## Flash Cards

to accompany

# A First Course in Linear Algebra

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Version 1.34

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The most recent version of this work can always be found at http://linear.ups.edu.

The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are <b>equal</b> , denoted $\alpha = \beta$ , if $a = c$ and $b = d$ .	Definition CNE Complex Number Equality 1	
	The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are <b>equal</b> , denoted $\alpha = \beta$ , if $a = c$ and $b = d$ .	

Definition CNA	Complex Number Addition	2
The <b>sum</b> of the con	applex numbers $\alpha = a + bi$ and $\beta = c + di$ , denoted $\alpha + \beta$ , is $(a + c) + (b + d)i$	į.
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#### Definition CNM Complex Number Multiplication

The **product** of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha\beta$ , is (ac - bd) + (ad + bc)i.

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Theorem PCNAProperties of Complex Number Arithmetic4The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha + \beta \in \mathbb{C}$ .
- MCCN Multiplicative Closure, Complex Numbers If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha\beta \in \mathbb{C}$ .
- CACN Commutativity of Addition, Complex Numbers For any  $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$ .
- CMCN Commutativity of Multiplication, Complex Numbers For any  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha\beta = \beta\alpha$ .
- AACN Additive Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}, \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .
- MACN Multiplicative Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ .
- DCN Distributivity, Complex Numbers For any  $\alpha$ ,  $\beta$ ,  $\gamma \in \mathbb{C}$ ,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ .
- ZCN Zero, Complex Numbers There is a complex number 0 = 0 + 0i so that for any  $\alpha \in \mathbb{C}, 0 + \alpha = \alpha$ .
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any  $\alpha \in \mathbb{C}$ ,  $1\alpha = \alpha$ .
- AICN Additive Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$  there exists  $-\alpha \in \mathbb{C}$  so that  $\alpha + (-\alpha) = 0$
- MICN Multiplicative Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$  there exists  $\frac{1}{\alpha} \in \mathbb{C}$  so that  $\alpha(\frac{1}{\alpha}) = 1$ .

Definition CCN	Conjugate of a Complex Number	5	
The <b>conjugate</b> of	the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$ .		
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Theorem CCRA	Complex Conjugation Respects Addition	6
Suppose that $c$ and $c$	d are complex numbers. Then $\overline{c+d} = \overline{c} + \overline{d}$ .	
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Theorem CCRM	Complex Conjugation Respects M	[ultiplication	7
Suppose that $c$ and $c$	$d$ are complex numbers. Then $\overline{cd} = \overline{cd}$ .		
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		32000, 2000	HODELU II. DECZEL

Theorem CCT Complex Conjugation Twice 8 Suppose that c is a complex number. Then  $\overline{\overline{c}} = c$ . ©2005, 2006 Robert A. Beezer

#### Definition MCN Modulus of a Complex Number

The **modulus** of the complex number  $c = a + bi \in \mathbb{C}$ , is the nonnegative real number

 $|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}.$ 

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Definition SET Set

A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write  $x \in S$ . If x is not in S, then we write  $x \notin S$ . We refer to the objects in a set as its elements.

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 $\mathbf{10}$ 

9

Definition SSET Subset	11
If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever x	$x \in S$ then $x \in T$ .
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Definition ES	Empty Set	12
The empty set is	the set with no elements. Its is denoted by $\emptyset$ .	
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#### Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$ . In this case, we write S = T.

Definition SE Set Equality

Definition C Cardinality	<b>14</b>
Suppose S is a finite set. Then the number of elements in S is called the ca of S, and is denoted $ S $ .	ardinality or size
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#### Definition SU Set Union

Suppose S and T are sets. Then the **union** of S and T, denoted  $S \cup T$ , is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$  if and only if  $x \in S$  or  $x \in T$ 

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 $\mathbf{15}$ 

 $\mathbf{16}$ 

Definition SI S	Set Intersection
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Suppose S and T are sets. Then the **intersection** of S and T, denoted  $S \cap T$ , is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$  if and only if  $x \in S$  and  $x \in T$ 

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#### Definition SC Set Complement

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted  $\overline{S}$ , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$  if and only if  $x \in U$  and  $x \notin S$ 

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Definition SLESystem of Linear Equations18A system of linear equations is a collection of m equations in the variable quantities<br/> $x_1, x_2, x_3, \dots, x_n$  of the form, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ <br/> $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ <br/> $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ <br/> $\vdots$ <br/> $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ where the values of  $a_{ij}, b_i$  and  $x_j$  are from the set of complex numbers,  $\mathbb{C}$ .©2005, 2006Robert A. Beezer

9

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#### Definition EO Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

#### $\mathbf{20}$

#### Theorem EOPSS Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

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#### Definition M Matrix

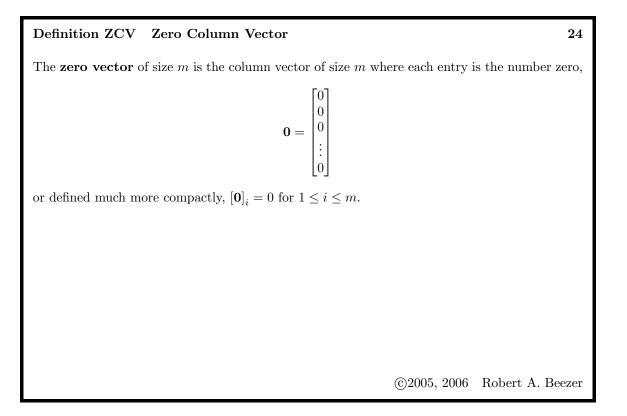
An  $m \times n$  matrix is a rectangular layout of numbers from  $\mathbb{C}$  having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation  $[A]_{ij}$  will refer to the complex number in row i and column j of A.

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#### $\mathbf{22}$

#### Definition CV Column Vector

A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ . Some books like to write vectors with arrows, such as  $\vec{u}$ . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component that is number i in the list that is the vector  $\mathbf{v}$  we write  $[\mathbf{v}]_i$ .



#### Definition CM Coefficient Matrix

For a system of linear equations,

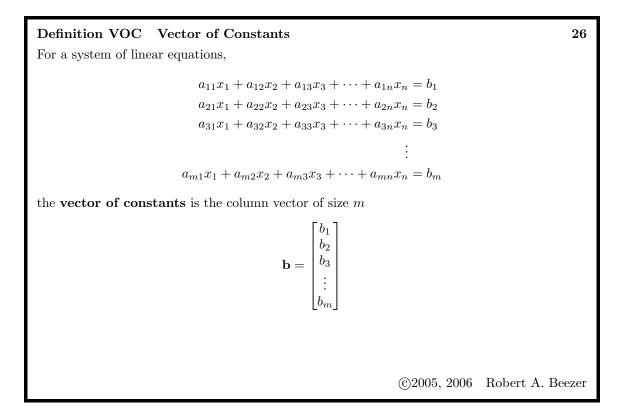
 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$   $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$   $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ :

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ 

the **coefficient matrix** is the  $m \times n$  matrix

	$a_{11}$	$a_{12}$	$a_{13} \\ a_{23} \\ a_{33}$	• • •	$a_{1n}$
	$a_{21}$	$a_{22}$	$a_{23}$		$a_{2n}$
A =	$a_{31}$	$a_{32}$	$a_{33}$		$a_{3n}$
	:				
	•				
	$a_{m1}$	$a_{m2}$	$a_{m3}$	• • •	$a_{mn}$





 $\mathbf{25}$ 

#### Definition SOLV Solution Vector

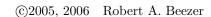
For a system of linear equations,

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$   $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$  $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ 

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ 

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



 $\mathbf{27}$ 

 $\mathbf{28}$ 

#### Definition LSMR Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and **b** is the vector of constants, then we will write  $\mathcal{LS}(A, \mathbf{b})$  as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

#### Definition AM Augmented Matrix

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants **b**. Then the **augmented matrix** of the system of equations is the  $m \times (n + 1)$  matrix whose first n columns are the columns of A and whose last column (number n + 1) is the column vector **b**. This matrix will be written as  $[A \mid \mathbf{b}]$ .

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#### Definition RO Row Operations

The following three operations will transform an  $m \times n$  matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1.  $R_i \leftrightarrow R_j$ : Swap the location of rows *i* and *j*.
- 2.  $\alpha R_i$ : Multiply row *i* by the nonzero scalar  $\alpha$ .
- 3.  $\alpha R_i + R_j$ : Multiply row *i* by the scalar  $\alpha$  and add to row *j*.

30

#### Definition REM Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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#### Theorem REMES Row-Equivalent Matrices represent Equivalent Systems 32

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

#### Definition RREF Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r. A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  where  $d_1 < d_2 < d_3 < \cdots < d_r$ , while the columns that are not pivot columns will be denoted as  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  where  $f_1 < f_2 < f_3 < \cdots < f_{n-r}$ .

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33

34

### **Theorem REMEF** Row-Equivalent Matrix in Echelon Form Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

#### Theorem RREFU Reduced Row-Echelon Form is Unique

Suppose that A is an  $m \times n$  matrix and that B and C are  $m \times n$  matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

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#### Definition RR Row-Reducing

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

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36

#### Definition CS Consistent System

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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#### Definition IDV Independent and Dependent Variables

Suppose A is the augmented matrix of a consistent system of linear equations and B is a rowequivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column). Then the variable  $x_j$  is **dependent**. A variable that is not dependent is called **independent** or **free**.

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37

38

#### Theorem RCLS Recognizing Consistency of a Linear System

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n + 1 of B.

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39

**40** 

**Theorem ISRN** Inconsistent Systems, r and n

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

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#### **Theorem CSRN** Consistent Systems, r and n

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then  $r \leq n$ . If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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#### Theorem FVCS Free Variables for Consistent Systems

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n - r free variables.

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41

 $\mathbf{42}$ 

#### Theorem PSSLS Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

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 $\mathbf{43}$ 

Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions 44

Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions.

#### Definition HS Homogeneous System

A system of linear equations,  $\mathcal{LS}(A, \mathbf{b})$  is **homogeneous** if the vector of constants is the zero vector, in other words,  $\mathbf{b} = \mathbf{0}$ .

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 $\mathbf{45}$ 

Theorem HSC Homogeneous Systems are Consistent	46
Suppose that a system of linear equations is homogeneous. Then the system is consistent.	
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Definition TSHSE	Trivial Solution to Homogeneous Systems	s of Equations	47
	bus system of linear equations has $n$ variables. e. $\mathbf{x} = 0$ is called the <b>trivial solution</b> .	The solution $x_1$ :	= 0,
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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 48

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

#### Definition NSM Null Space of a Matrix

The **null space** of a matrix A, denoted  $\mathcal{N}(A)$ , is the set of all the vectors that are solutions to the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$ .

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#### Definition SQM Square Matrix

A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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 $\mathbf{50}$ 

#### Definition NM Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations  $\mathcal{LS}(A, \mathbf{0})$  is  $\{\mathbf{0}\}$ , i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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Definition IMIdentity Matrix52The  $m \times m$  identity matrix,  $I_m$ , is defined by $[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$  $1 \le i, j \le m$ ©2005, 2006Robert A. Beezer

#### Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 53

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 5	64
	quare matrix. Then A is nonsingular if and only if the null space of A ne zero vector, i.e. $\mathcal{N}(A) = \{0\}.$	4,
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	©2005, 2006 Robert A. Beeze	51

#### Theorem NMUS Nonsingular Matrices and Unique Solutions

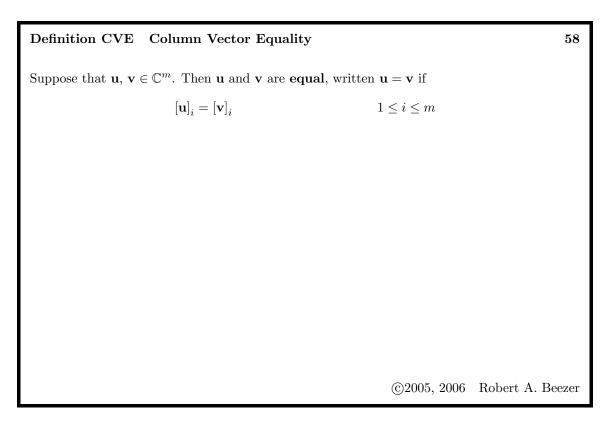
Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every choice of the constant vector  $\mathbf{b}$ .

