Theorem REMEF Row-Equivalent Matrix in Echelon Form Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

Theorem RREFU Reduced Row-Echelon Form is Unique

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

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Definition RR Row-Reducing

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

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Definition CS Consistent System

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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Definition IDV Independent and Dependent Variables

Suppose A is the augmented matrix of a consistent system of linear equations and B is a rowequivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column). Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

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Theorem RCLS Recognizing Consistency of a Linear System

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n + 1 of B.

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Theorem ISRN Inconsistent Systems, r and n

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

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Theorem CSRN Consistent Systems, r and n

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n - r free variables.

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Theorem PSSLS Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

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Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions 44

Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions.

Definition HS Homogeneous System

A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

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Theorem HSC Homogeneous Systems are Consistent	46
Suppose that a system of linear equations is homogeneous. Then the system is consistent.	
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Definition TSHSE	Trivial Solution to Homogeneous Systems	s of Equations	47
	bus system of linear equations has n variables. e. $\mathbf{x} = 0$ is called the trivial solution .	The solution x_1 :	= 0,
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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 48

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

Definition NSM Null Space of a Matrix

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

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Definition SQM Square Matrix

A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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Definition NM Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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Definition IMIdentity Matrix52The $m \times m$ identity matrix, I_m , is defined by $[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $1 \le i, j \le m$ ©2005, 2006Robert A. Beezer

Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 53

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 5	64
	quare matrix. Then A is nonsingular if and only if the null space of A ne zero vector, i.e. $\mathcal{N}(A) = \{0\}.$	4,
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Theorem NMUS Nonsingular Matrices and Unique Solutions

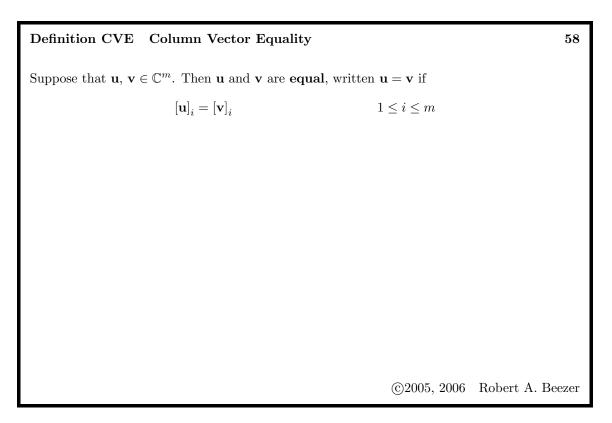
Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} .

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Theorem NME1	Nonsingular Matrix Equivalences, Round 1	56	
Suppose that A is a square matrix. The following are equivalent.			
1. A is nonsingul	1. A is nonsingular.		
2. A row-reduces	to the identity matrix.		
3. The null space	e of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$		
4. The linear sys	tem $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice	of b .	
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Definition VSCV	Vector Space of Column Vectors 57	7
The vector space \mathbb{C}^m the set of complex m	is the set of all column vectors (Definition CV) of size m with entries from umbers, \mathbb{C} .	n
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Definition CVA	Column Vector Addition		59
Suppose that \mathbf{u},\mathbf{v}	$\in \mathbb{C}^m$. The sum of u and v is the vector	$\mathbf{u}+\mathbf{v}$ defined b	ру
	$\left[\mathbf{u}+\mathbf{v}\right]_i=\left[\mathbf{u}\right]_i+\left[\mathbf{v}\right]_i$	$1 \leq i \leq m$	
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Definition CVSM	Column Vector Scalar Multipli	cation	60
Suppose $\mathbf{u} \in \mathbb{C}^m$ and	$\alpha \in \mathbb{C}$, then the scalar multiple of	u by α is the vec	tor $\alpha \mathbf{u}$ defined by
	$\left[\alpha\mathbf{u}\right]_{i}=\alpha\left[\mathbf{u}\right]_{i}$	$1 \leq i \leq m$	
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		C2005, 2006	Robert A. Beezer

Theorem VSPCV Vector Space Properties of Column Vectors 61 Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then • ACC Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v}$ $(\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- ZC Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- OC One Column Vectors If $\mathbf{u} \in \mathbb{C}^m$ then $\mathbf{1}_{\mathbf{u}}$

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Definition LCCV Linear Combination of Column Vectors Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if \mathbf{b} equals the linear combination of the columns of A formed with the entries of \mathbf{x} ,

 $[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n + 1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$ of size n by

$$[\mathbf{c}]_{i} = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, \, i = d_{k} \end{cases}$$

$$[\mathbf{u}_{j}]_{i} = \begin{cases} 1 & \text{if } i \in F, \, i = f_{j} \\ 0 & \text{if } i \in F, \, i \neq f_{j} \\ -[B]_{k, \, f_{i}} & \text{if } i \in D, \, i = d_{k} \end{cases}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

 $S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} \mid \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$

Theorem PSPHS Particular Solution Plus Homogeneous Solutions 65 Suppose that \mathbf{w} is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then \mathbf{y} is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$. 65

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Definition SSCV Span of a Set of Column Vectors

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n - r vectors \mathbf{z}_j , $1 \le j \le n - r$ of size n as

$$\left[\mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \} \rangle.$$

Definition RLDCV Relation of Linear Dependence for Column Vectors 68	
Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, a true statement of the form	
$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n = 0$	
is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the trivial relation of linear dependence on S.	
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Definition LICV Linear Independence of Column Vectors

The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems 70

Suppose that A is an $m \times n$ matrix and $S = {\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Theorem LIVRN Linearly Independent Vectors, r and n

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that $n > m$. Then S is a linearly dependent set.

Theorem MVSLD More Vectors than Size implies Linear Dependence

Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 73

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NME2 Nonsingular Matrix Equivalences	, Round 2	74		
Suppose that A is a square matrix. The following are equivalent.				
1. A is nonsingular.				
2. A row-reduces to the identity matrix.				
3. The null space of A contains only the zero vector, $\mathcal{N}($	$A) = \{0\}.$			
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for	every possible c	hoice of b .		
5. The columns of A form a linearly independent set.				
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Theorem BNS Basis for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n - r vectors \mathbf{z}_j , $1 \le j \le n - r$ of size n as

$$\begin{bmatrix} \mathbf{z}_j \end{bmatrix}_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -\begin{bmatrix} B \end{bmatrix}_{k, f_i} & \text{if } i \in D, \ i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

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Theorem DLDS Dependency in Linearly Dependent Sets

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

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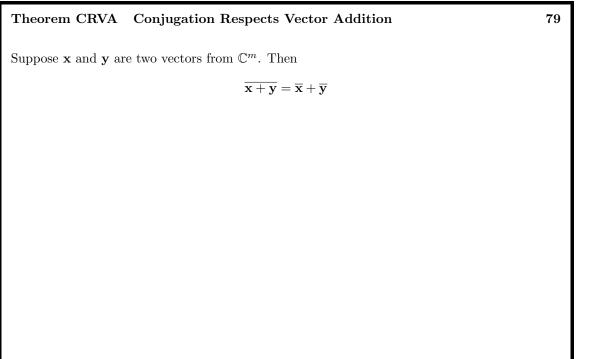
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Theorem BS Basis of a Span

Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = {d_1, d_2, d_3, \dots, d_r}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

Definition CCCV Complex Conjugate of a Column Vector			
Suppose that u is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by			
$\left[\overline{\mathbf{u}}\right]_i = \overline{\left[\mathbf{u}\right]_i} \qquad \qquad 1 \le i \le m$			
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Theorem CRSM	Conjugation Respects Vector Scalar Multiplication	80			
Suppose \mathbf{x} is a vector	Suppose x is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then				
	$\overline{\alpha \mathbf{x}} = \overline{\alpha} \overline{\mathbf{x}}$				
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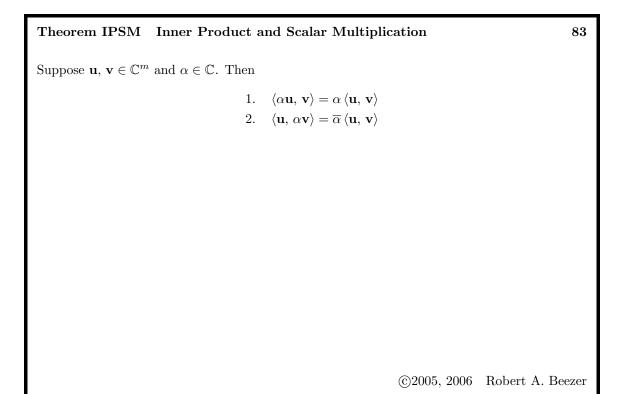
Definition IP Inner Product

