Flash Cards

to accompany

A First Course in Linear Algebra

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The most recent version of this work can always be found at `http://linear.ups.edu`.

Definition CNE Complex Number Equality **1**

The complex numbers $\alpha = a+bi$ and $\beta = c+di$ are **equal**, denoted $\alpha = \beta$, if $a = c$ and $b = d$.

Definition CNA Complex Number Addition **2**

The **sum** of the complex numbers $\alpha = a+bi$ and $\beta = c+di$, denoted $\alpha+\beta$, is $(a+c)+(b+d)i$.

Definition CNM Complex Number Multiplication 3

The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is $(ac - bd) + (ad + bc)i$.

Theorem PCNA Properties of Complex Number Arithmetic 4

The operations of addition and multiplication of complex numbers have the following properties.

- **ACCN Additive Closure, Complex Numbers** If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- **MCCN Multiplicative Closure, Complex Numbers** If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- **CACN Commutativity of Addition, Complex Numbers** For any $\alpha,\ \beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- **CMCN Commutativity of Multiplication, Complex Numbers** For any $\alpha,\ \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- **AACN Additive Associativity, Complex Numbers** For any $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- **MACN Multiplicative Associativity, Complex Numbers** For any $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $\alpha\left(\beta\gamma\right) = \left(\alpha\beta\right)\gamma$.
- **DCN Distributivity, Complex Numbers** For any $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- **ZCN Zero, Complex Numbers** There is a complex number $0 = 0 + 0i$ so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- **OCN One, Complex Numbers** There is a complex number $1 = 1 + 0i$ so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- **AICN Additive Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- **MICN Multiplicative Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

Definition CCN Conjugate of a Complex Number 5

The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

Theorem CCRA Complex Conjugation Respects Addition 6

Suppose that c and d are complex numbers. Then $\overline{c + d} = \overline{c} + \overline{d}$.

Theorem CCRM Complex Conjugation Respects Multiplication **7**

Suppose that c and d are complex numbers. Then $\overline{cd} = \overline{c}\,\overline{d}$.

Theorem CCT Complex Conjugation Twice **8**

Suppose that c is a complex number. Then $\overline{\overline{c}} = c$.

Definition MCN Modulus of a Complex Number

The **modulus** of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}.$$

Definition SET Set

A **set** is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \not\in S$. We refer to the objects in a set as its **elements**.

Definition SSET Subset 11

If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Definition ES Empty Set 12

The empty set is the set with no elements. Its is denoted by $\emptyset$.

Definition SE Set Equality **13**

Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write $S = T$.

Definition C Cardinality **14**

Suppose S is a finite set. Then the number of elements in S is called the **cardinality** or **size** of S, and is denoted $|S|$.

Definition SU   Set Union   **15**

Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

Definition SI   Set Intersection   **16**

Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$

Definition SC Set Complement 17

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted $\overline{S}$, is the set whose elements are those that are elements of U and not elements of S. More formally,

$$x \in \overline{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

Definition SLE System of Linear Equations 18

A **system of linear equations** is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Definition ESYS   Equivalent Systems **19**

Two systems of linear equations are **equivalent** if their solution sets are equal.

Definition EO   Equation Operations **20**

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Theorem EOPSS Equation Operations Preserve Solution Sets **21**

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M Matrix **22**

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, $C, \dots$) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Definition CV Column Vector **23**

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\underset{\sim}{u}$. To refer to the **entry** or **component** that is number i in the list that is the vector $\mathbf{v}$ we write $[\mathbf{v}]_i$.

Definition ZCV Zero Column Vector **24**

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $[\mathbf{0}]_i = 0$ for $1 \leq i \leq m$.