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 $\mathbf{55}$ 

Theorem NME1	Nonsingular Matrix Equivalences, Round 1	56
Suppose that $A$ is a	square matrix. The following are equivalent.	
1. $A$ is nonsingul	ar.	
2. $A$ row-reduces	to the identity matrix.	
3. The null space	e of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear sys	tem $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice	of <b>b</b> .
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Definition VSCV	Vector Space of Column Vectors 57	7
The vector space $\mathbb{C}^m$ the set of complex m	is the set of all column vectors (Definition CV) of size $m$ with entries from umbers, $\mathbb{C}$ .	n
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Definition CVA	Column Vector Addition		59
Suppose that $\mathbf{u},\mathbf{v}$	$\in \mathbb{C}^m$ . The <b>sum</b> of <b>u</b> and <b>v</b> is the vector	$\mathbf{u} + \mathbf{v}$ defined b	ру
	$\left[\mathbf{u}+\mathbf{v}\right]_i=\left[\mathbf{u}\right]_i+\left[\mathbf{v}\right]_i$	$1 \leq i \leq m$	
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Definition CVSM	Column Vector Scalar Multipli	cation	60
Suppose $\mathbf{u} \in \mathbb{C}^m$ and	$\alpha \in \mathbb{C}$ , then the scalar multiple of	<b>u</b> by $\alpha$ is the vec	tor $\alpha \mathbf{u}$ defined by
	$\left[\alpha\mathbf{u}\right]_{i}=\alpha\left[\mathbf{u}\right]_{i}$	$1 \leq i \leq m$	
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		C2005, 2006	Robert A. Beezer

#### Theorem VSPCV Vector Space Properties of Column Vectors 61 Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then • ACC Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ .

- SCC Scalar Closure, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha \mathbf{u} \in \mathbb{C}^m$ .
- CC Commutativity, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AAC Additive Associativity, Column Vectors If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v}$  $(\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- ZC Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in \mathbb{C}^m$ .
- AIC Additive Inverses, Column Vectors If  $\mathbf{u} \in \mathbb{C}^m$ , then there exists a vector  $-\mathbf{u} \in \mathbb{C}^m$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMAC Scalar Multiplication Associativity, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVAC Distributivity across Vector Addition, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSAC Distributivity across Scalar Addition, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .
- OC One Column Vectors If  $\mathbf{u} \in \mathbb{C}^m$  then  $\mathbf{1}_{\mathbf{u}}$

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62

# Definition LCCV Linear Combination of Column Vectors Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their linear combination is the vector $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$

#### Theorem SLSLC Solutions to Linear Systems are Linear Combinations

Denote the columns of the  $m \times n$  matrix A as the vectors  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ . Then  $\mathbf{x}$  is a solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$  if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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63

64

#### Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that  $[A \mid \mathbf{b}]$  is the augmented matrix for a consistent linear system  $\mathcal{LS}(A, \mathbf{b})$  of m equations in n variables. Let B be a row-equivalent  $m \times (n + 1)$  matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$ , and columns with leading 1's (pivot columns) having indices  $D = \{d_1, d_2, d_3, \ldots, d_r\}$ . Define vectors  $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$  of size n by

$$\begin{aligned} \left[ \mathbf{c} \right]_{i} &= \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, \, i = d_{k} \end{cases} \\ \left[ \mathbf{u}_{j} \right]_{i} &= \begin{cases} 1 & \text{if } i \in F, \, i = f_{j} \\ 0 & \text{if } i \in F, \, i \neq f_{j} \\ -[B]_{k,f_{j}} & \text{if } i \in D, \, i = d_{k} \end{cases} \end{aligned}$$

Then the set of solutions to the system of equations  $\mathcal{LS}(A, \mathbf{b})$  is

$$S = \{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \}$$

# **Theorem PSPHS** Particular Solution Plus Homogeneous Solutions 65 Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A, b)$ . Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$ .

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#### Definition SSCV Span of a Set of Column Vectors

Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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#### 66

#### Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the column indices where B has leading 1's (pivot columns) and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the set of column indices where B does not have leading 1's. Construct the n - r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n - r$  of size n as

$$\left[ \mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \} \rangle.$$

Definition RLDCV Relation of Linear Dependence for Column Vectors 68	
Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ , a true statement of the form	
$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n = 0$	
is a <b>relation of linear dependence</b> on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$ , then we say it is the <b>trivial relation of linear dependence</b> on S.	
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#### Definition LICV Linear Independence of Column Vectors

The set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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69

#### Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems 70

Suppose that A is an  $m \times n$  matrix and  $S = {\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  has a unique solution.

#### Theorem LIVRN Linearly Independent Vectors, r and n

Suppose that A is an  $m \times n$  matrix and  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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71

 $\mathbf{72}$ 

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is the set of vectors in $\mathbb{C}^m$ , and that $n > m$ . Then S is a linearly dependent set.

Theorem MVSLD More Vectors than Size implies Linear Dependence

#### Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 73

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NME2 Nonsingular Matrix Equivalences	, Round 2	74		
Suppose that $A$ is a square matrix. The following are equivalent.				
1. $A$ is nonsingular.				
2. $A$ row-reduces to the identity matrix.				
3. The null space of A contains only the zero vector, $\mathcal{N}($	$A) = \{0\}.$			
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for	every possible c	hoice of <b>b</b> .		
5. The columns of $A$ form a linearly independent set.				
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#### Theorem BNS Basis for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n - r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n - r$  of size n as

$$\begin{bmatrix} \mathbf{z}_j \end{bmatrix}_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -\begin{bmatrix} B \end{bmatrix}_{k, f_i} & \text{if } i \in D, \ i = d_k \end{cases}$$

Define the set  $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$ . Then

1.  $\mathcal{N}(A) = \langle S \rangle$ .

2. S is a linearly independent set.

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#### Theorem DLDS Dependency in Linearly Dependent Sets

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a set of vectors. Then S is a linearly dependent set if and only if there is an index  $t, 1 \le t \le n$  such that  $\mathbf{u}_t$  is a linear combination of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$ .

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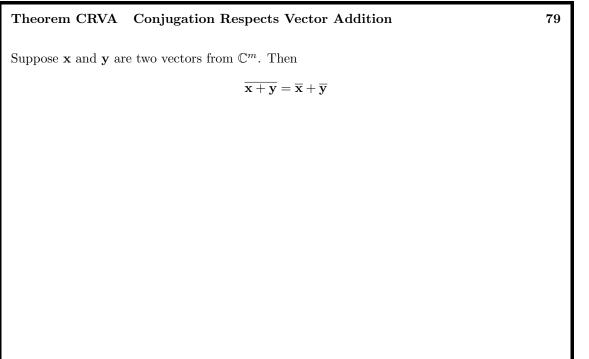
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#### Theorem BS Basis of a Span

Suppose that  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$  is a set of column vectors. Define  $W = \langle S \rangle$  and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with  $D = {d_1, d_2, d_3, \dots, d_r}$  the set of column indices corresponding to the pivot columns of B. Then

- 1.  $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}}$  is a linearly independent set.
- 2.  $W = \langle T \rangle$ .

Definition CCCV Complex Conjugate of a Column Vector			
Suppose that <b>u</b> is a vector from $\mathbb{C}^m$ . Then the conjugate of the vector, $\overline{\mathbf{u}}$ , is defined by			
$\left[\overline{\mathbf{u}}\right]_i = \overline{\left[\mathbf{u}\right]_i} \qquad \qquad 1 \le i \le m$			
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Theorem CRSM	Conjugation Respects Vector Scalar Multiplication	80			
Suppose $\mathbf{x}$ is a vector	Suppose <b>x</b> is a vector from $\mathbb{C}^m$ , and $\alpha \in \mathbb{C}$ is a scalar. Then				
	$\overline{\alpha \mathbf{x}} = \overline{\alpha}  \overline{\mathbf{x}}$				
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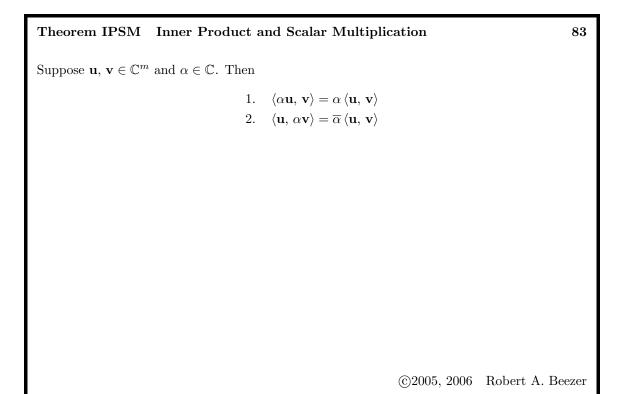
#### Definition IP Inner Product

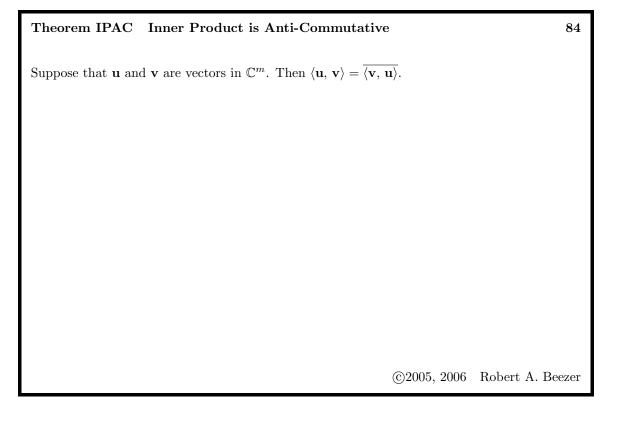
Given the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  the **inner product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity in  $\mathbb{C}$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_1 \overline{[\mathbf{v}]_1} + [\mathbf{u}]_2 \overline{[\mathbf{v}]_2} + [\mathbf{u}]_3 \overline{[\mathbf{v}]_3} + \dots + [\mathbf{u}]_m \overline{[\mathbf{v}]_m} = \sum_{i=1}^m [\mathbf{u}]_i \overline{[\mathbf{v}]_i}$$

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Theorem IPVA Inner Product and Vector Addition					
Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ . Then					
1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$					
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#### Definition NV Norm of a Vector

The  ${\bf norm}$  of the vector  ${\bf u}$  is the scalar quantity in  ${\mathbb C}$ 

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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Theorem IPN Inner Products and Norms	86
Suppose that <b>u</b> is a vector in $\mathbb{C}^m$ . Then $\ \mathbf{u}\ ^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ .	
Suppose that <b>u</b> is a vector in $\mathbb{C}^{-1}$ . Then $\ \mathbf{u}\  = \langle \mathbf{u}, \mathbf{u} \rangle$ .	
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85

Theorem PIP Positive Inner Products	87	
Suppose that <b>u</b> is a vector in $\mathbb{C}^m$ . Then $\langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ with equality if and only if $\mathbf{u} = 0$ .		
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Definition OV	Orthogonal Vectors	88
A pair of vectors, $\langle \mathbf{u}, \mathbf{v} \rangle = 0.$	$\mathbf{u}$ and $\mathbf{v},$ from $\mathbb{C}^m$ are <b>orthogonal</b> if their inner product is zero, that	is,
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#### Definition OSV Orthogonal Set of Vectors

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a set of vectors from  $\mathbb{C}^m$ . Then S is an **orthogonal** set if every pair of different vectors from S is orthogonal, that is  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  whenever  $i \neq j$ .

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89

90

Definition SUV Standard Unit Vectors Let  $\mathbf{e}_j \in \mathbb{C}^m$ ,  $1 \le j \le m$  denote the column vectors defined by  $\left[\mathbf{e}_j\right]_i = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}$ Then the set  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \le j \le m\}$ is the set of standard unit vectors in  $\mathbb{C}^m$ .

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## **Theorem OSLI** Orthogonal Sets are Linearly Independent Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

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 $\mathbf{91}$ 

 $\mathbf{92}$ 

#### Theorem GSP Gram-Schmidt Procedure

Suppose that  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$  is a linearly independent set of vectors in  $\mathbb{C}^m$ . Define the vectors  $\mathbf{u}_i, 1 \le i \le p$  by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if  $T = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$ , then T is an orthogonal set of non-zero vectors, and  $\langle T \rangle = \langle S \rangle$ .