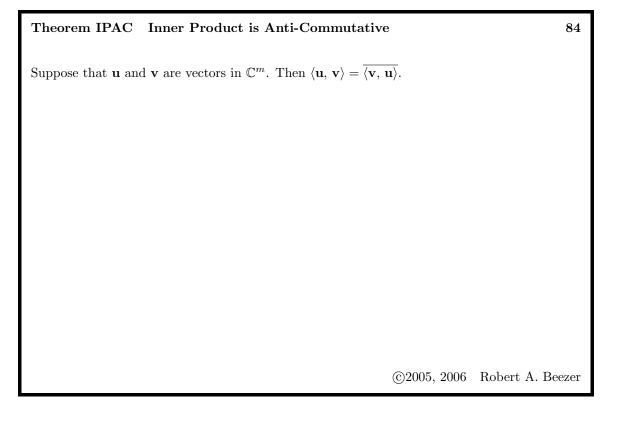
Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_1 \overline{[\mathbf{v}]_1} + [\mathbf{u}]_2 \overline{[\mathbf{v}]_2} + [\mathbf{u}]_3 \overline{[\mathbf{v}]_3} + \dots + [\mathbf{u}]_m \overline{[\mathbf{v}]_m} = \sum_{i=1}^m [\mathbf{u}]_i \overline{[\mathbf{v}]_i}$$

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Theorem IPVA Inner Product and Vector Addition					
Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then					
1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$					
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Definition NV Norm of a Vector

The ${\bf norm}$ of the vector ${\bf u}$ is the scalar quantity in ${\mathbb C}$

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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Theorem IPN Inner Products and Norms	86				
Suppose that u is a vector in \mathbb{C}^m . Then $\ \mathbf{u}\ ^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.					
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Theorem PIP Positive Inner Products	87	
Suppose that u is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ with equality if and only if $\mathbf{u} = 0$.		
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Definition OV	Orthogonal Vectors	88
A pair of vectors, $\langle \mathbf{u}, \mathbf{v} \rangle = 0.$	\mathbf{u} and $\mathbf{v},$ from \mathbb{C}^m are orthogonal if their inner product is zero, that	is,
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Definition OSV Orthogonal Set of Vectors

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors from \mathbb{C}^m . Then S is an **orthogonal** set if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

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Definition SUV Standard Unit Vectors Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \le j \le m$ denote the column vectors defined by $[\mathbf{e}_j]_i = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}$ Then the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \le j \le m\}$ is the set of standard unit vectors in \mathbb{C}^m .

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Theorem OSLI Orthogonal Sets are Linearly Independent Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

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Theorem GSP Gram-Schmidt Procedure

Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

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Definition ONS OrthoNormal Set Suppose $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

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Definition VSM Vector Space of $m \times n$ Matrices

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

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Definition ME Matrix Equality

The $m \times n$ matrices A and B are equal, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

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Definition MA Matrix Addition

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A+B]_{ii} = [A]_{ii} + [B]_{ii} \qquad 1 \le i \le m, \ 1 \le j \le n$$

Definition MSM Matrix Scalar Multiplication

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Theorem VSPM Vector Space Properties of Matrices

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One Matrices If $A \in M$ then 1A A

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Definition ZM Zero Matrix

The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$.

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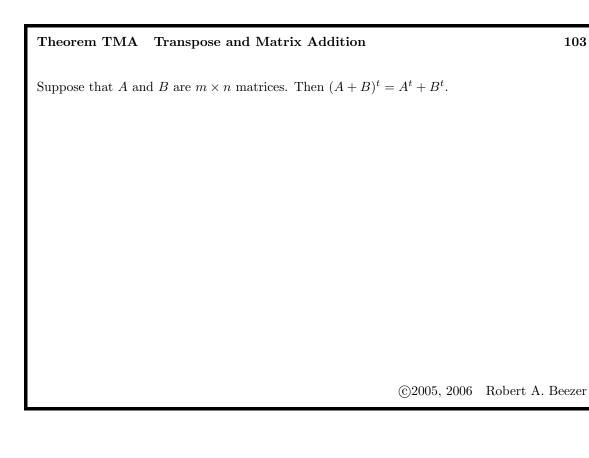
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Definition TM Transpose of a Matrix	100		
Given an $m \times n$ matrix A, its transpose is the $n \times m$ matrix A^t given by			
$\left[A^t\right]_{ij} = [A]_{ji}, 1 \le i \le n, \ 1 \le j \le m.$			
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The matrix A is symmetric if $A = A^t$.

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Theorem SMS	Symmetric Matrices are Square		102
	· ·		
Suppose that A is	a symmetric matrix. Then A is square.		
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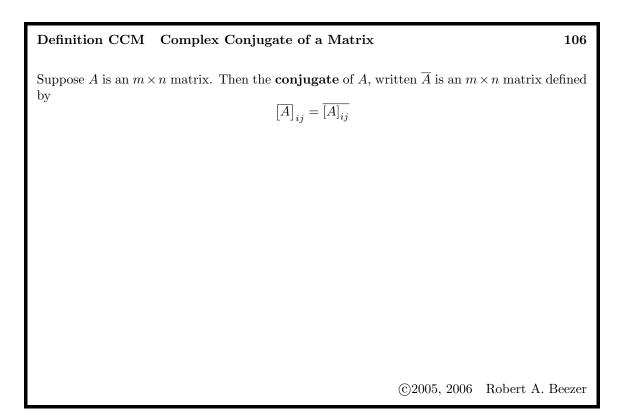


Theorem TMSM	Transpose and Matrix	Scalar Multiplication	104
Suppose that $\alpha \in \mathbb{C}$	and A is an $m \times n$ matrix.	Then $(\alpha A)^t = \alpha A^t$.	
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Theorem TT Transpose of a Transpose

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

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Theorem CRMA	Conjugation Respects Matrix Addition	107
Suppose that A and	B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.	
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Theorem CRMSM	Conjugation Respects Matrix Scalar Multiplication	108
Suppose that $\alpha \in \mathbb{C}$ and	nd A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.	
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Theorem CCM Conjugate of the Conjugate of a Matrix	109	
Suppose that A is an $m \times n$ matrix. Then $\overline{(\overline{A})} = A$.		
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 Theorem MCT
 Matrix Conjugation and Transposes
 110

 Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.
 (\overline{A}^t)

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Definition A Adjoint

If A is a matrix, then its **adjoint** is $A^* = (\overline{A})^t$.

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Theorem AMA Adjoint and Matrix Addition	112
Suppose A and B are matrices of the same size. Then $(A + B)^* = A^* + B^*$.	
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Theorem AMSM Adjoint and Matrix Scalar Multiplication	113
Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$.	
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Theorem AA Adjoint of an Adjoint	114
Suppose that A is a matrix. Then $(A^*)^* = A$	
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Definition MVP Matrix-Vector Product

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

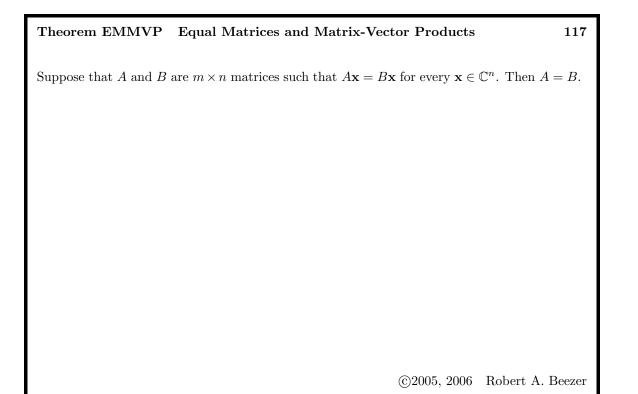
$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

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Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 116

The set of solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ equals the set of solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.