Definition CM Coefficient Matrix **25**

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants **26**

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Definition SOLV Solution Vector **27**

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n \end{bmatrix}$$

Definition LSMR Matrix Representation of a Linear System **28**

If A is the coefficient matrix of a system of linear equations and $\mathbf{b}$ is the vector of constants, then we will write $\mathcal{LS}(A,\,\mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

Definition AM Augmented Matrix **29**

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants $\mathbf{b}$. Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (number $n+1$) is the column vector $\mathbf{b}$. This matrix will be written as $[A \,|\, \mathbf{b}]$.

Definition RO Row Operations **30**

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
2. αR_i: Multiply row i by the nonzero scalar α.
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

Definition REM Row-Equivalent Matrices **31**

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Theorem REMES Row-Equivalent Matrices represent Equivalent Systems **32**

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

Definition RREF Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.
A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot colums will be denoted as $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Theorem REMEF Row-Equivalent Matrix in Echelon Form

Suppose A is a matrix. Then there is a matrix B so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

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Theorem RREFU Reduced Row-Echelon Form is Unique **35**

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Definition RR Row-Reducing **36**

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

Definition CS Consistent System **37**

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Definition IDV Independent and Dependent Variables **38**

Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column). Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem RCLS Recognizing Consistency of a Linear System **39**

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n+1$ of B.

Theorem ISRN Inconsistent Systems, r and n **40**

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Theorem CSRN Consistent Systems, r and n **41**

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Theorem FVCS Free Variables for Consistent Systems **42**

Suppose A is the augmented matrix of a *consistent* system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS Possible Solution Sets for Linear Systems **43**

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions 44

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Definition HS Homogeneous System **45**

A system of linear equations, $\mathcal{LS}(A,\,\mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

Theorem HSC Homogeneous Systems are Consistent **46**

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE Trivial Solution to Homogeneous Systems of Equations **47**

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions **48**

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Definition NSM Null Space of a Matrix **49**

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition SQM Square Matrix **50**

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM Nonsingular Matrix **51**

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A,\,\mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Definition IM Identity Matrix **52**

The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i,\, j \leq m$$

Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 53

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS Nonsingular Matrices have Trivial Null Spaces 54

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Theorem NMUS Nonsingular Matrices and Unique Solutions **55**

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Theorem NME1 Nonsingular Matrix Equivalences, Round 1 **56**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

Definition VSCV Vector Space of Column Vectors **57**

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE Column Vector Equality **58**

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Then $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Definition CVA Column Vector Addition **59**

Suppose that $\mathbf{u},\,\mathbf{v}\in\mathbb{C}^m$. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u}+\mathbf{v}$ defined by

$$\left[\mathbf{u}+\mathbf{v}\right]_i=\left[\mathbf{u}\right]_i+\left[\mathbf{v}\right]_i \qquad\qquad 1\leq i\leq m$$

Definition CVSM Column Vector Scalar Multiplication **60**

Suppose $\mathbf{u}\in\mathbb{C}^m$ and $\alpha\in\mathbb{C}$, then the **scalar multiple** of $\mathbf{u}$ by α is the vector $\alpha\mathbf{u}$ defined by

$$\left[\alpha\mathbf{u}\right]_i=\alpha\left[\mathbf{u}\right]_i \qquad\qquad 1\leq i\leq m$$

Theorem VSPCV Vector Space Properties of Column Vectors **61**

Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC Additive Closure, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- **SCC Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.
- **CC Commutativity, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **AAC Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **ZC Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- **AIC Additive Inverses, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- **SMAC Scalar Multiplication Associativity, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- **DVAC Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- **DSAC Distributivity across Scalar Addition, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- **OC One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Definition LCCV Linear Combination of Column Vectors **62**

Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n$$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations **63**

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x}$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems **64**

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}$, $\mathbf{u}_j$, $1 \le j \le n-r$ of size n by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D,\ i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r} \,\middle|\, x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \right\}$$

Theorem PSPHS Particular Solution Plus Homogeneous Solutions **65**

Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A,\,b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\,b)$ if and only if $\mathbf{y}=\mathbf{w}+\mathbf{z}$ for some vector $\mathbf{z}\in\mathcal{N}(A)$.