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# Definition ONS OrthoNormal Set Suppose $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$ . Then S is an **orthonormal** set of vectors.

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**Definition VSM** Vector Space of  $m \times n$  Matrices

The vector space  $M_{mn}$  is the set of all  $m \times n$  matrices with entries from the set of complex numbers.

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#### Definition ME Matrix Equality

The  $m \times n$  matrices A and B are equal, written A = B provided  $[A]_{ij} = [B]_{ij}$  for all  $1 \le i \le m$ ,  $1 \le j \le n$ .

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95

96

#### Definition MA Matrix Addition

Given the  $m \times n$  matrices A and B, define the **sum** of A and B as an  $m \times n$  matrix, written A + B, according to

$$[A+B]_{ii} = [A]_{ii} + [B]_{ii} \qquad 1 \le i \le m, \ 1 \le j \le n$$

#### Definition MSM Matrix Scalar Multiplication

Given the  $m \times n$  matrix A and the scalar  $\alpha \in \mathbb{C}$ , the scalar multiple of A is an  $m \times n$  matrix, written  $\alpha A$  and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Theorem VSPM Vector Space Properties of Matrices

Suppose that  $M_{mn}$  is the set of all  $m \times n$  matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If  $A, B \in M_{mn}$ , then  $A + B \in M_{mn}$ .
- SCM Scalar Closure, Matrices If  $\alpha \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha A \in M_{mn}$ .
- CM Commutativity, Matrices If  $A, B \in M_{mn}$ , then A + B = B + A.
- AAM Additive Associativity, Matrices If  $A, B, C \in M_{mn}$ , then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix,  $\mathcal{O}$ , called the zero matrix, such that  $A + \mathcal{O} = A$  for all  $A \in M_{mn}$ .
- AIM Additive Inverses, Matrices If  $A \in M_{mn}$ , then there exists a matrix  $-A \in M_{mn}$  so that  $A + (-A) = \mathcal{O}$ .
- SMAM Scalar Multiplication Associativity, Matrices If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha(\beta A) = (\alpha \beta)A$ .
- DMAM Distributivity across Matrix Addition, Matrices If  $\alpha \in \mathbb{C}$  and  $A, B \in M_{mn}$ , then  $\alpha(A+B) = \alpha A + \alpha B$ .
- DSAM Distributivity across Scalar Addition, Matrices If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $(\alpha + \beta)A = \alpha A + \beta A$ .
- OM One Matrices If  $A \in M$  then 1A A

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#### Definition ZM Zero Matrix

The  $m \times n$  zero matrix is written as  $\mathcal{O} = \mathcal{O}_{m \times n}$  and defined by  $[\mathcal{O}]_{ij} = 0$ , for all  $1 \le i \le m$ ,  $1 \le j \le n$ .

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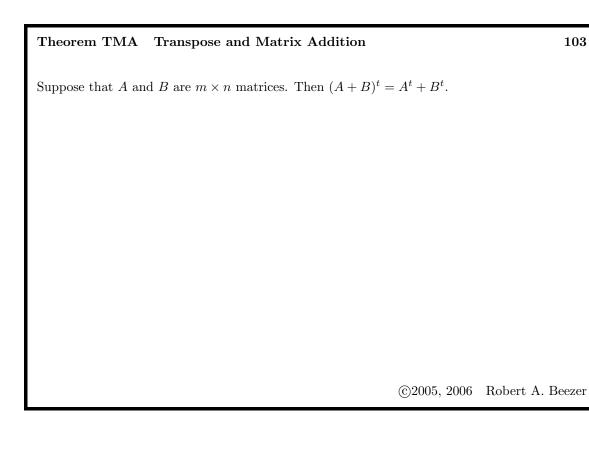
99

Definition TM Transpose of a Matrix	100		
Given an $m \times n$ matrix A, its <b>transpose</b> is the $n \times m$ matrix $A^t$ given by			
$\left[A^t\right]_{ij} = [A]_{ji},  1 \le i \le n, \ 1 \le j \le m.$			
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The matrix A is symmetric if  $A = A^t$ .

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Theorem SMS	Symmetric Matrices are Square		102
	· ·		
Suppose that $A$ is	a symmetric matrix. Then $A$ is square.		
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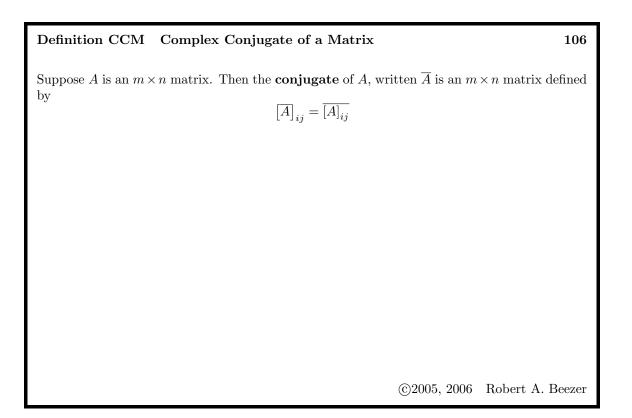


Theorem TMSM	Transpose and Matrix	Scalar Multiplication	104
Suppose that $\alpha \in \mathbb{C}$	and A is an $m \times n$ matrix.	Then $(\alpha A)^t = \alpha A^t$ .	
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#### Theorem TT Transpose of a Transpose

Suppose that A is an  $m \times n$  matrix. Then  $(A^t)^t = A$ .

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Theorem CRMA	Conjugation Respects Matrix Addition	107
Suppose that $A$ and	$B$ are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$ .	
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Theorem CRMSM	Conjugation Respects Matrix Scalar Multiplication	108
Suppose that $\alpha \in \mathbb{C}$ and	nd A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$ .	
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Theorem CCM Conjugate of the Conjugate of a Matrix	109	
Suppose that A is an $m \times n$ matrix. Then $\overline{(\overline{A})} = A$ .		
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 Theorem MCT
 Matrix Conjugation and Transposes
 110

 Suppose that A is an  $m \times n$  matrix. Then  $\overline{(A^t)} = (\overline{A})^t$ .
 ( $\overline{A}^t$ )

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### Definition A Adjoint

If A is a square matrix, then its **adjoint** is  $A^* = (\overline{A})^t$ .

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Theorem AMA Adjoint and Matrix Addition	on	112
Suppose $A$ and $B$ are matrices of the same size. The	then $(A+B)^* = A^* + B^*$ .	
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Theorem AMSM Adjoint and Matrix Scalar Multiplication	113
Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$ .	
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Theorem AA Adjoint of an Adjoint	114
Suppose that A is a matrix. Then $(A^*)^* = A$	
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#### Definition MVP Matrix-Vector Product

Suppose A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$  and  $\mathbf{u}$  is a vector of size n. Then the **matrix-vector product** of A with  $\mathbf{u}$  is the linear combination

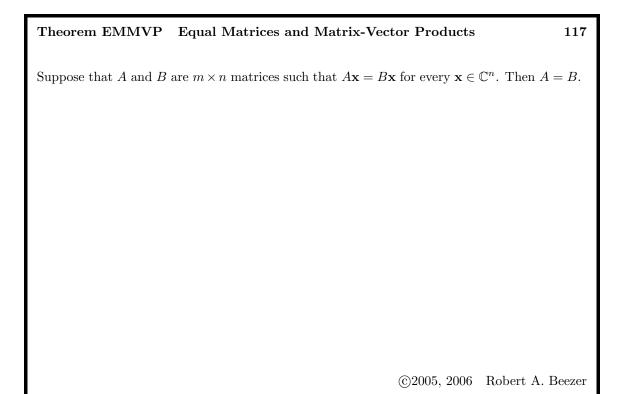
$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

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115

Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 116

The set of solutions to the linear system  $\mathcal{LS}(A, \mathbf{b})$  equals the set of solutions for  $\mathbf{x}$  in the vector equation  $A\mathbf{x} = \mathbf{b}$ .



Definition MM Matrix Multiplication

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix with columns  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ . Then the **matrix product** of A with B is the  $m \times p$  matrix where column i is the matrix-vector product  $A\mathbf{B}_i$ . Symbolically,

 $AB = A \left[ \mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[ A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$ 

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#### Theorem EMP Entries of Matrix Products

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then for  $1 \le i \le m, 1 \le j \le p$ , the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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Theorem MMZM	Matrix Multiplication and the Zero Matrix	120
Suppose A is an $m \times 1$ . 1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$ 2. $\mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$		
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#### Theorem MMIM Matrix Multiplication and Identity Matrix

Suppose A is an  $m \times n$  matrix. Then 1.  $AI_n = A$ 2.  $I_m A = A$ 

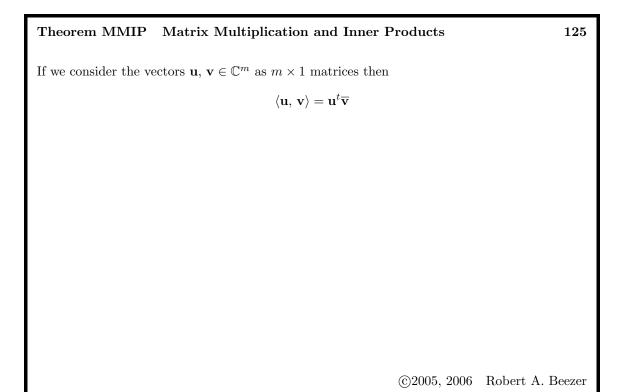
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Theorem MMDAA Matrix Multiplication Distributes Across Addition	122
<b>Theorem MMDAA</b> Matrix Multiplication Distributes Across Addition Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. 1. $A(B+C) = AB + AC$ 2. $(B+C)D = BD + CD$	
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## Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 123

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Let  $\alpha$  be a scalar. Then  $\alpha(AB) = (\alpha A)B = A(\alpha B)$ .

Theorem MMA	Matrix Multiplication is Associative	124
Suppose $A$ is an $m$ (AB)D.	$\times  n$ matrix, $B$ is an $n \times p$ matrix and $D$ is a $p \times s$ matrix. Then $A(BD$	) =
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 Theorem MMCC
 Matrix Multiplication and Complex Conjugation
 126

 Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then  $\overline{AB} = \overline{AB}$ .
  $\overline{B}$  

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Theorem MMT	Matrix Multiplication and Transposes	127
Suppose $A$ is an $m$	$\times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$ .	
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Theorem MMAD	Matrix Multiplication and Adjoints	128
Suppose A is an $m \times$	<i>n</i> matrix and <i>B</i> is an $n \times p$ matrix. Then $(AB)^* = B^*A^*$ .	
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#### Theorem AIP Adjoint and Inner Product

Suppose that A is an  $m \times n$  matrix and  $\mathbf{x} \in \mathbb{C}^n$ ,  $\mathbf{y} \in \mathbb{C}^m$ . Then  $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$ .

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Definition HM	Hermitian Matrix	130
The square matrix	A is <b>Hermitian</b> (or <b>self-adjoint</b> ) if $A = A^*$ .	
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Theorem HMIP	Hermitian Matrices and Inner Products	131
Suppose that $A$ is a s for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ .	square matrix of size <i>n</i> . Then <i>A</i> is Hermitian if and only if $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{x} \rangle$	$4\mathbf{y}\rangle$
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Definition MI Ma	atrix Inverse	132
	e square matrices of size $n$ such that $AB = I_n$ and the <b>inverse</b> of $A$ . In this situation, we write $B = A^{-1}$	
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Theorem TTMI Two-by-Two Matrix Inverse

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if  $ad - bc \neq 0$ . When A is invertible, then

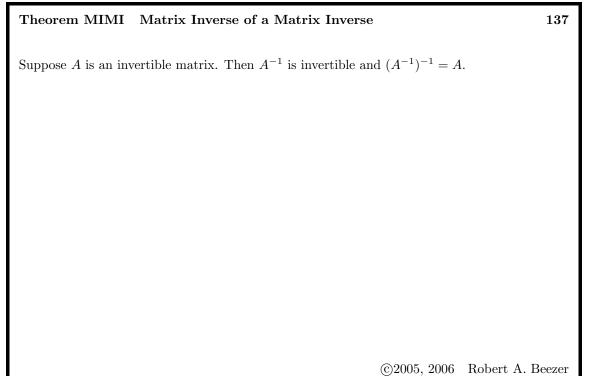
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Theorem CINM	Computing the I	nverse of a Nonsir	ngular Matrix	x 134
Suppose A is a nonsin $n \times n$ identity matrix to M and in reduced columns of N. Then	$I_n$ to the right of the row-echelon form.	he matrix $A$ . Let $N \mid$	be a matrix tha	at is row-equivalent
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Theorem MIU	Matrix Inverse is Unique	135
Suppose the squar	re matrix A has an inverse. Then $A^{-1}$ is unique.	
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Theorem SS Socks and Shoes	136
Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB invertible matrix.	' is an
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Theorem MIT	Matrix Inverse of a Transpose	138
Suppose $A$ is an in	nvertible matrix. Then $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$ .	
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 Theorem MISM
 Matrix Inverse of a Scalar Multiple
 139

 Suppose A is an invertible matrix and  $\alpha$  is a nonzero scalar. Then  $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$  and  $\alpha A$  is invertible.