Definition MM Matrix Multiplication

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

 $AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$

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Theorem EMP Entries of Matrix Products

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \le i \le m, 1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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Theorem MMZM	Matrix Multiplication and the Zero Matrix	120
Suppose A is an $m \times 1$. 1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$ 2. $\mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$		
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Theorem MMIM Matrix Multiplication and Identity Matrix

Suppose A is an $m \times n$ matrix. Then 1. $AI_n = A$ 2. $I_m A = A$

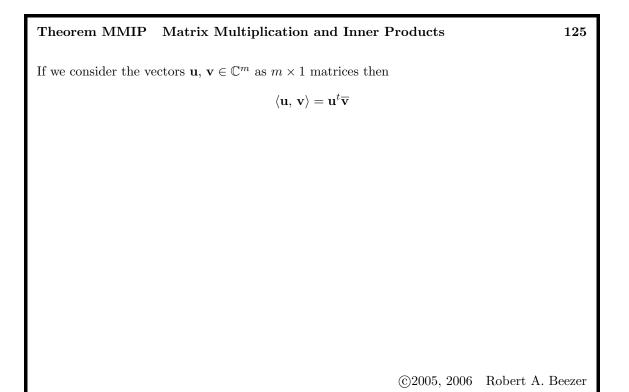
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Theorem MMDAA Matrix Multiplication Distributes Across Addition	122
Theorem MMDAA Matrix Multiplication Distributes Across Addition Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. 1. $A(B+C) = AB + AC$ 2. $(B+C)D = BD + CD$	
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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 123

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA	Matrix Multiplication is Associative	124
Suppose A is an m (AB)D.	$\times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD$) =
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 Theorem MMCC
 Matrix Multiplication and Complex Conjugation
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 Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{AB}$.
 \overline{B}

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Theorem MMT	Matrix Multiplication and Transposes	127
Suppose A is an m	$\times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.	
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Theorem MMAD	Matrix Multiplication and Adjoints	128
Suppose A is an $m \times$	<i>n</i> matrix and <i>B</i> is an $n \times p$ matrix. Then $(AB)^* = B^*A^*$.	
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Theorem AIP Adjoint and Inner Product

Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$.

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Definition HM	Hermitian Matrix	130
The square matrix	A is Hermitian (or self-adjoint) if $A = A^*$.	
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Theorem HMIP	Hermitian Matrices and Inner Products	131
Suppose that A is a s for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.	square matrix of size <i>n</i> . Then <i>A</i> is Hermitian if and only if $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{x} \rangle$	$4\mathbf{y}\rangle$
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Definition MI Ma	atrix Inverse	132
	e square matrices of size n such that $AB = I_n$ and the inverse of A . In this situation, we write $B = A^{-1}$	
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Theorem TTMI Two-by-Two Matrix Inverse

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

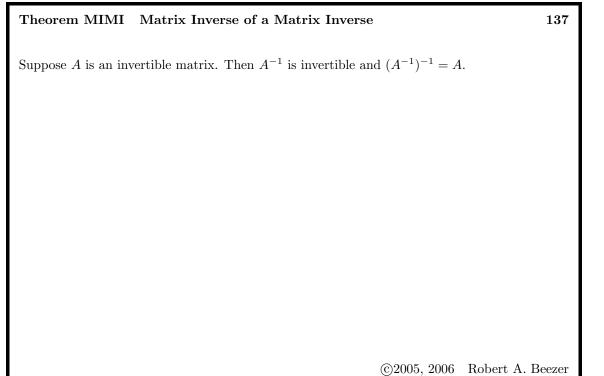
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Theorem CINM	Computing the I	nverse of a Nonsir	ngular Matrix	x 134
Suppose A is a nonsin $n \times n$ identity matrix to M and in reduced columns of N. Then	I_n to the right of the row-echelon form.	he matrix A . Let $N \mid$	be a matrix tha	at is row-equivalent
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Theorem MIU	Matrix Inverse is Unique	135
Suppose the square	re matrix A has an inverse. Then A^{-1} is unique.	
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Theorem SS Socks and Shoes	136
Suppose A and B are invertible matrices of size n. Then AB is an invertible matrix $(AB)^{-1} = B^{-1}A^{-1}$.	and



Theorem MIT	Matrix Inverse of a Transpose	138
Suppose A is an in	nvertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.	
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 Theorem MISM
 Matrix Inverse of a Scalar Multiple
 139

 Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and αA is invertible.

Theorem NPNT	Nonsingular Product has Nonsingular Terms 140)
Suppose that A and if A and B are both	B are square matrices of size n . The product AB is nonsingular if and only nonsingular.	y
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Theorem OSIS	One-Sided Inverse is Sufficient	141
Suppose A and B	B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.	
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 Theorem NI Nonsingularity is Invertibility
 142

 Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.
 Image: Comparison of the second secon

Theorem NME3 Nonsingular Matrix Equivalences, Round 3
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system LS(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.

Theorem SNCM	Solution	with Nonsing	gular Coeffic	ient Matrix	144
Suppose that A is no	onsingular.	Then the uniq	ue solution to ,	$\mathcal{LS}(A, \mathbf{b})$ is A^-	⁻¹ b.
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Definition UM Unitary Matrices	145
Suppose that U is a square matrix of size n such that $U^*U = I_n$. Then we say U is unita	ry.
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Theorem UMI	Unitary I	Matrices are Invertible	146
Suppose that U is	s a unitary n	natrix of size <i>n</i> . Then <i>U</i> is nonsingular, and $U^{-1} = U^*$.	
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Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets 147
Suppose that A is a square matrix of size n with columns $S = {\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n}$. Then A is a unitary matrix if and only if S is an orthonormal set.
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Theorem UMPIP	Unitary Matrices Pre	eserve Inner I	Products		148
Suppose that U is a u	unitary matrix of size n as	nd u and v are	two vectors fr	om \mathbb{C}^n . Then	
$\langle U {f u}, U$	$\langle \mathbf{v} angle = \langle \mathbf{u}, \mathbf{v} angle$	and	$\ U\mathbf{v}\ =$	$\ \mathbf{v}\ $	
			©2005, 2006	Robert A. B	eezer

Definition CSM Column Space of a Matrix

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS	Column Spaces and Consistent Systems 150)
Suppose A is an n $\mathcal{LS}(A, \mathbf{b})$ is consist	$n \times n$ matrix and b is a vector of size m . Then $\mathbf{b} \in \mathcal{C}(A)$ if and only its ent.	f
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Theorem BCS Basis of the Column Space

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $\mathcal{C}(A) = \langle T \rangle$.

Theorem CSNM	Column Space of a Nonsingular Matrix	152
Suppose A is a square	re matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.	
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Theorem NME4 Nonsingular Matrix Equivalences, Round 4
153
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system LS(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is Cⁿ, C(A) = Cⁿ.

Definition RSM Row Space of a Matrix

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

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Theorem REMRS	Row-Equivalent Matrices have eq	ual Row Spa	aces 155
Suppose A and B are	row-equivalent matrices. Then $\mathcal{R}(A) =$	$\mathcal{R}(B).$	
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Theorem BRS Basis for the Row Space	156
Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form S be the set of nonzero columns of B^t . Then	. Let
1. $\mathcal{R}(A) = \langle S \rangle.$	
2. S is a linearly independent set.	
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Theorem CSRST	Column Space, Row Space, Transpose	157	
Suppose A is a matri	x. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.		