Definition SSCV Span of a Set of Column Vectors **66**

Given a set of vectors $S=\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_p\}$, their **span**, $\langle S\rangle$, is the set of all possible linear combinations of $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S\rangle &= \{\,\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\cdots+\alpha_p\mathbf{u}_p \mid \alpha_i\in\mathbb{C},\,1\leq i\leq p\}\\ &= \left\{\sum_{i=1}^{p}\alpha_i\mathbf{u}_i \,\middle|\, \alpha_i\in\mathbb{C},\,1\leq i\leq p\right\}\end{aligned}$$

Theorem SSNS Spanning Sets for Null Spaces **67**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the $n-r$ vectors $\mathbf{z}_j$, $1 \leq j \leq n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\, i = f_j \\ 0 & \text{if } i \in F,\, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\} \rangle .$$

Definition RLDCV Relation of Linear Dependence for Column Vectors **68**

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, a true statement of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is the **trivial relation of linear dependence** on S.

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Definition LICV **Linear Independence of Column Vectors** **69**

The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Theorem LIVHS **Linearly Independent Vectors and Homogeneous Systems** **70**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Theorem LIVRN Linearly Independent Vectors, r and n **71**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Theorem MVSLD More Vectors than Size implies Linear Dependence **72**

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 73

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NME2 Nonsingular Matrix Equivalences, Round 2 74

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces **75**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ and $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n-r$ vectors $\mathbf{z}_j$, $1 \leq j \leq n-r$ of size n as

$$\left[\mathbf{z}_j\right]_i = \begin{cases} 1 & \text{if } i \in F,\, i = f_j \\ 0 & \text{if } i \in F,\, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D,\, i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1,\, \mathbf{z}_2,\, \mathbf{z}_3,\, \ldots,\, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.
2. S is a linearly independent set.

Theorem DLDS Dependency in Linearly Dependent Sets **76**

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \leq t \leq n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_{t-1},\, \mathbf{u}_{t+1},\, \ldots,\, \mathbf{u}_n$.

Theorem BS Basis of a Span 77

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \ldots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots \mathbf{v}_{d_r}\}$ is a linearly independent set.
2. $W = \langle T \rangle$.

Definition CCCV Complex Conjugate of a Column Vector 78

Suppose that $\mathbf{u}$ is a vector from $\mathbb{C}^m$. Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \qquad\qquad 1 \leq i \leq m$$

Theorem CRVA Conjugation Respects Vector Addition **79**

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x}+\mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Theorem CRSM Conjugation Respects Vector Scalar Multiplication **80**

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP Inner Product **81**

Given the vectors $\mathbf{u},\,\mathbf{v}\in\mathbb{C}^m$ the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle\mathbf{u},\,\mathbf{v}\rangle=\left[\mathbf{u}\right]_1\overline{\left[\mathbf{v}\right]_1}+\left[\mathbf{u}\right]_2\overline{\left[\mathbf{v}\right]_2}+\left[\mathbf{u}\right]_3\overline{\left[\mathbf{v}\right]_3}+\cdots+\left[\mathbf{u}\right]_m\overline{\left[\mathbf{v}\right]_m}=\sum_{i=1}^{m}\left[\mathbf{u}\right]_i\overline{\left[\mathbf{v}\right]_i}$$

Theorem IPVA Inner Product and Vector Addition **82**

Suppose $\mathbf{u},\,\mathbf{v},\,\mathbf{w}\in\mathbb{C}^m$. Then

1. $\langle\mathbf{u}+\mathbf{v},\,\mathbf{w}\rangle=\langle\mathbf{u},\,\mathbf{w}\rangle+\langle\mathbf{v},\,\mathbf{w}\rangle$
2. $\langle\mathbf{u},\,\mathbf{v}+\mathbf{w}\rangle=\langle\mathbf{u},\,\mathbf{v}\rangle+\langle\mathbf{u},\,\mathbf{w}\rangle$