Theorem NPNT	Nonsingular Product has Nonsingular Terms 140	)
Suppose that $A$ and if $A$ and $B$ are both	B are square matrices of size $n$ . The product $AB$ is nonsingular if and only nonsingular.	y
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Theorem OSIS	One-Sided Inverse is Sufficient	141
Suppose $A$ and $B$	B are square matrices of size $n$ such that $AB = I_n$ . Then $BA = I_n$ .	
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 Theorem NI Nonsingularity is Invertibility
 142

 Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.
 Image: Comparison of the second secon

Theorem NME3 Nonsingular Matrix Equivalences, Round 3
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system LS(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.

Theorem SNCM	Solution	with Nonsing	gular Coeffic	ient Matrix	144
Suppose that $A$ is no	onsingular.	Then the uniq	ue solution to ,	$\mathcal{LS}(A, \mathbf{b})$ is $A^-$	<sup>-1</sup> b.
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Definition UM Unitary Matrices	145
Suppose that U is a square matrix of size n such that $U^*U = I_n$ . Then we say U is <b>unita</b>	ry.
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Theorem UMI	Unitary I	Matrices are Invertible	146
Suppose that $U$ is	s a unitary n	natrix of size <i>n</i> . Then <i>U</i> is nonsingular, and $U^{-1} = U^*$ .	
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Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets 147
Suppose that A is a square matrix of size n with columns $S = {\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n}$ . Then A is a unitary matrix if and only if S is an orthonormal set.
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Theorem UMPIP	Unitary Matrices Pre	eserve Inner I	Products		148
Suppose that $U$ is a u	unitary matrix of size $n$ as	nd <b>u</b> and <b>v</b> are	two vectors fr	om $\mathbb{C}^n$ . Then	
$\langle U {f u}, U$	$\langle \mathbf{v}  angle = \langle \mathbf{u},  \mathbf{v}  angle$	and	$\ U\mathbf{v}\  =$	$\ \mathbf{v}\ $	
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# Definition CSM Column Space of a Matrix

Suppose that A is an  $m \times n$  matrix with columns  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ . Then the **column space** of A, written  $\mathcal{C}(A)$ , is the subset of  $\mathbb{C}^m$  containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS	Column Spaces and Consistent Systems 150	)
Suppose $A$ is an $n$ $\mathcal{LS}(A, \mathbf{b})$ is consist	$n \times n$ matrix and <b>b</b> is a vector of size $m$ . Then $\mathbf{b} \in \mathcal{C}(A)$ if and only its ent.	f
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# Theorem BCS Basis of the Column Space

Suppose that A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the set of column indices where B has leading 1's. Let  $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$ . Then

- 1. T is a linearly independent set.
- 2.  $\mathcal{C}(A) = \langle T \rangle$ .

Theorem CSNM	Column Space of a Nonsingular Matrix	152
Suppose $A$ is a square	re matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$ .	
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Theorem NME4 Nonsingular Matrix Equivalences, Round 4
153
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system LS(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is C<sup>n</sup>, C(A) = C<sup>n</sup>.

Definition RSM Row Space of a Matrix

Suppose A is an  $m \times n$  matrix. Then the **row space** of A,  $\mathcal{R}(A)$ , is the column space of  $A^t$ , i.e.  $\mathcal{R}(A) = \mathcal{C}(A^t)$ .

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Theorem REMRS	Row-Equivalent Matrices have eq	ual Row Spa	aces 155
Suppose $A$ and $B$ are	row-equivalent matrices. Then $\mathcal{R}(A) =$	$\mathcal{R}(B).$	
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Theorem BRS Basis for the Row Space	156
Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form S be the set of nonzero columns of $B^t$ . Then	. Let
1. $\mathcal{R}(A) = \langle S \rangle.$	
2. $S$ is a linearly independent set.	
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Theorem CSRST	Column Space, Row Space, Transpose	157	
Suppose $A$ is a matri	x. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$ .		

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 Definition LNS
 Left Null Space
 158

 Suppose A is an  $m \times n$  matrix. Then the left null space is defined as  $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$ .

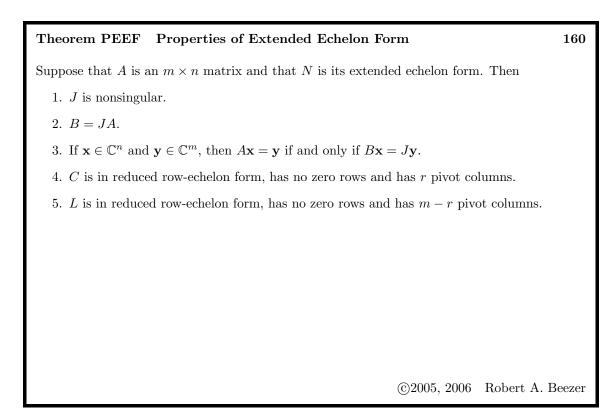
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### Definition EEF Extended Echelon Form

Suppose A is an  $m \times n$  matrix. Add m new columns to A that together equal an  $m \times m$  identity matrix to form an  $m \times (n+m)$  matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the  $m \times n$  matrix formed from the first n columns of N and let J denote the  $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the  $r \times n$  matrix formed from all of the non-zero rows of B. Let K be the  $r \times m$  matrix formed from the first r rows of J, while L will be the  $(m - r) \times m$ matrix formed from the bottom m - r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$



### Theorem FS Four Subsets

Suppose A is an  $m \times n$  matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m - r rows. Then

- 1. The null space of A is the null space of C,  $\mathcal{N}(A) = \mathcal{N}(C)$ .
- 2. The row space of A is the row space of C,  $\mathcal{R}(A) = \mathcal{R}(C)$ .
- 3. The column space of A is the null space of L,  $C(A) = \mathcal{N}(L)$ .
- 4. The left null space of A is the row space of L,  $\mathcal{L}(A) = \mathcal{R}(L)$ .

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### Definition VS Vector Space

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space if the following ten properties hold.

- AC Additive Closure If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} \in V$ .
- SC Scalar Closure If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha \mathbf{u} \in V$ .
- C Commutativity If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AA Additive Associativity If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- Z Zero Vector There is a vector, 0, called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in V$ .
- AI Additive Inverses If  $\mathbf{u} \in V$ , then there exists a vector  $-\mathbf{u} \in V$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMA Scalar Multiplication Associativity If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVA Distributivity across Vector Addition If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSA Distributivity across Scalar Addition If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .
- O One If  $\mathbf{u} \in V$  then  $1\mathbf{u} = \mathbf{u}$

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

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Theorem ZVU Zero Vector is Unique	163
Suppose that V is a vector space. The zero vector, $0$ , is unique.	
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Theorem AIU Additiv	e Inverses are Unique	164
Suppose that $V$ is a vector	space. For each $\mathbf{u} \in V$ , the additive inverse, $-\mathbf{u}$ , is unique.	
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Theorem ZSSM	Zero Scalar in Scalar Multiplicatio	n	165
Suppose that $V$ is a	vector space and $\mathbf{u} \in V$ . Then $0\mathbf{u} = 0$ .		
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Theorem ZVSM	Zero Vector in Scalar Multiplication	166
Suppose that $V$ is a	vector space and $\alpha \in \mathbb{C}$ . Then $\alpha 0 = 0$ .	
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Theorem SMEZV	Scalar Multiplication Equals the Zero Vector	168
Suppose that $V$ is a v	vector space and $\alpha \in \mathbb{C}$ . If $\alpha \mathbf{u} = 0$ , then either $\alpha = 0$ or $\mathbf{u} = 0$ .	
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# Definition S Subspace

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of  $V, W \subseteq V$ . Then W is a **subspace** of V.

Theorem TSS Testing Subsets for Subspaces	170
Suppose that V is a vector space and W is a subset of V, $W \subseteq V$ . Endow W with the second operations as V. Then W is a subspace if and only if three conditions are met	ame
1. W is non-empty, $W \neq \emptyset$ .	
2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$ , then $\mathbf{x} + \mathbf{y} \in W$ .	
3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$ , then $\alpha \mathbf{x} \in W$ .	
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Definition TS	Trivial Subspaces				171
Given the vector	space $V$ , the subspaces	$V$ and $\{0\}$ are each	ı called a <b>trivia</b>	l subspace.	
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Theorem NSMS	Null Space of a Matrix is a Subspace	172
Suppose that $A$ is a	n $m \times n$ matrix. Then the null space of $A$ , $\mathcal{N}(A)$ , is a subspace of $\mathbb{C}^n$ .	
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# Definition LC Linear Combination

Suppose that V is a vector space. Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their **linear combination** is the vector

 $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\cdots+\alpha_n\mathbf{u}_n.$ 

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### Definition SS Span of a Set

Suppose that V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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### 174

# Theorem SSS Span of a Set is a Subspace

Suppose V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t} \subseteq V$ , their span,  $\langle S \rangle$ , is a subspace.

Theorem CSMS Column Space of a Matrix is a Subspace		176
Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$		
0000	0000	
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Theorem RSMS	Row Space of a Matrix is a Subspace	177
Suppose that $A$ is a	n $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$ .	
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Theorem LNSMS	Left Null Space of a Matrix is a Subspace	178
Suppose that $A$ is an	$m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$ .	
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### Definition RLD Relation of Linear Dependence

180

Suppose that V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ , an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is a trivial relation of linear dependence on S.

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### Definition LI Linear Independence

Suppose that V is a vector space. The set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

# Definition TSVS To Span a Vector Space

Suppose V is a vector space. A subset S of V is a **spanning set** for V if  $\langle S \rangle = V$ . In this case, we also say S **spans** V.

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181

182

### Theorem VRRB Vector Representation Relative to a Basis

Suppose that V is a vector space and  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$  is a linearly independent set that spans V. Let **w** be any vector in V. Then there exist *unique* scalars  $a_1, a_2, a_3, \dots, a_m$  such that

 $\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$ 

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# Definition B Basis

Suppose V is a vector space. Then a subset  $S \subseteq V$  is a **basis** of V if it is linearly independent and spans V.

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Theorem SUVB Standard Unit Vectors are a Basis	184
The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$ .	$\mathbf{e}_2,\mathbf{e}_3,\ldots,\mathbf{e}_m\} =$
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# Theorem CNMB Columns of Nonsingular Matrix are a Basis

Suppose that A is a square matrix of size m. Then the columns of A are a basis of  $\mathbb{C}^m$  if and only if A is nonsingular.