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 Definition LNS
 Left Null Space
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 Suppose A is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

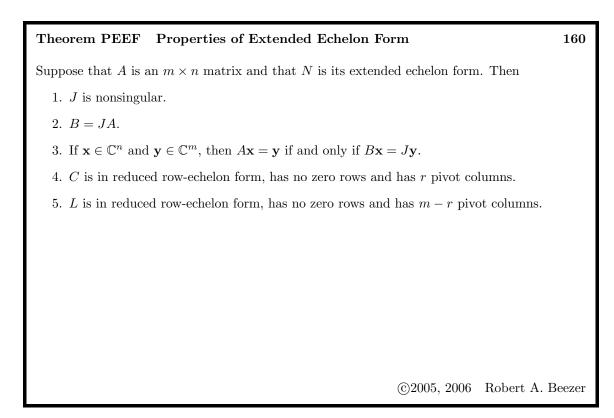
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Definition EEF Extended Echelon Form

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m - r) \times m$ matrix formed from the bottom m - r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$



Theorem FS Four Subsets

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m - r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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Definition VS Vector Space

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space over \mathbb{C} if the following ten properties hold.

- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- Z Zero Vector There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One If $\mathbf{u} \in V$ then $1\mathbf{u} = \mathbf{u}$

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

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Theorem ZVU Zero Vector is Unique	163
Suppose that V is a vector space. The zero vector, 0 , is unique.	
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Theorem AIU Additiv	e Inverses are Unique	164
Suppose that V is a vector	space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.	
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Theorem ZSSM	Zero Scalar in Scalar Multiplicatio	n	165
Suppose that V is a	vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = 0$.		
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Theorem ZVSM	Zero Vector in Scalar Multiplication	166
Suppose that V is a	vector space and $\alpha \in \mathbb{C}$. Then $\alpha 0 = 0$.	
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Theorem SMEZV	Scalar Multiplication Equals the Zero Vector	168
Suppose that V is a v	vector space and $\alpha \in \mathbb{C}$. If $\alpha \mathbf{u} = 0$, then either $\alpha = 0$ or $\mathbf{u} = 0$.	
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Definition S Subspace

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of $V, W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS Testing Subsets for Subspaces	170
Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the second operations as V. Then W is a subspace if and only if three conditions are met	ame
1. W is non-empty, $W \neq \emptyset$.	
2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.	
3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.	
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Definition TS	Trivial Subspaces				171
Given the vector	space V , the subspaces	V and $\{0\}$ are each	ı called a trivia	l subspace.	
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Theorem NSMS	Null Space of a Matrix is a Subspace	172
Suppose that A is a	n $m \times n$ matrix. Then the null space of A , $\mathcal{N}(A)$, is a subspace of \mathbb{C}^n .	
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Definition LC Linear Combination Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n.$

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Definition SS Span of a Set

Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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Theorem SSS Span of a Set is a Subspace

Suppose V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Theorem CSMS Column Space of a Matrix is a Subspace		176
Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m		
0000	0000	
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Theorem RSMS	Row Space of a Matrix is a Subspace	177
Suppose that A is a	n $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .	
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Theorem LNSMS	Left Null Space of a Matrix is a Subspace	178
Suppose that A is an	$m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .	
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Definition RLD Relation of Linear Dependence

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Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is a trivial relation of linear dependence on S.

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Definition LI Linear Independence

Suppose that V is a vector space. The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Definition TSVS To Span a Vector Space

Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

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Theorem VRRB Vector Representation Relative to a Basis

Suppose that V is a vector space and $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ is a linearly independent set that spans V. Let **w** be any vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \dots, a_m$ such that

 $\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$

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Definition B Basis

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

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Theorem SUVB Standard Unit Vectors are a Basis	184
The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .	$\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m \} \; = \;$
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Theorem CNMB Columns of Nonsingular Matrix are a Basis

Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

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Theorem NME5	Nonsingular Matrix Equivalences, Round 5	186
Suppose that A is a square matrix of size n . The following are equivalent.		
1. A is nonsingul	ar.	
2. A row-reduces	to the identity matrix.	
3. The null space	e of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear sys	tem $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of	of A are a linearly independent set.	
6. A is invertible		
7. The column sp	pace of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of	of A are a basis for \mathbb{C}^n .	
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Theorem COBCoordinates and Orthonormal Bases187Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For
any $\mathbf{w} \in W$,
 $\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \dots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$ $\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \dots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$ $(\mathbb{C}2005, 2006$ Robert A. Beezer

Theorem UMCOB Unitary Matrices Convert Orthonormal Bases 188	8
Let A be an $n \times n$ matrix and $B = {\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n}$ be an orthonormal basis of \mathbb{C}^n . Define	.e
$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$	
Then A is a unitary matrix if and only if C is an orthonormal basis of \mathbb{C}^n .	
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Definition D Dimension

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD	Spanning Sets and Linear Dependence	190
Suppose that $S = \{$ Then any set of $t +$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space - 1 or more vectors from V is linearly dependent.	e V.
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Theorem BIS Bases have Identical Sizes 191 Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size. Image: Comparison of the same size of the same si

Theorem DCM Dir	$\qquad \qquad $		192
The dimension of \mathbb{C}^m (1)	Example VSCV) is m .		
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Theorem DP Dimension of P_n		193
The dimension of P_n (Example VSP) is $n + 1$.		
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Theorem DM	Dimension of M_{mn}		194
The dimension of	M_{mn} (Example VSM) is mn .		
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Definition NOM Nullity Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

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Definition ROM Rank Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim (\mathcal{C}(A))$.

Theorem CRN Computing Rank and Nullity

197

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

Theorem RPNC	Rank Plus Nullity is Columns	198
Suppose that A is a	$m m \times n$ matrix. Then $r(A) + n(A) = n$.	
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Theorem RNNM Rank and Nullity of a Nonsingular Matrix

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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Theorem NME6 Nonsingular Matrix Equivalences, Round 6 Suppose that A is a square matrix of size n . The following are equivalent.	200
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of A are a basis for \mathbb{C}^n .	
9. The rank of A is $n, r(A) = n$.	
10. The nullity of A is zero, $n(A) = 0$.	
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Theorem ELIS Extending Linearly Independent Sets

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

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Theorem G Goldilocks 202	2
Suppose that V is a vector space of dimension t. Let $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ be a set of vectors from V. Then	f
1. If $m > t$, then S is linearly dependent.	ļ
2. If $m < t$, then S does not span V.	
3. If $m = t$ and S is linearly independent, then S spans V.	ļ
4. If $m = t$ and S spans V, then S is linearly independent.	ļ
	ļ
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Theorem PSSDProper Subspaces have Smaller Dimension203
Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then dim $(U) < \dim(V)$.

Theorem EDYES	Equal Dimensions Yields Equal Subspaces	204
Suppose that U and dim (V) . Then $U = V$	V are subspaces of the vector space W, such that $U \subseteq V$ a V.	nd $\dim(U) =$
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 Theorem RMRT
 Rank of a Matrix is the Rank of the Transpose
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 Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.
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Theorem DFS Dimensions of Four Subspaces

Suppose that A is an $m\times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

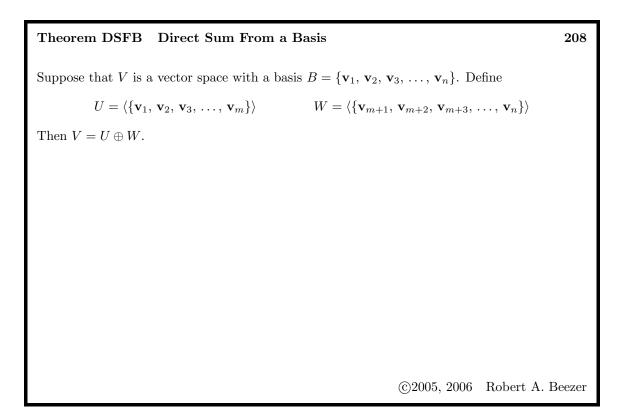
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Definition DS Direct Sum

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Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

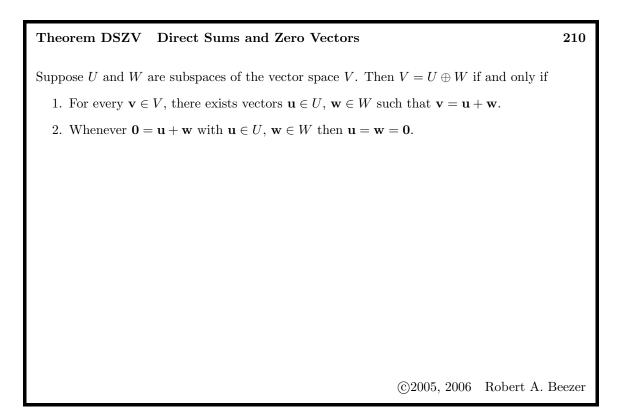
- 1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.
- Then V is the **direct sum** of U and W and we write $V = U \oplus W$.