Theorem IPSM Inner Product and Scalar Multiplication **83**

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

1. $\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
2. $\langle \mathbf{u}, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$

Theorem IPAC Inner Product is Anti-Commutative **84**

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

Definition NV Norm of a Vector **85**

The **norm** of the vector $\mathbf{u}$ is the scalar quantity in $\mathbb{C}$

$$\|\mathbf{u}\| = \sqrt{\left|[\mathbf{u}]_1\right|^2 + \left|[\mathbf{u}]_2\right|^2 + \left|[\mathbf{u}]_3\right|^2 + \cdots + \left|[\mathbf{u}]_m\right|^2} = \sqrt{\sum_{i=1}^{m} \left|[\mathbf{u}]_i\right|^2}$$

Theorem IPN Inner Products and Norms **86**

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

Theorem PIP Positive Inner Products 87

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV Orthogonal Vectors 88

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u},\, \mathbf{v} \rangle = 0$.

Definition OSV Orthogonal Set of Vectors **89**

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then S is an **orthogonal set** if every pair of different vectors from S is orthogonal, that is $\langle\mathbf{u}_i,\, \mathbf{u}_j\rangle = 0$ whenever $i \neq j$.

Definition SUV Standard Unit Vectors **90**

Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$\left[\mathbf{e}_j\right]_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \ldots,\, \mathbf{e}_m\} = \{\, \mathbf{e}_j \mid 1 \leq j \leq m\}$$

is the set of **standard unit vectors** in $\mathbb{C}^m$.

Theorem OSLI Orthogonal Sets are Linearly Independent **91**

Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSP Gram-Schmidt Procedure **92**

Suppose that $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3,\, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1},\, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Definition ONS OrthoNormal Set **93**

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

Definition VSM Vector Space of $m \times n$ Matrices **94**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality **95**

The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition MA Matrix Addition **96**

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad\qquad 1 \leq i \leq m,\ 1 \leq j \leq n$$

Definition MSM Matrix Scalar Multiplication 97

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad\qquad 1 \leq i \leq m,\ 1 \leq j \leq n$$

Theorem VSPM Vector Space Properties of Matrices 98

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM Additive Closure, Matrices** If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- **SCM Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- **CM Commutativity, Matrices** If $A, B \in M_{mn}$, then $A + B = B + A$.
- **AAM Additive Associativity, Matrices** If $A, B, C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.
- **ZM Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- **AIM Additive Inverses, Matrices** If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- **SMAM Scalar Multiplication Associativity, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.
- **DMAM Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.
- **DSAM Distributivity across Scalar Addition, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- **OM One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM Zero Matrix **99**

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m\times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM Transpose of a Matrix **100**

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,, \quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix **101**

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS Symmetric Matrices are Square **102**

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA Transpose and Matrix Addition **103**

Suppose that A and B are $m \times n$ matrices. Then $(A+B)^t = A^t + B^t$.

Theorem TMSM Transpose and Matrix Scalar Multiplication **104**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Theorem TT Transpose of a Transpose **105**

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Definition CCM Complex Conjugate of a Matrix **106**

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

Theorem CRMA Conjugation Respects Matrix Addition **107**

Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication **108**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Theorem CCM Conjugate of the Conjugate of a Matrix 109

Suppose that A is an $m \times n$ matrix. Then $\overline{\left(\overline{A}\right)} = A$.

Theorem MCT Matrix Conjugation and Transposes 110

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.

Definition A Adjoint **111**

If A is a square matrix, then its **adjoint** is $A^* = \left(\overline{A}\right)^t$.

Theorem AMA Adjoint and Matrix Addition **112**

Suppose A and B are matrices of the same size. Then $(A+B)^* = A^* + B^*$.