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 $\mathbf{185}$ 

Theorem NME5	Nonsingular Matrix Equivalences, Round 5	186
Suppose that $A$ is a	square matrix of size $n$ . The following are equivalent.	
1. $A$ is nonsingul	ar.	
2. $A$ row-reduces	to the identity matrix.	
3. The null space	e of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear sys	tem $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$ .	
5. The columns of	of $A$ are a linearly independent set.	
6. $A$ is invertible		
7. The column sp	pace of A is $\mathbb{C}^n$ , $\mathcal{C}(A) = \mathbb{C}^n$ .	
8. The columns of	of A are a basis for $\mathbb{C}^n$ .	
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# Theorem COB Coordinates and Orthonormal Bases

187

Suppose that  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$  is an orthonormal basis of the subspace W of  $\mathbb{C}^m$ . For any  $\mathbf{w} \in W$ ,  $\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \dots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$ 

Theorem UMCOB	Unitary Matrices Convert Orthonormal Bases	188
Let A be an $n \times n$ mat $C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, $ basis of $\mathbb{C}^n$ .	rix and $B = {\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n}$ be an orthonormal basis of $\mathbb{C}^n$ . De $\dots, A\mathbf{x}_n$ . Then A is a unitary matrix if and only if C is an orthonormal basis of $\mathbb{C}^n$ .	efine rmal
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# Definition D Dimension

Suppose that V is a vector space and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$  is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD	Spanning Sets and Linear Dependence	190
Suppose that $S = \{$ Then any set of $t +$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space - 1 or more vectors from V is linearly dependent.	e V.
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# Theorem BIS Bases have Identical Sizes 191 Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size. Image: Comparison of the same size of the same si

Theorem DCM Dir	$\qquad \qquad $		192
The dimension of $\mathbb{C}^m$ (1)	Example VSCV) is $m$ .		
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<b>Theorem DP</b> Dimension of $P_n$		193
The dimension of $P_n$ (Example VSP) is $n + 1$ .		
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Theorem DM	<b>Dimension of</b> $M_{mn}$		194
The dimension of	$M_{mn}$ (Example VSM) is $mn$ .		
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# Definition NOM Nullity Of a Matrix

Suppose that A is an  $m \times n$  matrix. Then the **nullity** of A is the dimension of the null space of A,  $n(A) = \dim(\mathcal{N}(A))$ .

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195

196

# Definition ROM Rank Of a Matrix

Suppose that A is an  $m \times n$  matrix. Then the **rank** of A is the dimension of the column space of A,  $r(A) = \dim (\mathcal{C}(A))$ .

# Theorem CRN Computing Rank and Nullity

### 197

Suppose that A is an  $m \times n$  matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

Theorem RPNC	Rank Plus Nullity is Columns	198
Suppose that $A$ is a	$m m \times n$ matrix. Then $r(A) + n(A) = n$ .	
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# Theorem RNNM Rank and Nullity of a Nonsingular Matrix

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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<b>Theorem NME6</b> Nonsingular Matrix Equivalences, Round 6 Suppose that $A$ is a square matrix of size $n$ . The following are equivalent.	200
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$ .	
5. The columns of $A$ are a linearly independent set.	
6. A is invertible.	
7. The column space of A is $\mathbb{C}^n$ , $\mathcal{C}(A) = \mathbb{C}^n$ .	
8. The columns of A are a basis for $\mathbb{C}^n$ .	
9. The rank of A is $n, r(A) = n$ .	
10. The nullity of A is zero, $n(A) = 0$ .	
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# Theorem ELIS Extending Linearly Independent Sets

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose  $\mathbf{w}$  is a vector such that  $\mathbf{w} \notin \langle S \rangle$ . Then the set  $S' = S \cup \{\mathbf{w}\}$  is linearly independent.

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Theorem G Goldilocks 202	2
Suppose that V is a vector space of dimension t. Let $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ be a set of vectors from V. Then	f
1. If $m > t$ , then S is linearly dependent.	ļ
2. If $m < t$ , then S does not span V.	
3. If $m = t$ and S is linearly independent, then S spans V.	ļ
4. If $m = t$ and S spans V, then S is linearly independent.	ļ
	ļ
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Theorem PSSDProper Subspaces have Smaller Dimension203
Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$ . Then dim $(U) < \dim(V)$ .

Theorem EDYES	Equal Dimensions Yields Equal Subspaces	<b>204</b>
Suppose that $U$ and dim $(V)$ . Then $U = V$	V are subspaces of the vector space W, such that $U \subseteq V$ a V.	nd $\dim(U) =$
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 Theorem RMRT
 Rank of a Matrix is the Rank of the Transpose
 205

 Suppose A is an  $m \times n$  matrix. Then  $r(A) = r(A^t)$ .
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### Theorem DFS Dimensions of Four Subspaces

Suppose that A is an  $m\times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim  $(\mathcal{N}(A)) = n r$
- 2. dim  $(\mathcal{C}(A)) = r$
- 3. dim  $(\mathcal{R}(A)) = r$
- 4. dim  $(\mathcal{L}(A)) = m r$

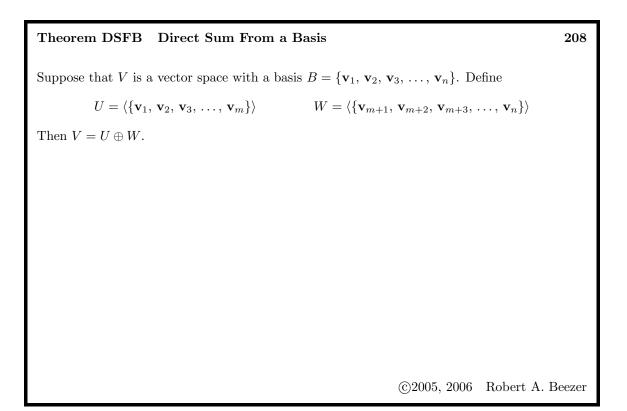
206

### Definition DS Direct Sum

207

Suppose that V is a vector space with two subspaces U and W such that for every  $\mathbf{v} \in V$ ,

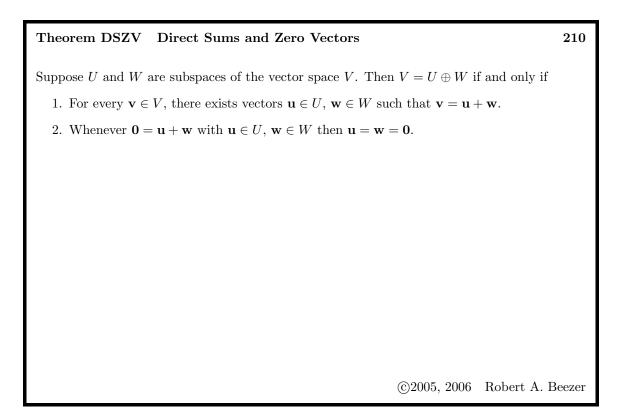
- 1. There exists vectors  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  such that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If  $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$  and  $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$  where  $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$  then  $\mathbf{u}_1 = \mathbf{u}_2$  and  $\mathbf{w}_1 = \mathbf{w}_2$ .
- Then V is the **direct sum** of U and W and we write  $V = U \oplus W$ .

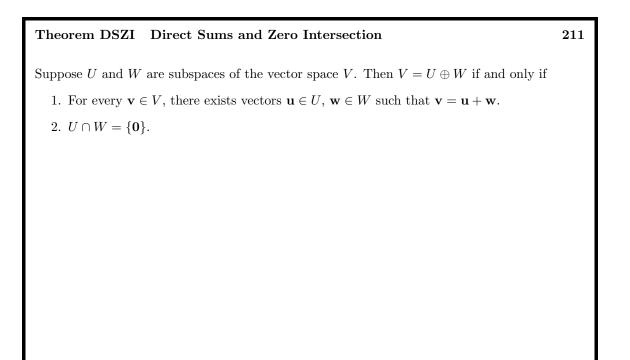


# Theorem DSFOS Direct Sum From One Subspace

209

Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that  $V = U \oplus W$ .





Theorem DSLI Direct Sums and Linear Independence	212
Suppose U and W are subspaces of the vector space V with $V = U \oplus W$ . Suppose the linearly independent subset of U and S is a linearly independent subset of W. Then R linearly independent subset of V.	
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### Theorem DSD Direct Sums and Dimension

Suppose U and W are subspaces of the vector space V with  $V = U \oplus W$ . Then dim  $(V) = \dim(U) + \dim(W)$ .

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Theorem RDS	Repeated Direct Sums
-------------	----------------------

Suppose V is a vector space with subspaces U and W with  $V = U \oplus W$ . Suppose that X and Y are subspaces of W with  $W = X \oplus Y$ . Then  $V = U \oplus X \oplus Y$ .

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 $\mathbf{214}$ 

### **Definition ELEM** Elementary Matrices

1. For  $i \neq j$ ,  $E_{i,j}$  is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq \\ 1 & k \neq i, k \neq j, \ell = \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

 $k \\ k$ 

2. For  $\alpha \neq 0$ ,  $E_i(\alpha)$  is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For  $i \neq j$ ,  $E_{i,j}(\alpha)$  is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = j \end{cases}$$

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### Theorem EMDRO Elementary Matrices Do Row Operations

Suppose that A is an  $m \times n$  matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

1. If the row operation swaps rows i and j, then  $B = E_{i,j}A$ .

2. If the row operation multiplies row *i* by  $\alpha$ , then  $B = E_i(\alpha) A$ .

3. If the row operation multiplies row i by  $\alpha$  and adds the result to row j, then  $B = E_{i,j}(\alpha) A$ .

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 $\mathbf{215}$ 

 $\mathbf{216}$ 

Theorem EMN Elementary Matrices are Nonsingular					
If $E$ is an elementary matrix, then $E$ is nonsingular.					

Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices 218

Suppose that A is a nonsingular matrix. Then there exists elementary matrices  $E_1, E_2, E_3, \ldots, E_t$  so that  $A = E_1 E_2 E_3 \ldots E_t$ .

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### Definition SM SubMatrix

Definition DM Determinant of a Matrix

Suppose that A is an  $m \times n$  matrix. Then the **submatrix** A(i|j) is the  $(m-1) \times (n-1)$  matrix obtained from A by removing row i and column j.

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Suppose A is a square matrix. Then its **determinant**, det (A) = |A|, is an element of  $\mathbb{C}$  defined recursively by: If A is a 1 × 1 matrix, then det  $(A) = [A]_{11}$ . If A is a matrix of size n with  $n \ge 2$ , then det  $(A) = [A]_{11} \det (A(1|1)) - [A]_{12} \det (A(1|2)) + [A]_{13} \det (A(1|3)) - [A]_{14} \det (A(1|4)) + \dots + (-1)^{n+1} [A]_{1n} \det (A(1|n))$ (©2005, 2006 Robert A. Beezer

### 219

 $\mathbf{220}$ 

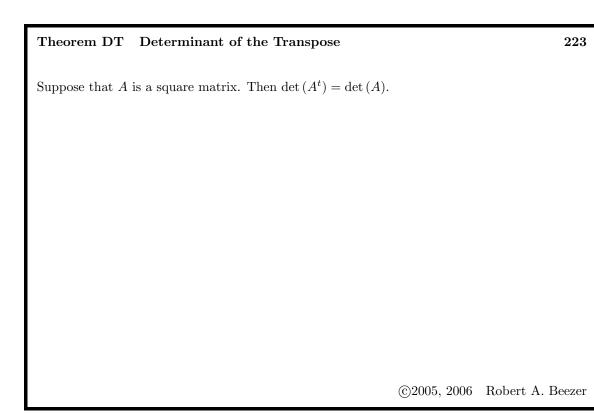
Theorem DMST Determinant of Matrices of Size Two

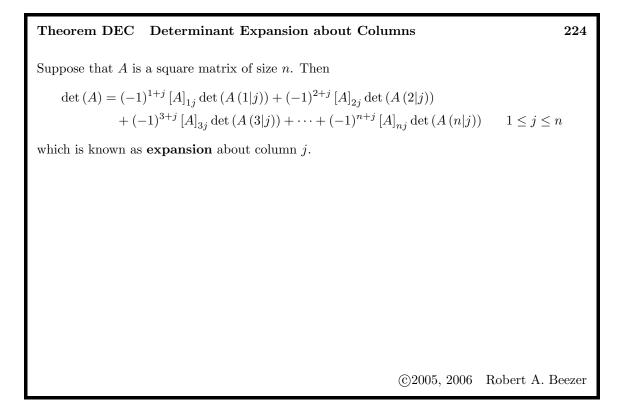
Suppose that 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Then det  $(A) = ad - bc$ 

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 $\mathbf{221}$ 

Theorem DER Determinant Expansion about Rows	222
Suppose that $A$ is a square matrix of size $n$ . Then	
$\det (A) = (-1)^{i+1} [A]_{i1} \det (A(i 1)) + (-1)^{i+2} [A]_{i2} \det (A(i 2)) + (-1)^{i+3} [A]_{i3} \det (A(i 3)) + \dots + (-1)^{i+n} [A]_{in} \det (A(i n)) \qquad 1 \le i \le n$	ı
which is known as <b>expansion</b> about row $i$ .	
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Theorem DZRC Determinant with Zero Row or Column	225
Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then det $(A) = 0$ .	here

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 $\mathbf{226}$ 

### Theorem DRCS Determinant for Row or Column Swap

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then  $\det(B) = -\det(A)$ .