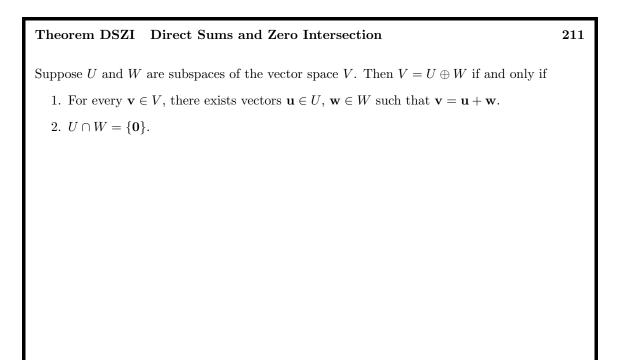


Theorem DSFOS Direct Sum From One Subspace

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Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.





Theorem DSLI Direct Sums and Linear Independence	212
Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose the linearly independent subset of U and S is a linearly independent subset of W. Then R linearly independent subset of V.	
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Theorem DSD Direct Sums and Dimension

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then dim $(V) = \dim(U) + \dim(W)$.

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Theorem RDS	Repeated Direct Sums
-------------	----------------------

Suppose V is a vector space with subspaces U and W with $V = U \oplus W$. Suppose that X and Y are subspaces of W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

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Definition ELEM Elementary Matrices

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq \\ 1 & k \neq i, k \neq j, \ell = \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

 $k \\ k$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = j \end{cases}$$

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Theorem EMDRO Elementary Matrices Do Row Operations

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.

2. If the row operation multiplies row *i* by α , then $B = E_i(\alpha) A$.

3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.

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Theorem EMN Elementary Matrices are Nonsingular					
If E is an elementary matrix, then E is nonsingular.					

Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices 218

Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \ldots, E_t$ so that $A = E_1 E_2 E_3 \ldots E_t$.

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Definition SM SubMatrix

Definition DM Determinant of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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Suppose A is a square matrix. Then its **determinant**, det (A) = |A|, is an element of \mathbb{C} defined recursively by: If A is a 1 × 1 matrix, then det $(A) = [A]_{11}$. If A is a matrix of size n with $n \ge 2$, then det $(A) = [A]_{11} \det (A(1|1)) - [A]_{12} \det (A(1|2)) + [A]_{13} \det (A(1|3)) - [A]_{14} \det (A(1|4)) + \dots + (-1)^{n+1} [A]_{1n} \det (A(1|n))$ (©2005, 2006 Robert A. Beezer

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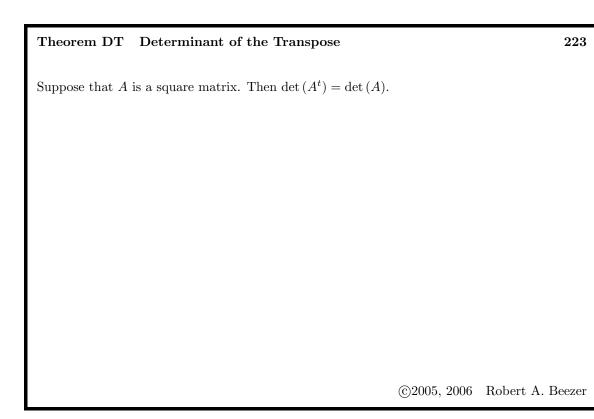
Theorem DMST Determinant of Matrices of Size Two

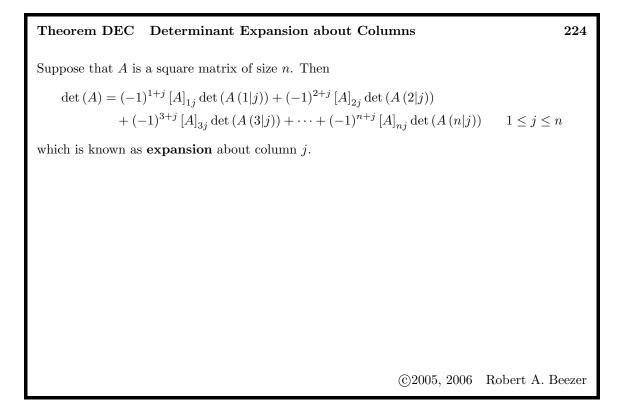
Suppose that
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Then det $(A) = ad - bc$

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Theorem DER Determinant Expansion about Rows	222			
Suppose that A is a square matrix of size n . Then				
$\det (A) = (-1)^{i+1} [A]_{i1} \det (A(i 1)) + (-1)^{i+2} [A]_{i2} \det (A(i 2)) + (-1)^{i+3} [A]_{i3} \det (A(i 3)) + \dots + (-1)^{i+n} [A]_{in} \det (A(i n)) \qquad 1 \le i \le n$	ı			
which is known as expansion about row i .				
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Theorem DZRC Determinant with Zero Row or Column	225
Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then det $(A) = 0$.	here

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Theorem DRCS Determinant for Row or Column Swap

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM Determinant for Row or Column Multiples

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α , or by multiplying a single column by the scalar α . Then det $(B) = \alpha \det(A)$.

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Theorem DERC	Determinant with Equal Rows or Columns 22	28
Suppose that A is a s	square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$	0.
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Theorem DRCMA Determinant for Row or Column Multiples and Addition 229

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then det $(B) = \det(A)$.

Theorem DIM	Determinant of the Identity Matrix	230
For every $n \ge 1$, d	$\det\left(I_n\right) = 1.$	
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Theorem DEM Determinants of Elementary Matrices

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. det $(E_{i,j}) = -1$
- 2. det $(E_i(\alpha)) = \alpha$
- 3. det $(E_{i,j}(\alpha)) = 1$

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Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 232

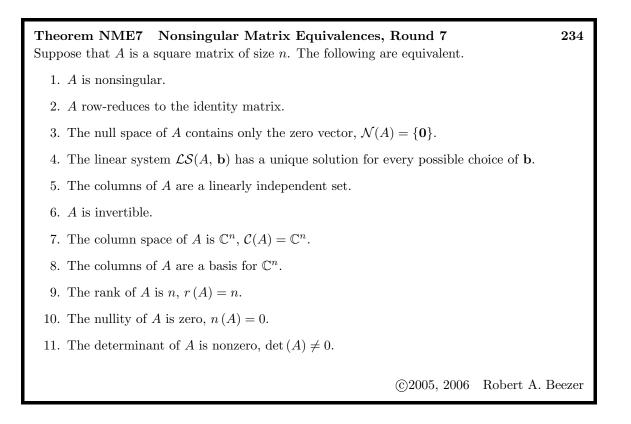
Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

 $\det (EA) = \det (E) \det (A)$

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Theorem SMZD Singular Matrices have Zero Determinants Let A be a square matrix. Then A is singular if and only if det(A) = 0. ©2005, 2006 Robert A. Beezer

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Theorem DRMM Determinant Respects Matrix Multiplication

Suppose that A and B are square matrices of the same size. Then det (AB) = det (A) det (B).

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Definition EEM Eigenvalues and Eigenvectors of a Matrix	236		
Suppose that A is a square matrix of size $n, \mathbf{x} \neq 0$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an eigenvector of A with eigenvalue λ if			
$A\mathbf{x} = \lambda \mathbf{x}$			
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Theorem EMHE Every Matrix Has an Eigenvalue	237			
Suppose A is a square matrix. Then A has at least one eigenvalue.				
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Definition CP Characteristic Polynomial 238 Suppose that A is a square matrix of size n. Then the characteristic polynomial of A is the polynomial $p_A(x)$ defined by $p_A(x) = \det(A - xI_n)$

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Theorem EMRCP	Eigenvalues of a Matrix are Roots of Characteristic Polynomi-
als	239

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

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Definition EM Eigenspace of a Matrix

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.