Theorem AMSM Adjoint and Matrix Scalar Multiplication **113**

Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$.

Theorem AA Adjoint of an Adjoint **114**

Suppose that A is a matrix. Then $(A^*)^* = A$

Definition MVP Matrix-Vector Product

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\ldots$, $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \mathbf{A}_n$$

Theorem SLEMM Systems of Linear Equations as Matrix Multiplication

The set of solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ equals the set of solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Theorem EMMVP Equal Matrices and Matrix-Vector Products **117**

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Definition MM Matrix Multiplication **118**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Theorem EMP Entries of Matrix Products **119**

Suppose A is an $m \times n$ matrix and $B =$ is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of AB are given by

$$\begin{aligned}
\left[AB\right]_{ij} &= \left[A\right]_{i1}\left[B\right]_{1j} + \left[A\right]_{i2}\left[B\right]_{2j} + \left[A\right]_{i3}\left[B\right]_{3j} + \cdots + \left[A\right]_{in}\left[B\right]_{nj} \\
&= \sum_{k=1}^{n} \left[A\right]_{ik}\left[B\right]_{kj}
\end{aligned}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix **120**

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
2. $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

Theorem MMIM Matrix Multiplication and Identity Matrix 121

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Theorem MMDAA Matrix Multiplication Distributes Across Addition 122

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B+C) = AB + AC$
2. $(B+C)D = BD + CD$

Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication **123**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA Matrix Multiplication is Associative **124**

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

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Theorem MMIP Matrix Multiplication and Inner Products **125**

If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Theorem MMCC Matrix Multiplication and Complex Conjugation **126**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT Matrix Multiplication and Transposes **127**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Theorem MMAD Matrix Multiplication and Adjoints **128**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^* = B^* A^*$.

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Theorem AIP Adjoint and Inner Product **129**

Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x},\, \mathbf{y}\rangle = \langle \mathbf{x},\, A^*\mathbf{y}\rangle$.

Definition HM Hermitian Matrix **130**

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = A^*$.

Theorem HMIP Hermitian Matrices and Inner Products **131**

Suppose that A is a square matrix of size n. Then A is Hermitian if and only if $\langle A\mathbf{x},\, \mathbf{y}\rangle = \langle \mathbf{x},\, A\mathbf{y}\rangle$ for all $\mathbf{x},\, \mathbf{y} \in \mathbb{C}^n$.

Definition MI Matrix Inverse **132**

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Theorem TTMI Two-by-Two Matrix Inverse **133**

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem CINM Computing the Inverse of a Nonsingular Matrix **134**

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Theorem MIU Matrix Inverse is Unique **135**

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS Socks and Shoes **136**

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI Matrix Inverse of a Matrix Inverse **137**

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Theorem MIT Matrix Inverse of a Transpose **138**

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Theorem MISM Matrix Inverse of a Scalar Multiple **139**

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem NPNT Nonsingular Product has Nonsingular Terms **140**

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Theorem OSIS One-Sided Inverse is Sufficient **141**

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Theorem NI Nonsingularity is Invertibility **142**

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NME3 Nonsingular Matrix Equivalences, Round 3 **143**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.

Theorem SNCM Solution with Nonsingular Coefficient Matrix **144**

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A,\, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Definition UM Unitary Matrices 145

Suppose that U is a square matrix of size n such that $U^*U = I_n$. Then we say U is **unitary**.

Theorem UMI Unitary Matrices are Invertible 146

Suppose that U is a unitary matrix of size n. Then U is nonsingular, and $U^{-1} = U^*$.

Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets **147**

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.

Theorem UMPIP Unitary Matrices Preserve Inner Products **148**

Suppose that U is a unitary matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle U\mathbf{u},\, U\mathbf{v}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle \qquad\text{and}\qquad \|U\mathbf{v}\| = \|\mathbf{v}\|$$

Definition CSM Column Space of a Matrix **149**

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle\{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n\}\rangle$$

Theorem CSCS Column Spaces and Consistent Systems **150**

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A,\,\mathbf{b})$ is consistent.