### Theorem DRCM Determinant for Row or Column Multiples

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar  $\alpha$ , or by multiplying a single column by the scalar  $\alpha$ . Then det  $(B) = \alpha \det(A)$ .

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227

Theorem DERC	Determinant with Equal Rows or Columns 22	28
Suppose that $A$ is a s	square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$	0.
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### Theorem DRCMA Determinant for Row or Column Multiples and Addition 229

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar  $\alpha$  and then adding it to another row, or by multiplying a column by the scalar  $\alpha$  and then adding it to another column. Then det  $(B) = \det(A)$ .

Theorem DIM	Determinant of the Identity Matrix	230
For every $n \ge 1$ , d	$\det\left(I_n\right) = 1.$	
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### Theorem DEM Determinants of Elementary Matrices

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. det  $(E_{i,j}) = -1$
- 2. det  $(E_i(\alpha)) = \alpha$
- 3. det  $(E_{i,j}(\alpha)) = 1$

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231

Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 232

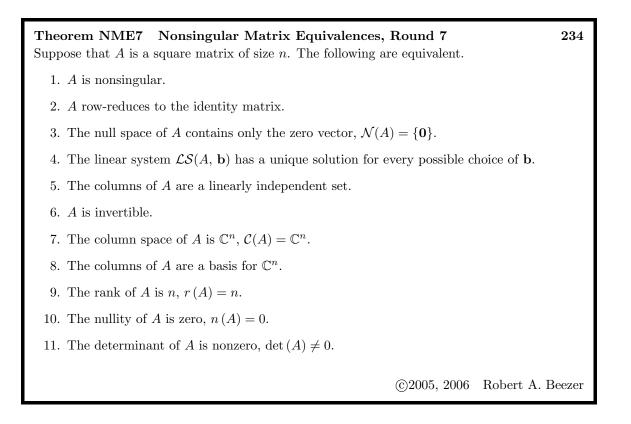
Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

 $\det (EA) = \det (E) \det (A)$ 

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### Theorem SMZD Singular Matrices have Zero Determinants Let A be a square matrix. Then A is singular if and only if det(A) = 0. ©2005, 2006 Robert A. Beezer

 $\mathbf{233}$ 



### Theorem DRMM Determinant Respects Matrix Multiplication

Suppose that A and B are square matrices of the same size. Then  $\det(AB) = \det(A) \det(B)$ .

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 $\mathbf{235}$ 

Definition EEM Eigenvalues and Eigenvectors of a Matrix	236		
Suppose that A is a square matrix of size $n, \mathbf{x} \neq 0$ is a vector in $\mathbb{C}^n$ , and $\lambda$ is a scalar in $\mathbb{C}$ . Then we say $\mathbf{x}$ is an <b>eigenvector</b> of A with <b>eigenvalue</b> $\lambda$ if			
$A\mathbf{x} = \lambda \mathbf{x}$			
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Theorem EMHE Every Matrix Has an Eigenvalue	237			
Suppose $A$ is a square matrix. Then $A$ has at least one eigenvalue.				
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Definition CP Characteristic Polynomial 238 Suppose that A is a square matrix of size n. Then the characteristic polynomial of A is the polynomial  $p_A(x)$  defined by  $p_A(x) = \det(A - xI_n)$ 

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Theorem EMRCP	Eigenvalues of a Matrix are Roots of Characteristic Polynomi-
als	239

Suppose A is a square matrix. Then  $\lambda$  is an eigenvalue of A if and only if  $p_A(\lambda) = 0$ .

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 $\mathbf{240}$ 

### Definition EM Eigenspace of a Matrix

Suppose that A is a square matrix and  $\lambda$  is an eigenvalue of A. Then the **eigenspace** of A for  $\lambda$ ,  $\mathcal{E}_A(\lambda)$ , is the set of all the eigenvectors of A for  $\lambda$ , together with the inclusion of the zero vector.

Theorem EMS	Eigenspace for a Matrix is a Subspace 2	241
	uare matrix of size $n$ and $\lambda$ is an eigenvalue of $A$ . Then the eigenspace $\mathcal{E}_A$ he vector space $\mathbb{C}^n$ .	$(\lambda)$
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 Theorem EMNS
 Eigenspace of a Matrix is a Null Space
 242

 Suppose A is a square matrix of size n and  $\lambda$  is an eigenvalue of A. Then
  $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$   $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$  

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### Definition AME Algebraic Multiplicity of an Eigenvalue

 $\mathbf{243}$ 

Suppose that A is a square matrix and  $\lambda$  is an eigenvalue of A. Then the **algebraic multiplicity** of  $\lambda$ ,  $\alpha_A(\lambda)$ , is the highest power of  $(x - \lambda)$  that divides the characteristic polynomial,  $p_A(x)$ .

Definition GME	Geometric Multiplicity of an Eigenvalue 24	14		
Suppose that A is a square matrix and $\lambda$ is an eigenvalue of A. Then the <b>geometric maplicity</b> of $\lambda$ , $\gamma_A(\lambda)$ , is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$ .				
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### Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 245

Suppose that A is an  $n \times n$  square matrix and  $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$  is a set of eigenvectors with eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$  such that  $\lambda_i \neq \lambda_j$  whenever  $i \neq j$ . Then S is a linearly independent set.

Theorem SMZE Singular Matrices have Zero Eigenvalues					
Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.					
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Theorem NME8Nonsingular Matrix Equivalences, Round 82Suppose that A is a square matrix of size n. The following are equivalent.2		
1. A is nonsingular.		
2. A row-reduces to the identity matrix.		
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$		
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$ .		
5. The columns of $A$ are a linearly independent set.		
6. A is invertible.		
7. The column space of A is $\mathbb{C}^n$ , $\mathcal{C}(A) = \mathbb{C}^n$ .		
8. The columns of A are a basis for $\mathbb{C}^n$ .		
9. The rank of A is $n, r(A) = n$ .		
10. The nullity of A is zero, $n(A) = 0$ .		
11. The determinant of A is nonzero, $det(A) \neq 0$ .		
12. $\lambda = 0$ is not an eigenvalue of A.		
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Theorem ESMM	Eigenvalues of a Scalar Multiple of a Matrix	248
Suppose $A$ is a square	re matrix and $\lambda$ is an eigenvalue of $A$ . Then $\alpha\lambda$ is an eigenvalue of $\alpha\lambda$	4.
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Theorem EOMP Eigenvalues Of Matrix Powers	249
Suppose A is a square matrix, $\lambda$ is an eigenvalue of A, and $s \ge 0$ is an integer. Then $\lambda^s$ eigenvalue of $A^s$ .	' is an
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Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then  $q(\lambda)$  is an eigenvalue of the matrix q(A). ©2005, 2006 Robert A. Beezer

 $\mathbf{250}$ 

Theorem EPM Eigenvalues of the Polynomial of a Matrix

# Theorem EIM Eigenvalues of the Inverse of a Matrix 251 Suppose A is a square nonsingular matrix and $\lambda$ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $A^{-1}$ . 1

Theorem ETM	Eigenvalues of the Transpose of a Matrix 252	2
Suppose $A$ is a squ $A^t$ .	hare matrix and $\lambda$ is an eigenvalue of $A$ . Then $\lambda$ is an eigenvalue of the matrix	x
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## **Theorem ERMCP** Eigenvalues of Real Matrices come in Conjugate Pairs 253 Suppose A is a square matrix with real entries and **x** is an eigenvector of A for the eigenvalue $\lambda$ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$ .

Theorem DCP	Degree of the Charact	eristic Polynon	nial	254
Suppose that $A$ is has degree $n$ .	a square matrix of size $n$ .	Then the charac	teristic polyno	omial of $A$ , $p_A(x)$ ,
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### Theorem NEM Number of Eigenvalues of a Matrix

 $\mathbf{255}$ 

Suppose that A is a square matrix of size n with distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ . Then

$$\sum_{i=1}^{k} \alpha_A \left( \lambda_i \right) =$$

n

Theorem ME Multiplicities of an Eigenvalue	256
Suppose that A is a square matrix of size n and $\lambda$ is an eigenvalue. Then	
$1 \le \gamma_A\left(\lambda\right) \le \alpha_A\left(\lambda\right) \le n$	
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Theorem MNEM Maximum Number of Eigenvalues of a Matrix 25	57
Suppose that $A$ is a square matrix of size $n$ . Then $A$ cannot have more than $n$ distinct eige values.	n-

Theorem HMRE	Hermitian Matrices have Real Eigenvalues	<b>258</b>
Suppose that $A$ is a $\mathcal{I}$	Hermitian matrix and $\lambda$ is an eigenvalue of $A$ . Then $\lambda \in \mathbb{R}$ .	
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Theorem HMOE	Hermitian Matrices have Orthogonal Eigenve	ectors 259
	a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors and $\mathbf{y}$ are orthogonal vectors.	ors of $A$ for different
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### Definition SIM Similar Matrices

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that  $A = S^{-1}BS$ .

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 $\mathbf{260}$ 

### Theorem SER Similarity is an Equivalence Relation

Suppose A, B and C are square matrices of size n. Then

- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE Similar Mat	rices have Equal I	Eigenvalues	262
Suppose A and B are similar mat equal, that is, $p_A(x) = p_B(x)$ .	rices. Then the char	acteristic polynomia	ls of $A$ and $B$ are
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 $\mathbf{261}$ 

Definition DIM	Diagonal Matrix	263
Suppose that $A$ is a	a square matrix. Then A is a <b>diagonal matrix</b> if $[A]_{ij} = 0$ whenever	$i \neq j.$
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Definition DZM Diagonalizable Matrix	<b>264</b>
Suppose $A$ is a square matrix. Then $A$ is <b>diagonalizable</b> if $A$ is simil	ar to a diagonal matrix.
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Theorem DC	Diagonalization Characterization	265		
Suppose $A$ is a square matrix of size $n$ . Then $A$ is diagonalizable if and only if there exists a linearly independent set $S$ that contains $n$ eigenvectors of $A$ .				
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Theorem DMFE	Diagonalizable Matrices have Full Eigenspaces 2	66
Suppose $A$ is a square eigenvalue $\lambda$ of $A$ .	re matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for ever	əry

### Theorem DED Distinct Eigenvalues implies Diagonalizable

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

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267

 $\mathbf{268}$ 

### Definition LT Linear Transformation

A linear transformation,  $T: U \mapsto V$ , is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1.  $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$  for all  $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2.  $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$  for all  $\mathbf{u} \in U$  and all  $\alpha \in \mathbb{C}$

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Theorem LTTZZ Linear Transformations Take Zero to Zero	269
Suppose $T: U \mapsto V$ is a linear transformation. Then $T(0) = 0$ .	
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Theorem MBLT	Matrices Build Linear Transformations	270
Suppose that $A$ is an a linear transformat	an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$ . Then tion.	ı T is
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### Theorem MLTCV Matrix of a Linear Transformation, Column Vectors 271

Suppose that  $T: \mathbb{C}^n \mapsto \mathbb{C}^m$  is a linear transformation. Then there is an  $m \times n$  matrix A such that  $T(\mathbf{x}) = A\mathbf{x}$ .

Theorem LTLC         Linear Transformations and Linear Combinations         272			
Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from $U$ and $a_1, a_2, a_3, \ldots, a_t$ are scalars from $\mathbb{C}$ . Then			
$T(a_{1}\mathbf{u}_{1} + a_{2}\mathbf{u}_{2} + a_{3}\mathbf{u}_{3} + \dots + a_{t}\mathbf{u}_{t}) = a_{1}T(\mathbf{u}_{1}) + a_{2}T(\mathbf{u}_{2}) + a_{3}T(\mathbf{u}_{3}) + \dots + a_{t}T(\mathbf{u}_{t})$			
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### Theorem LTDB Linear Transformation Defined on a Basis

Suppose  $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a basis for the vector space U and  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$  is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation,  $T: U \mapsto V$ , such that  $T(\mathbf{u}_i) = \mathbf{v}_i, 1 \leq i \leq n$ .