Theorem EMS	Eigenspace for a Matrix is a Subspace 2	241
	uare matrix of size n and λ is an eigenvalue of A . Then the eigenspace \mathcal{E}_A he vector space \mathbb{C}^n .	(λ)
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 Theorem EMNS
 Eigenspace of a Matrix is a Null Space
 242

 Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then
 $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$ $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$

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Definition AME Algebraic Multiplicity of an Eigenvalue

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME	Geometric Multiplicity of an Eigenvalue 24	44		
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the geometric muplicity of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.				
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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 245

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE Singular Matrices have Zero Eigenvalues					
Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.					
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Theorem NME8Nonsingular Matrix Equivalences, Round 82Suppose that A is a square matrix of size n. The following are equivalent.2		
1. A is nonsingular.		
2. A row-reduces to the identity matrix.		
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$		
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .		
5. The columns of A are a linearly independent set.		
6. A is invertible.		
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.		
8. The columns of A are a basis for \mathbb{C}^n .		
9. The rank of A is $n, r(A) = n$.		
10. The nullity of A is zero, $n(A) = 0$.		
11. The determinant of A is nonzero, $det(A) \neq 0$.		
12. $\lambda = 0$ is not an eigenvalue of A.		
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Theorem ESMM	Eigenvalues of a Scalar Multiple of a Matrix	248
Suppose A is a square	re matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of $\alpha\lambda$	4.
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Theorem EOMP Eigenvalues Of Matrix Powers	249
Suppose A is a square matrix, λ is an eigenvalue of A, and $s \ge 0$ is an integer. Then λ^s eigenvalue of A^s .	' is an
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Suppose A is a square matrix and λ is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix q(A). ©2005, 2006 Robert A. Beezer

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Theorem EPM Eigenvalues of the Polynomial of a Matrix

Theorem EIM Eigenvalues of the Inverse of a Matrix 251 Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1} . 1

Theorem ETM	Eigenvalues of the Transpose of a Matrix 252	2
Suppose A is a squ A^t .	hare matrix and λ is an eigenvalue of A . Then λ is an eigenvalue of the matrix	x
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Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs 253 Suppose A is a square matrix with real entries and **x** is an eigenvector of A for the eigenvalue λ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP	Degree of the Charact	eristic Polynon	nial	254
Suppose that A is has degree n .	a square matrix of size n .	Then the charac	teristic polyno	omial of A , $p_A(x)$,
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Theorem NEM Number of Eigenvalues of a Matrix

 $\mathbf{255}$

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) =$$

n

Theorem ME Multiplicities of an Eigenvalue	256
Suppose that A is a square matrix of size n and λ is an eigenvalue. Then	
$1 \le \gamma_A\left(\lambda\right) \le \alpha_A\left(\lambda\right) \le n$	
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Theorem MNEM Maximum Number of Eigenvalues of a Matrix 25	57
Suppose that A is a square matrix of size n . Then A cannot have more than n distinct eige values.	n-

Theorem HMRE	Hermitian Matrices have Real Eigenvalues	258
Suppose that A is a \mathcal{I}	Hermitian matrix and λ is an eigenvalue of A . Then $\lambda \in \mathbb{R}$.	
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Theorem HMOE	Hermitian Matrices have Orthogonal Eigenve	ectors 259
	a Hermitian matrix and \mathbf{x} and \mathbf{y} are two eigenvectors and \mathbf{y} are orthogonal vectors.	ors of A for different
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Definition SIM Similar Matrices

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

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Theorem SER Similarity is an Equivalence Relation

Suppose A, B and C are square matrices of size n. Then

- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE Similar Mat	rices have Equal I	Eigenvalues	262
Suppose A and B are similar mat equal, that is, $p_A(x) = p_B(x)$.	rices. Then the char	acteristic polynomia	ls of A and B are
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Definition DIM	Diagonal Matrix	263
Suppose that A is a	a square matrix. Then A is a diagonal matrix if $[A]_{ij} = 0$ whenever	$i \neq j.$
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Definition DZM Diagonalizable Matrix	264
Suppose A is a square matrix. Then A is diagonalizable if A is simil	ar to a diagonal matrix.
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Theorem DC	Diagonalization Characterization	265		
Suppose A is a square matrix of size n . Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A .				
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Theorem DMFE	Diagonalizable Matrices have Full Eigenspaces 2	66
Suppose A is a square eigenvalue λ of A .	re matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for ever	əry

Theorem DED Distinct Eigenvalues implies Diagonalizable

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

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Definition LT Linear Transformation

A linear transformation, $T: U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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Theorem LTTZZ Linear Transformations Take Zero to Zero	269
Suppose $T: U \mapsto V$ is a linear transformation. Then $T(0) = 0$.	
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Theorem MBLT	Matrices Build Linear Transformations	270
Suppose that A is an a linear transformat	an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then tion.	ı T is
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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors 271

Suppose that $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations 272			
Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then			
$T(a_{1}\mathbf{u}_{1} + a_{2}\mathbf{u}_{2} + a_{3}\mathbf{u}_{3} + \dots + a_{t}\mathbf{u}_{t}) = a_{1}T(\mathbf{u}_{1}) + a_{2}T(\mathbf{u}_{2}) + a_{3}T(\mathbf{u}_{3}) + \dots + a_{t}T(\mathbf{u}_{t})$			
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Theorem LTDB Linear Transformation Defined on a Basis

Suppose $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a basis for the vector space U and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation, $T: U \mapsto V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i, 1 \leq i \leq n$.

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Definition PI Pre-Image

Suppose that $T: U \mapsto V$ is a linear transformation. For each **v**, define the **pre-image** of **v** to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{\mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v}\right\}$$

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Definition LTA Linear Transformation Addition

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Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S: U \mapsto V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 276

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S: U \mapsto V$ is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication

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Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the scalar multiple is the function $\alpha T: U \mapsto V$ whose outputs are defined by

$$\left(\alpha T\right)\left(\mathbf{u}\right) = \alpha T\left(\mathbf{u}\right)$$

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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 278

Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T): U \mapsto V$ is a linear transformation.

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Theorem VSLT Vector Space of Linear Transformations

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{L}T(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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Definition LTC Linear Transformation Composition

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T): U \mapsto W$ whose outputs are defined by

 $(S \circ T) (\mathbf{u}) = S (T (\mathbf{u}))$

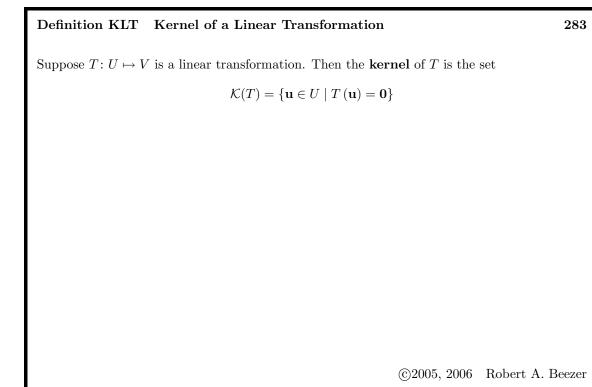
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Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 281

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then $(S \circ T): U \mapsto W$ is a linear transformation.

Definition ILT Injective Linear Transformation	282
Suppose $T: U \mapsto V$ is a linear transformation. Then T is injective if wheneve then $\mathbf{x} = \mathbf{y}$.	$\operatorname{er} T\left(\mathbf{x}\right) = T\left(\mathbf{y}\right),$
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Theorem KLTS Kernel of a Linear Transformation is a Subspace 284 Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of $T, \mathcal{K}(T)$, is a subspace of U.

Theorem KPI Kernel and Pre-Image

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

 $T^{-1}\left(\mathbf{v}\right) = \left\{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\right\} = \mathbf{u} + \mathcal{K}(T)$

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Theorem KILT	Kernel of an Injective Line	ear Transformation	286
Suppose that $T: U$ of T is trivial, $\mathcal{K}(T)$	$Y \mapsto V$ is a linear transformation. $Y = \{0\}.$. Then T is injective if an	d only if the kernel
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Theorem ILTLI Injective Linear Transformations and Linear Independence $\mathbf{287}$

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

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Theorem ILTB	Injective Linear Transformations and Bases	288
	$U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis productive if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linear of V .	
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Theorem ILTD Injective Linear Transformations and Dimension 289 Suppose that $T: U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$. (©2005, 2006) Robert A. Beezer

Theorem CILTI Composition of Injective Linear Transformations is Injective 290

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are injective linear transformations. Then $(S \circ T): U \mapsto W$ is an injective linear transformation.