Theorem BCS Basis of the Column Space **151**

Suppose that A is an $m\times n$ matrix with columns $\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D=\{d_1,\,d_2,\,d_3,\,\ldots,\,d_r\}$ be the set of column indices where B has leading 1's. Let $T=\{\mathbf{A}_{d_1},\,\mathbf{A}_{d_2},\,\mathbf{A}_{d_3},\,\ldots,\,\mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.
2. $\mathcal{C}(A)=\langle T\rangle$.

Theorem CSNM Column Space of a Nonsingular Matrix **152**

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A)=\mathbb{C}^n$.

Theorem NME4 Nonsingular Matrix Equivalences, Round 4 **153**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

Definition RSM Row Space of a Matrix **154**

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Theorem REMRS Row-Equivalent Matrices have equal Row Spaces **155**

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Theorem BRS Basis for the Row Space **156**

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \langle S \rangle$.
2. S is a linearly independent set.

Theorem CSRST Column Space, Row Space, Transpose **157**

Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Definition LNS Left Null Space **158**

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Definition EEF Extended Echelon Form 159

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.
Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom $m-r$ rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array}\right]$$

Theorem PEEF Properties of Extended Echelon Form 160

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. J is nonsingular.
2. $B = JA$.
3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
5. L is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

Theorem FS Four Subsets **161**

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m - r$ rows. Then

1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
3. The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.
4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS Vector Space **162**

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- **AC Additive Closure** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- **SC Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.
- **C Commutativity** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **AA Additive Associativity** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- **AI Additive Inverses** If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- **SMA Scalar Multiplication Associativity** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- **DVA Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- **DSA Distributivity across Scalar Addition** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- **O One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU Zero Vector is Unique **163**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU Additive Inverses are Unique **164**

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

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Theorem ZSSM Zero Scalar in Scalar Multiplication **165**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Theorem ZVSM Zero Vector in Scalar Multiplication **166**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM Additive Inverses from Scalar Multiplication **167**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Theorem SMEZV Scalar Multiplication Equals the Zero Vector **168**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Definition S Subspace 169

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS Testing Subsets for Subspaces 170

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.
2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Definition TS Trivial Subspaces **171**

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS Null Space of a Matrix is a Subspace **172**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC Linear Combination 173

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Definition SS Span of a Set 174

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\langle S\rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\begin{aligned}\langle S\rangle &= \{\,\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \le i \le t\} \\ &= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}\end{aligned}$$

Theorem SSS Span of a Set is a Subspace 175

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t\} \subseteq V$, their span, $\langle S\rangle$, is a subspace.

Theorem CSMS Column Space of a Matrix is a Subspace 176

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Theorem RSMS Row Space of a Matrix is a Subspace 177

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Theorem LNSMS Left Null Space of a Matrix is a Subspace 178

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$.

Definition RLD Relation of Linear Dependence **179**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LI Linear Independence **180**

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Definition TSVS To Span a Vector Space 181

Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

Theorem VRRB Vector Representation Relative to a Basis 182

Suppose that V is a vector space and $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let $\mathbf{w}$ be any vector in V. Then there exist *unique* scalars $a_1,\, a_2,\, a_3,\, \ldots,\, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Definition B   Basis

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem SUVB   Standard Unit Vectors are a Basis

The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Theorem CNMB Columns of Nonsingular Matrix are a Basis **185**

Suppose that A is a square matrix of size m. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Theorem NME5 Nonsingular Matrix Equivalences, Round 5 **186**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for $\mathbb{C}^n$.