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 $\mathbf{273}$ 

 $\mathbf{274}$ 

### Definition PI Pre-Image

Suppose that  $T: U \mapsto V$  is a linear transformation. For each **v**, define the **pre-image** of **v** to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{ \mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v} \right\}$$

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### Definition LTA Linear Transformation Addition

 $\mathbf{275}$ 

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are two linear transformations with the same domain and codomain. Then their **sum** is the function  $T + S: U \mapsto V$  whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 276

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are two linear transformations with the same domain and codomain. Then  $T + S: U \mapsto V$  is a linear transformation.

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### Definition LTSM Linear Transformation Scalar Multiplication

277

Suppose that  $T: U \mapsto V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then the scalar multiple is the function  $\alpha T: U \mapsto V$  whose outputs are defined by

$$\left(\alpha T\right)\left(\mathbf{u}\right) = \alpha T\left(\mathbf{u}\right)$$

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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 278

Suppose that  $T: U \mapsto V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then  $(\alpha T): U \mapsto V$  is a linear transformation.

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### Theorem VSLT Vector Space of Linear Transformations

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V,  $\mathcal{L}T(U, V)$  is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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### Definition LTC Linear Transformation Composition

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations. Then the **composition** of S and T is the function  $(S \circ T): U \mapsto W$  whose outputs are defined by

 $(S \circ T) (\mathbf{u}) = S (T (\mathbf{u}))$ 

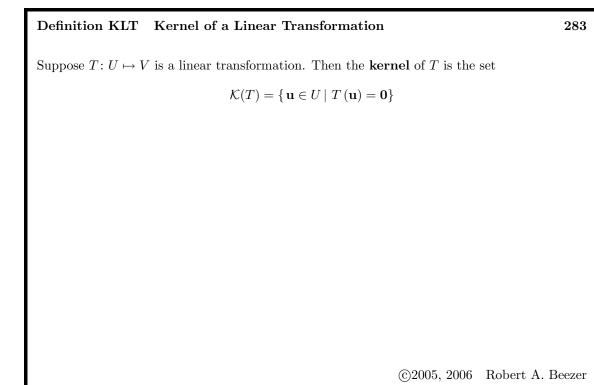
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 $\mathbf{280}$ 

### Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 281

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations. Then  $(S \circ T): U \mapsto W$  is a linear transformation.

Definition ILT Injective Linear Transformation	282
Suppose $T: U \mapsto V$ is a linear transformation. Then $T$ is <b>injective</b> if wheneve then $\mathbf{x} = \mathbf{y}$ .	$\operatorname{er} T\left(\mathbf{x}\right) = T\left(\mathbf{y}\right),$
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**Theorem KLTS** Kernel of a Linear Transformation is a Subspace 284 Suppose that  $T: U \mapsto V$  is a linear transformation. Then the kernel of  $T, \mathcal{K}(T)$ , is a subspace of U.

### Theorem KPI Kernel and Pre-Image

Suppose  $T: U \mapsto V$  is a linear transformation and  $\mathbf{v} \in V$ . If the preimage  $T^{-1}(\mathbf{v})$  is non-empty, and  $\mathbf{u} \in T^{-1}(\mathbf{v})$  then

 $T^{-1}\left(\mathbf{v}\right) = \left\{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\right\} = \mathbf{u} + \mathcal{K}(T)$ 

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Theorem KILT Kernel of an Injective Linear Tra	ansformation	286
Suppose that $T: U \mapsto V$ is a linear transformation. Then of T is trivial, $\mathcal{K}(T) = \{0\}.$	T is injective if an	d only if the kernel
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### $\mathbf{285}$

### Theorem ILTLI Injective Linear Transformations and Linear Independence $\mathbf{287}$

Suppose that  $T: U \mapsto V$  is an injective linear transformation and  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$  is a linearly independent subset of U. Then  $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$  is a linearly independent subset of V.

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Theorem ILTB	Injective Linear Transformations and Bases	288
	$U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis productive if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linear of $V$ .	
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# Theorem ILTD Injective Linear Transformations and Dimension 289 Suppose that $T: U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$ . (©2005, 2006) Robert A. Beezer

Theorem CILTI Composition of Injective Linear Transformations is Injective 290

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are injective linear transformations. Then  $(S \circ T): U \mapsto W$  is an injective linear transformation.

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### Definition SLT Surjective Linear Transformation 291 Suppose $T: U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$ . ©2005, 2006 Robert A. Beezer

Definition RLT	Range of a Linear Transformation	292
Suppose $T \colon U \mapsto V$	is a linear transformation. Then the <b>range</b> of $T$ is the set	
	$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$	
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### Theorem RLTS Range of a Linear Transformation is a Subspace

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the range of T,  $\mathcal{R}(T)$ , is a subspace of V.

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Theorem RSLT	Range of a Surjective Linear Transformation   29	94
Suppose that $T: U$ of $T$ equals the code	$H \mapsto V$ is a linear transformation. Then T is surjective if and only if the range domain, $\mathcal{R}(T) = V$ .	.ge
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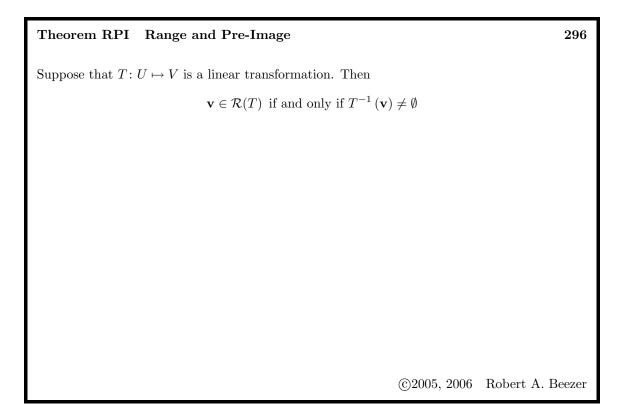
### Theorem SSRLT Spanning Set for Range of a Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation and  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$  spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

spans  $\mathcal{R}(T)$ .

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### Theorem SLTB Surjective Linear Transformations and Bases

Suppose that  $T: U \mapsto V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is surjective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a spanning set for V.

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Theorem SLTD	Surjective Linear Transformations and Dimension	298
Suppose that $T: U$	$F \mapsto V$ is a surjective linear transformation. Then dim $(U) \ge \dim (V)$ .	
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### Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 299

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are surjective linear transformations. Then  $(S \circ T): U \mapsto W$  is a surjective linear transformation.

Definition IDLT Identity	<sup>7</sup> Linear Transfo	rmation	300
The <b>identity linear transfo</b>	rmation on the v	ector space $W$ is defined as	
	$I_W \colon W \mapsto W,$	$I_{W}\left(\mathbf{w}\right)=\mathbf{w}$	
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Definition IVLTInvertible Linear Transformations301Suppose that  $T: U \mapsto V$  is a linear transformation. If there is a function  $S: V \mapsto U$  such that $S \circ T = I_U$  $T \circ S = I_V$ then T is invertible. In this case, we call S the inverse of T and write  $S = T^{-1}$ .

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Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 302

Suppose that  $T: U \mapsto V$  is an invertible linear transformation. Then the function  $T^{-1}: V \mapsto U$  is a linear transformation.

### Theorem IILT Inverse of an Invertible Linear Transformation

Suppose that  $T: U \mapsto V$  is an invertible linear transformation. Then  $T^{-1}$  is an invertible linear transformation and  $(T^{-1})^{-1} = T$ .

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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective304

Suppose  $T: U \mapsto V$  is a linear transformation. Then T is invertible if and only if T is injective and surjective.

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Theorem CIVLT Composition of Invertible Linear Transformations	305
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. composition, $(S \circ T): U \mapsto W$ is an invertible linear transformation.	Then the

Theorem ICLT	Inverse of a Composition of Linear Transformations	306
Suppose that $T: U$ invertible and $(S \circ$	$V \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. $(T)^{-1} = T^{-1} \circ S^{-1}.$	Then $S \circ T$ is
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### Definition IVS Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain  $V, T: U \mapsto V$ . In this case, we write  $U \cong V$ , and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED	Isomorphic Vector Spaces have Equal Dimension 30	)8
Suppose $U$ and $V$ ar	re isomorphic vector spaces. Then $\dim(U) = \dim(V)$ .	
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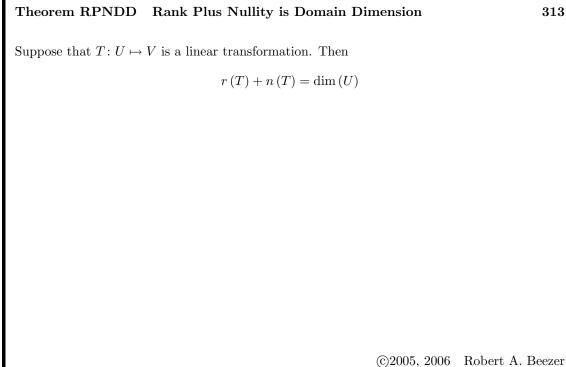
### Definition ROLT Rank Of a Linear Transformation 309 Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T, r(T), is the dimension of the range of T, $r(T) = \dim(\mathcal{R}(T))$ ©2005, 2006 Robert A. Beezer

 Definition NOLT
 Nullity Of a Linear Transformation
 310

 Suppose that  $T: U \mapsto V$  is a linear transformation. Then the nullity of T, n(T), is the dimension of the kernel of T,
  $n(T) = \dim(\mathcal{K}(T))$ 
 $n(T) = \dim(\mathcal{K}(T))$   $n(T) = \dim(\mathcal{K}(T))$ 

## **Theorem ROSLTRank Of a Surjective Linear Transformation311**Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V,<br/> $r(T) = \dim(V)$ , if and only if T is surjective.

Theorem NOILT Nullity Of an Injective Linear Trans	formation	312
Suppose that $T: U \mapsto V$ is an injective linear transformation. n(T) = 0, if and only if T is injective.	Then the n	ullity of $T$ is zero,
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### Definition VR Vector Representation

Suppose that V is a vector space with a basis  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ . Define a function  $\rho_B: V \mapsto \mathbb{C}^n$  as follows. For  $\mathbf{w} \in V$  define the column vector  $\rho_B(\mathbf{w}) \in \mathbb{C}^n$  by

 $\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \dots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$ 

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 $\mathbf{314}$ 

Theorem VRLT	Vector Representation is a Linear Transformation 3	15
The function $\rho_B$ (D	Definition VR) is a linear transformation.	
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	<b>,</b> , , , , , , , , , , , , , , , , , ,	

Theorem VRI Vector Representation is Injective	316
The function $\rho_B$ (Definition VR) is an injective linear transformation.	
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Theorem VRS Vector Representation is Surjective	317
The function $\rho_B$ (Definition VR) is a surjective linear transformation.	
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Theorem VRILT Vector Representation is an Invertible Linear Transformation 318

The function  $\rho_B$  (Definition VR) is an invertible linear transformation.