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Definition SLT Surjective Linear Transformation 291 Suppose $T: U \mapsto V$ is a linear transformation. Then T is surjective if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$. (©2005, 2006) Robert A. Beezer

Definition RLTRange of a Linear Transformation292Suppose $T: U \mapsto V$ is a linear transformation. Then the range of T is the set $\mathcal{R}(T) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$ $\mathcal{R}(T) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$ ©2005, 2006Robert A. Beezer

Theorem RLTS Range of a Linear Transformation is a Subspace

Suppose that $T: U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

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Theorem RSLT	Range of a Surjective Linear Transformation 29	94
Suppose that $T: U$ of T equals the code	$H \mapsto V$ is a linear transformation. Then T is surjective if and only if the range domain, $\mathcal{R}(T) = V$.	.ge
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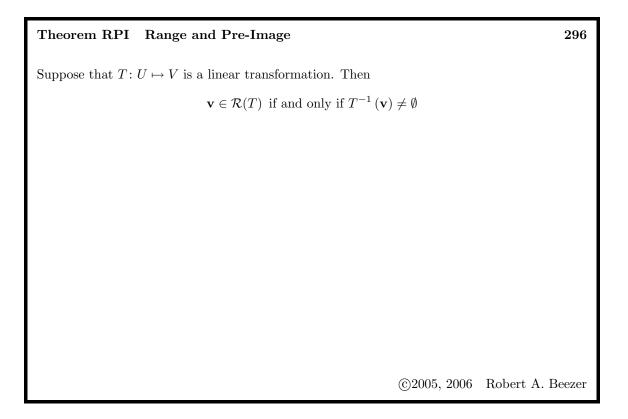
Theorem SSRLT Spanning Set for Range of a Linear Transformation

Suppose that $T: U \mapsto V$ is a linear transformation and $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$ spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

spans $\mathcal{R}(T)$.

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Theorem SLTB Surjective Linear Transformations and Bases

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.

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Theorem SLTD	Surjective Linear Transformations and Dimension	298
Suppose that $T: U$	$F \mapsto V$ is a surjective linear transformation. Then dim $(U) \ge \dim (V)$.	
The second s	· · · · · · · · · · · · · · · · · · ·	
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Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 299

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are surjective linear transformations. Then $(S \circ T): U \mapsto W$ is a surjective linear transformation.

Definition IDLT Identity	⁷ Linear Transfo	rmation	300
The identity linear transfo	rmation on the v	ector space W is defined as	
	$I_W \colon W \mapsto W,$	$I_{W}\left(\mathbf{w}\right)=\mathbf{w}$	
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Definition IVLTInvertible Linear Transformations301Suppose that $T: U \mapsto V$ is a linear transformation. If there is a function $S: V \mapsto U$ such that $S \circ T = I_U$ $T \circ S = I_V$ then T is invertible. In this case, we call S the inverse of T and write $S = T^{-1}$.

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Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 302

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}: V \mapsto U$ is a linear transformation.

Theorem IILT Inverse of an Invertible Linear Transformation

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$.

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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective304

Suppose $T: U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

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Theorem CIVLT Composition of Invertible Linear Transformations	305
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. composition, $(S \circ T): U \mapsto W$ is an invertible linear transformation.	Then the

Theorem ICLT	Inverse of a Composition of Linear Transformations	306
Suppose that $T: U$ invertible and $(S \circ$	$V \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. $(T)^{-1} = T^{-1} \circ S^{-1}.$	Then $S \circ T$ is
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Definition IVS Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain $V, T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED	Isomorphic Vector Spaces have Equal Dimension 30)8
Suppose U and V ar	re isomorphic vector spaces. Then $\dim(U) = \dim(V)$.	
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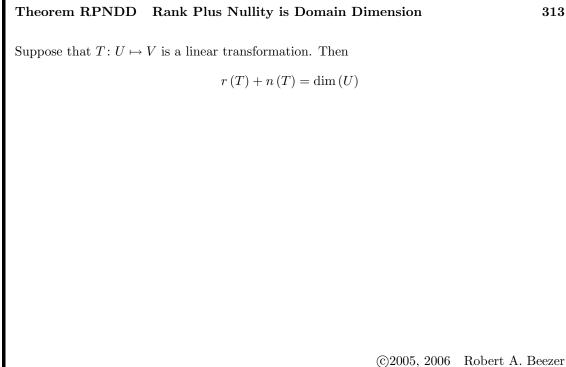
Definition ROLT Rank Of a Linear Transformation 309 Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T, r(T), is the dimension of the range of T, $r(T) = \dim(\mathcal{R}(T))$ ©2005, 2006 Robert A. Beezer

 Definition NOLT
 Nullity Of a Linear Transformation
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 Suppose that $T: U \mapsto V$ is a linear transformation. Then the nullity of T, n(T), is the dimension of the kernel of T,
 $n(T) = \dim(\mathcal{K}(T))$
 $n(T) = \dim(\mathcal{K}(T))$ $n(T) = \dim(\mathcal{K}(T))$

Theorem ROSLT Rank Of a Surjective Linear Transformation311Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V,
 $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT	Nullity Of an Injective Linear Transformation312
Suppose that $T: U \vdash$ and only if T is inject	$\rightarrow V$ is a linear transformation. Then the nullity of T is zero, $n(T) = 0$, if tive.
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Definition VR Vector Representation

Suppose that V is a vector space with a basis $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$. Define a function $\rho_B: V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by

 $\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \dots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$

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Theorem VRLT	Vector Representation is a Linear Transformation 3	15
The function ρ_B (D	Definition VR) is a linear transformation.	
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Theorem VRI Vector Representation is Injective	316
The function ρ_B (Definition VR) is an injective linear transformation.	
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Theorem VRS Vector Representation is Surjective	317
The function ρ_B (Definition VR) is a surjective linear transformation.	
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Theorem VRILT Vector Representation is an Invertible Linear Transformation 318

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS	Characterization of Finite Dimensional Vector Spaces	319
Suppose that V is a	vector space with dimension n . Then V is isomorphic to \mathbb{C}^n .	
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Suppose U and V are both finite-dimensional vector spaces. and only if dim $(U) = \dim(V)$.	Then U and V are isomorphic if

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces

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Theorem CLI Coordinatization and Linear Independence

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n .

oordinatization and Spanning Sets	322
vector space with a basis <i>B</i> of size <i>n</i> . Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} $ $\langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle.$) if
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V	ector space with a basis <i>B</i> of size <i>n</i> . Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \\ \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle.$

Definition MR Matrix Representation

Suppose that $T: U \mapsto V$ is a linear transformation, $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

 $M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations325

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then M_{R}^{T+S} S T

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 326

Suppose that $T: U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then $M_{\Sigma}^{\alpha T}$ T

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 327

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

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Theorem KNS	Kernel an	d Null Space	Isomorphism

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}\big(M_{B,C}^T\big)$$

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Theorem IMR Invertible Matrix Representations

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix. When T is invertible,

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^{T}\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

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Theorem NME9 Nonsingular Matrix Equivalences, Round 9 Suppose that A is a square matrix of size n . The following are equivalent.	332
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of A are a basis for \mathbb{C}^n .	
9. The rank of A is $n, r(A) = n$.	
10. The nullity of A is zero, $n(A) = 0$.	
11. The determinant of A is nonzero, $\det(A) \neq 0$.	
12. $\lambda = 0$ is not an eigenvalue of A.	
13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.	
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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation 333 Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

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Definition CBM Change-of-Basis Matrix

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

= $[\rho_C (I_V (\mathbf{v}_1)) | \rho_C (I_V (\mathbf{v}_2)) | \rho_C (I_V (\mathbf{v}_3)) | \dots | \rho_C (I_V (\mathbf{v}_n))]$
= $[\rho_C (\mathbf{v}_1) | \rho_C (\mathbf{v}_2) | \rho_C (\mathbf{v}_3) | \dots | \rho_C (\mathbf{v}_n)]$

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Theorem CB Change-of-Basis

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

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Theorem ICBM Inverse of Char	-of-Basis Matrix 336
Suppose that V is a vector space, and I $C_{B,C}$ is nonsingular and	and C are bases of V. Then the change-of-basis matrix $C_{B,C}^{-1} = C_{C,B}$
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Theorem MRCB Matrix Representation and Change of Basis 337 Suppose that $T: U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then $M_{B,D}^T = C_{E,D}M_{C,E}^TC_{B,C}$ $M_{B,D}^T = C_{E,D}M_{C,E}^TC_{B,C}$ $(\bigcirc 2005, 2006)$ Robert A. Beezer

 Theorem SCB
 Similarity and Change of Basis
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 Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then
 $M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$
 $M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$ $(C_{B,C}, C_{B,C}, C_{B,C})$

 (Countries)
 $(C_{B,C}, C_{B,C}, C_{B,C})$

Theorem EER Eigenvalues, Eigenvectors, Representations

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Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

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Definition NLT Nilpotent Linear Transformation

Suppose that $T: V \mapsto V$ is a linear transformation such that there is an integer p > 0 such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.

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Definition JB Jordan Block

Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$\left[J_n\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$

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Theorem NJB	Nilpotent Jordan Blocks		342
The Jordan block	$J_{n}(0)$ is nilpotent of index n .		
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Theorem ENLT	Eigenvalues of Nilpotent Linear Transformations 3	343
Suppose that $T: V$ $\lambda = 0.$	$\mapsto V$ is a nilpotent linear transformation and λ is an eigenvalue of T . The second secon	nen
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Theorem DNLT	Diagonalizable Nilpotent Linear	r Transformation	ns 344
Suppose the linear t T is the zero linear	transformation $T \colon V \mapsto V$ is nilpotent transformation.	. Then T is diagon	alizable if and only
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Theorem KPLT Kernels of Powers of Linear Transformations

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Suppose $T: V \mapsto V$ is a linear transformation, where dim (V) = n. Then there is an integer m, $0 \le m \le n$, such that

 $\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$

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Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 346

Suppose $T: V \mapsto V$ is a nilpotent linear transformation with index p and dim (V) = n. Then $0 \le p \le n$ and

 $\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$

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Theorem CFNLT Canonical Form for Nilpotent Linear Transformations 347

Suppose that $T: V \mapsto V$ is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation, $M_{B,B}^T$, is block diagonal with each block being a Jordan block, $J_n(0)$. The size of the largest block is the index p, and the total number of blocks is the nullity of T, n(T).

Definition IS Invariant Subspace	348
Suppose that $T: V \mapsto V$ is a linear transformation and W is a subspace of V , that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$. Then W is an invariant subspace of V is	
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Theorem EIS Eigenspaces are Invariant Subspaces

Suppose that $T: V \mapsto V$ is a linear transformation with eigenvalue λ and associated eigenspace $\mathcal{E}_T(\lambda)$. Let W be any subspace of $\mathcal{E}_T(\lambda)$. Then W is an invariant subspace of V relative to T.

Theorem KPIS	Kernels of Powers are Invariant Subspaces350
	$V \mapsto V$ is a linear transformation. Then $\mathcal{K}(T^k)$ is an invariant subspace of
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Definition GEV Generalized Eigenvector

Suppose that $T: V \mapsto V$ is a linear transformation. Suppose further that for $\mathbf{x} \neq \mathbf{0}$, $(T - \lambda I_V)^k (\mathbf{x}) = \mathbf{0}$ for some k > 0. Then \mathbf{x} is a **generalized eigenvector** of T with eigenvalue λ .

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Definition GES Generalized Eigenspace

Suppose that $T: V \mapsto V$ is a linear transformation. Define the **generalized eigenspace** of T for λ as

$$\mathcal{G}_{T}\left(\lambda\right) = \left\{ \mathbf{x} \mid \left(T - \lambda I_{V}\right)^{k}\left(\mathbf{x}\right) = \mathbf{0} \text{ for some } k \ge 0 \right\}$$

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Theorem GESIS	Generalized Eigenspace is an Invariant Subspace 353
Suppose that $T: V \vdash$ an invariant subspace	V is a linear transformation. Then the generalized eigenspace $\mathcal{G}_{T}(\lambda)$ is of V relative to T.
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Theorem GEK	Generalized Eigenspace as a Kernel	354
Suppose that $T: V$ Then $\mathcal{G}_T(\lambda) = \mathcal{K}(\mathbf{r})$	$V \mapsto V$ is a linear transformation, dim $(V) = n$, and λ is an eigen $(T - \lambda I_V)^n$.	envalue of T .
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Definition LTR Linear Transformation Restriction

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Suppose that $T: V \mapsto V$ is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by

$$T|_{U} \colon U \mapsto U \qquad \qquad T|_{U} \left(\mathbf{u}\right) = T\left(\mathbf{u}\right)$$

Theorem RGEN	Restriction to Generalized Eigenspace is Nilpotent	356
Suppose $T: V \mapsto V$ is $T _{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is a	s a linear transformation with eigenvalue λ . Then the linear transfor nilpotent.	mation
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Definition IE Index of an Eigenvalue

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Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then the **index** of λ , $\iota_T(\lambda)$, is the index of the nilpotent linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$.

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Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace 358

Suppose that $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then there is a basis of the the generalized eigenspace $\mathcal{G}_T(\lambda)$ such that the restriction $T|_{\mathcal{G}_T(\lambda)}: \mathcal{G}_T(\lambda) \mapsto \mathcal{G}_T(\lambda)$ has a matrix representation that is block diagonal where each block is a Jordan block of the form $J_n(\lambda)$.

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Theorem GESD	Generalized Eigenspace Decomposition	359
Suppose that $T: V$ Then	$\mapsto V$ is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$	$, \lambda_m.$
	$V = \mathcal{G}_{T}(\lambda_{1}) \oplus \mathcal{G}_{T}(\lambda_{2}) \oplus \mathcal{G}_{T}(\lambda_{3}) \oplus \cdots \oplus \mathcal{G}_{T}(\lambda_{m})$	

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Theorem DGES Dimension of Generalized Eigenspaces	360
Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then the dimension generalized eigenspace for λ is the algebraic multiplicity of λ , dim $(\mathcal{G}_T(\lambda_i)) = \alpha_T(\lambda_i)$.	of the
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Definition JCF Jordan Canonical Form

A square matrix is in **Jordan canonical form** if it meets the following requirements:

- 1. The matrix is block diagonal.
- 2. Each block is a Jordan block.
- 3. If $\rho < \lambda$ then the block $J_k(\rho)$ occupies rows with indices greater than the indices of the rows occupied by $J_\ell(\lambda)$.
- 4. If $\rho = \lambda$ and $\ell < k$, then the block $J_{\ell}(\lambda)$ occupies rows with indices greater than the indices of the rows occupied by $J_k(\lambda)$.

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Theorem JCFLT Jordan Canonical Form for a Linear Transformation

Suppose $T: V \mapsto V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

- 1. The matrix representation is in Jordan canonical form.
- 2. If $J_k(\lambda)$ is one of the Jordan blocks, then λ is an eigenvalue of T.
- 3. For a fixed value of λ , the largest block of the form $J_k(\lambda)$ has size equal to the index of λ , $\iota_T(\lambda)$.
- 4. For a fixed value of λ , the number of blocks of the form $J_k(\lambda)$ is the geometric multiplicity of λ , $\gamma_T(\lambda)$.
- 5. For a fixed value of λ , the number of rows occupied by blocks of the form $J_k(\lambda)$ is the algebraic multiplicity of λ , $\alpha_T(\lambda)$.

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Theorem CHT Cayley-Hamilton Theorem 363 Suppose A is a square matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = \mathcal{O}$.