Theorem CFDVS	Characterization of Finite Dimensional Vector Spaces	319
Suppose that $V$ is a	vector space with dimension $n$ . Then $V$ is isomorphic to $\mathbb{C}^n$ .	
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Suppose U and V are both finite-dimensional vector spaces. and only if dim $(U) = \dim(V)$ .	Then $U$ and $V$ are isomorphic if

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces

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### Theorem CLI Coordinatization and Linear Independence

Suppose that U is a vector space with a basis B of size n. Then  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$  is a linearly independent subset of U if and only if  $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$  is a linearly independent subset of  $\mathbb{C}^n$ .

oordinatization and Spanning Sets	322
vector space with a basis <i>B</i> of size <i>n</i> . Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} $ $\langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle.$	) if
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V	ector space with a basis <i>B</i> of size <i>n</i> . Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \\ \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle.$

### Definition MR Matrix Representation

Suppose that  $T: U \mapsto V$  is a linear transformation,  $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the  $m \times n$  matrix,

 $M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$ 

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### Theorem MRSLT Matrix Representation of a Sum of Linear Transformations325

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are linear transformations, B is a basis of U and C is a basis of V. Then  $M_{R}^{T+S}$ S T

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 326

Suppose that  $T: U \mapsto V$  is a linear transformation,  $\alpha \in \mathbb{C}$ , B is a basis of U and C is a basis of V. Then  $M_{\Sigma}^{\alpha T}$ T

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

### Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 327

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

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 $\mathbf{328}$ 

Theorem KNS	Kernel an	d Null Space	Isomorphism

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of  $M_{B,C}^T$ ,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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### Theorem RCSI Range and Column Space Isomorphism

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of  $M_{B,C}^T$ ,

$$\mathcal{R}(T) \cong \mathcal{C}\big(M_{B,C}^T\big)$$

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329

330

### Theorem IMR Invertible Matrix Representations

Suppose that  $T: U \mapsto V$  is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C,  $M_{B,C}^T$  is an invertible matrix, and  $M_{B,C}^{T-1} = (M_{B,C}^T)^{-1}$ 

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^-$$

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### Theorem IMILT Invertible Matrices, Invertible Linear Transformation

Suppose that A is a square matrix of size n and  $T: \mathbb{C}^n \mapsto \mathbb{C}^n$  is the linear transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ . Then A is invertible matrix if and only if T is an invertible linear transformation.

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 $\mathbf{331}$ 

<b>Theorem NME9</b> Nonsingular Matrix Equivalences, Round 9 Suppose that $A$ is a square matrix of size $n$ . The following are equivalent.	332
1. $A$ is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$ .	
5. The columns of $A$ are a linearly independent set.	
6. A is invertible.	
7. The column space of A is $\mathbb{C}^n$ , $\mathcal{C}(A) = \mathbb{C}^n$ .	
8. The columns of A are a basis for $\mathbb{C}^n$ .	
9. The rank of A is $n, r(A) = n$ .	
10. The nullity of A is zero, $n(A) = 0$ .	
11. The determinant of A is nonzero, $\det(A) \neq 0$ .	
12. $\lambda = 0$ is not an eigenvalue of A.	
13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.	
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### **Definition EELT** Eigenvalue and Eigenvector of a Linear Transformation 333 Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue $\lambda$ if $T(\mathbf{v}) = \lambda \mathbf{v}$ .

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**33**4

### Definition CBM Change-of-Basis Matrix

Suppose that V is a vector space, and  $I_V: V \mapsto V$  is the identity linear transformation on V. Let  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$  and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of  $I_V$  relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$
  
=  $[\rho_C (I_V (\mathbf{v}_1)) | \rho_C (I_V (\mathbf{v}_2)) | \rho_C (I_V (\mathbf{v}_3)) | \dots | \rho_C (I_V (\mathbf{v}_n))]$   
=  $[\rho_C (\mathbf{v}_1) | \rho_C (\mathbf{v}_2) | \rho_C (\mathbf{v}_3) | \dots | \rho_C (\mathbf{v}_n)]$ 

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### Theorem CB Change-of-Basis

Suppose that  $\mathbf{v}$  is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

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Theorem ICBM Inverse of Char	-of-Basis Matrix 336
Suppose that V is a vector space, and $I$ $C_{B,C}$ is nonsingular and	and C are bases of V. Then the change-of-basis matrix $C_{B,C}^{-1} = C_{C,B}$
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# Theorem MRCB Matrix Representation and Change of Basis 337 Suppose that $T: U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then $M_{B,D}^T = C_{E,D}M_{C,E}^TC_{B,C}$ $M_{B,D}^T = C_{E,D}M_{C,E}^TC_{B,C}$ $(\bigcirc 2005, 2006)$ Robert A. Beezer

 Theorem SCB
 Similarity and Change of Basis
 338

 Suppose that  $T: V \mapsto V$  is a linear transformation and B and C are bases of V. Then
  $M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$ 
 $M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$   $(C_{B,C}, C_{B,C}, C_{B,C})$  

 (Countries)
  $(C_{B,C}, C_{B,C}, C_{B,C})$ 

### Theorem EER Eigenvalues, Eigenvectors, Representations

**340** 

Suppose that  $T: V \mapsto V$  is a linear transformation and B is a basis of V. Then  $\mathbf{v} \in V$  is an eigenvector of T for the eigenvalue  $\lambda$  if and only if  $\rho_B(\mathbf{v})$  is an eigenvector of  $M_{B,B}^T$  for the eigenvalue  $\lambda$ .

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### Definition NLT Nilpotent Linear Transformation

Suppose that  $T: V \mapsto V$  is a linear transformation such that there is an integer p > 0 such that  $T^p(\mathbf{v}) = \mathbf{0}$  for every  $\mathbf{v} \in V$ . The smallest p for which this condition is met is called the **index** of T.

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### Definition JB Jordan Block

Given the scalar  $\lambda \in \mathbb{C}$ , the Jordan block  $J_n(\lambda)$  is the  $n \times n$  matrix defined by

$$\left[J_n\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$

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Theorem NJB	Nilpotent Jordan Blocks		342
The Jordan block	$J_{n}(0)$ is nilpotent of index $n$ .		
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Theorem ENLT	Eigenvalues of Nilpotent Linear Transformations 3	343
Suppose that $T: V$ $\lambda = 0.$	$\mapsto V$ is a nilpotent linear transformation and $\lambda$ is an eigenvalue of $T$ . The second secon	nen
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Theorem DNLT	Diagonalizable Nilpotent Linear	r Transformation	ns 344
Suppose the linear t $T$ is the zero linear	transformation $T \colon V \mapsto V$ is nilpotent transformation.	. Then $T$ is diagon	alizable if and only
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### Theorem KPLT Kernels of Powers of Linear Transformations

**345** 

Suppose  $T: V \mapsto V$  is a linear transformation, where dim (V) = n. Then there is an integer m,  $0 \le m \le n$ , such that

 $\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$ 

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Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 346

Suppose  $T: V \mapsto V$  is a nilpotent linear transformation with index p and dim (V) = n. Then  $0 \le p \le n$  and

 $\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$ 

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### Theorem CFNLT Canonical Form for Nilpotent Linear Transformations 347

Suppose that  $T: V \mapsto V$  is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation,  $M_{B,B}^T$ , is block diagonal with each block being a Jordan block,  $J_n(0)$ . The size of the largest block is the index p, and the total number of blocks is the nullity of T, n(T).

Definition IS Invariant Subspace	348
Suppose that $T: V \mapsto V$ is a linear transformation and $W$ is a subspace of $V$ , that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$ . Then $W$ is an <b>invariant subspace</b> of $V$ is	
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### Theorem EIS Eigenspaces are Invariant Subspaces

Suppose that  $T: V \mapsto V$  is a linear transformation with eigenvalue  $\lambda$  and associated eigenspace  $\mathcal{E}_T(\lambda)$ . Let W be any subspace of  $\mathcal{E}_T(\lambda)$ . Then W is an invariant subspace of V relative to T.

Theorem KPIS	Kernels of Powers are Invariant Subspaces350
	$V \mapsto V$ is a linear transformation. Then $\mathcal{K}(T^k)$ is an invariant subspace of
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### Definition GEV Generalized Eigenvector

Suppose that  $T: V \mapsto V$  is a linear transformation. Suppose further that for  $\mathbf{x} \neq \mathbf{0}$ ,  $(T - \lambda I_V)^k (\mathbf{x}) = \mathbf{0}$  for some k > 0. Then  $\mathbf{x}$  is a **generalized eigenvector** of T with eigenvalue  $\lambda$ .

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351

 $\mathbf{352}$ 

### Definition GES Generalized Eigenspace

Suppose that  $T: V \mapsto V$  is a linear transformation. Define the **generalized eigenspace** of T for  $\lambda$  as

$$\mathcal{G}_{T}\left(\lambda\right) = \left\{ \mathbf{x} \mid \left(T - \lambda I_{V}\right)^{k}\left(\mathbf{x}\right) = \mathbf{0} \text{ for some } k \ge 0 \right\}$$

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Theorem GESIS	Generalized Eigenspace is an Invariant Subspace 353
Suppose that $T: V \vdash$ an invariant subspace	V is a linear transformation. Then the generalized eigenspace $\mathcal{G}_{T}(\lambda)$ is of V relative to T.
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Theorem GEK	Generalized Eigenspace as a Kernel	354
Suppose that $T: V$ Then $\mathcal{G}_T(\lambda) = \mathcal{K}(\mathbf{r})$	$V \mapsto V$ is a linear transformation, dim $(V) = n$ , and $\lambda$ is an eigen $(T - \lambda I_V)^n$ .	envalue of $T$ .
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### Definition LTR Linear Transformation Restriction

355

Suppose that  $T: V \mapsto V$  is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by

$$T|_{U} \colon U \mapsto U \qquad \qquad T|_{U} \left(\mathbf{u}\right) = T\left(\mathbf{u}\right)$$

Theorem RGEN	Restriction to Generalized Eigenspace is Nilpotent	356
Suppose $T: V \mapsto V$ is $T _{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is a	s a linear transformation with eigenvalue $\lambda$ . Then the linear transfor nilpotent.	mation
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### Definition IE Index of an Eigenvalue

357

Suppose  $T: V \mapsto V$  is a linear transformation with eigenvalue  $\lambda$ . Then the **index** of  $\lambda$ ,  $\iota_T(\lambda)$ , is the index of the nilpotent linear transformation  $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ .

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Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace 358

Suppose that  $T: V \mapsto V$  is a linear transformation with eigenvalue  $\lambda$ . Then there is a basis of the the generalized eigenspace  $\mathcal{G}_T(\lambda)$  such that the restriction  $T|_{\mathcal{G}_T(\lambda)}: \mathcal{G}_T(\lambda) \mapsto \mathcal{G}_T(\lambda)$  has a matrix representation that is block diagonal where each block is a Jordan block of the form  $J_n(\lambda)$ .

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Theorem GESD	Generalized Eigenspace Decomposition	359
Suppose that $T(V)$ Then	V is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$	, $\lambda_m$ .
	$V = \mathcal{G}_{T}(\lambda_{1}) \oplus \mathcal{G}_{T}(\lambda_{2}) \oplus \mathcal{G}_{T}(\lambda_{3}) \oplus \cdots \oplus \mathcal{G}_{T}(\lambda_{m})$	

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Theorem DGES Dimension of Generalized Eigenspaces	360
Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue $\lambda$ . Then the dimension generalized eigenspace for $\lambda$ is the algebraic multiplicity of $\lambda$ , dim $(\mathcal{G}_T(\lambda_i)) = \alpha_T(\lambda_i)$ .	n of the
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### Definition JCF Jordan Canonical Form

A square matrix is in **Jordan canonical form** if it meets the following requirements:

- 1. The matrix is block diagonal.
- 2. Each block is a Jordan block.
- 3. If  $\rho < \lambda$  then the block  $J_k(\rho)$  occupies rows with indices greater than the indices of the rows occupied by  $J_\ell(\lambda)$ .
- 4. If  $\rho = \lambda$  and  $\ell < k$ , then the block  $J_{\ell}(\lambda)$  occupies rows with indices greater than the indices of the rows occupied by  $J_k(\lambda)$ .

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### Theorem JCFLT Jordan Canonical Form for a Linear Transformation

Suppose  $T: V \mapsto V$  is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

- 1. The matrix representation is in Jordan canonical form.
- 2. If  $J_k(\lambda)$  is one of the Jordan blocks, then  $\lambda$  is an eigenvalue of T.
- 3. For a fixed value of  $\lambda$ , the largest block of the form  $J_k(\lambda)$  has size equal to the index of  $\lambda$ ,  $\iota_T(\lambda)$ .
- 4. For a fixed value of  $\lambda$ , the number of blocks of the form  $J_k(\lambda)$  is the geometric multiplicity of  $\lambda$ ,  $\gamma_T(\lambda)$ .
- 5. For a fixed value of  $\lambda$ , the number of rows occupied by blocks of the form  $J_k(\lambda)$  is the algebraic multiplicity of  $\lambda$ ,  $\alpha_T(\lambda)$ .

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## Theorem CHT Cayley-Hamilton Theorem 363 Suppose A is a square matrix with characteristic polynomial $p_A(x)$ . Then $p_A(A) = \mathcal{O}$ .