Theorem COB Coordinates and Orthonormal Bases **187**

Suppose that $B = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w},\,\mathbf{v}_1\rangle\,\mathbf{v}_1 + \langle \mathbf{w},\,\mathbf{v}_2\rangle\,\mathbf{v}_2 + \langle \mathbf{w},\,\mathbf{v}_3\rangle\,\mathbf{v}_3 + \cdots + \langle \mathbf{w},\,\mathbf{v}_p\rangle\,\mathbf{v}_p$$

Theorem UMCOB Unitary Matrices Convert Orthonormal Bases **188**

Let A be an $n \times n$ matrix and $B = \{\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_3,\,\ldots,\,\mathbf{x}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Define $C = \{A\mathbf{x}_1,\,A\mathbf{x}_2,\,A\mathbf{x}_3,\,\ldots,\,A\mathbf{x}_n\}$. Then A is a unitary matrix if and only if C is an orthonormal basis of $\mathbb{C}^n$.

Definition D Dimension 189

Suppose that V is a vector space and $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD Spanning Sets and Linear Dependence 190

Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t+1$ or more vectors from V is linearly dependent.

Theorem BIS Bases have Identical Sizes **191**

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM Dimension of $\mathbb{C}^m$ **192**

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP Dimension of P_n 193

The dimension of P_n (Example VSP) is $n + 1$.

Theorem DM Dimension of M_{mn} 194

The dimension of M_{mn} (Example VSM) is mn.

Definition NOM Nullity Of a Matrix **195**

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM Rank Of a Matrix **196**

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Theorem CRN Computing Rank and Nullity **197**

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Theorem RPNC Rank Plus Nullity is Columns **198**

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Theorem RNNM Rank and Nullity of a Nonsingular Matrix **199**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. The rank of A is n, $r(A) = n$.
3. The nullity of A is zero, $n(A) = 0$.

Theorem NME6 Nonsingular Matrix Equivalences, Round 6 **200**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.

Theorem ELIS Extending Linearly Independent Sets **201**

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks **202**

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.
2. If $m < t$, then S does not span V.
3. If $m = t$ and S is linearly independent, then S spans V.
4. If $m = t$ and S spans V, then S is linearly independent.

Theorem PSSD Proper Subspaces have Smaller Dimension **203**

Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.

Theorem EDYES Equal Dimensions Yields Equal Subspaces **204**

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Theorem RMRT Rank of a Matrix is the Rank of the Transpose 205

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem DFS Dimensions of Four Subspaces 206

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim(\mathcal{N}(A)) = n - r$
2. $\dim(\mathcal{C}(A)) = r$
3. $\dim(\mathcal{R}(A)) = r$
4. $\dim(\mathcal{L}(A)) = m - r$

Definition DS Direct Sum **207**

Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U$, $\mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

Then V is the **direct sum** of U and W and we write $V = U \oplus W$.

Theorem DSFB Direct Sum From a Basis **208**

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define

$$U = \langle\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}\rangle \qquad W = \langle\{\mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \mathbf{v}_{m+3}, \ldots, \mathbf{v}_n\}\rangle$$

Then $V = U \oplus W$.

Theorem DSFOS Direct Sum From One Subspace **209**

Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.

Theorem DSZV Direct Sums and Zero Vectors **210**

Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
2. Whenever $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$, $\mathbf{w} \in W$ then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

Theorem DSZI Direct Sums and Zero Intersection **211**

Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
2. $U \cap W = \{\mathbf{0}\}$.

Theorem DSLI Direct Sums and Linear Independence **212**

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose that R is a linearly independent subset of U and S is a linearly independent subset of W. Then $R \cup S$ is a linearly independent subset of V.

Theorem DSD Direct Sums and Dimension **213**

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then $\dim(V) = \dim(U) + \dim(W)$.

Theorem RDS Repeated Direct Sums **214**

Suppose V is a vector space with subspaces U and W with $V = U \oplus W$. Suppose that X and Y are subspaces of W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

Definition ELEM Elementary Matrices **215**

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with
$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with
$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with
$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Theorem EMDRO Elementary Matrices Do Row Operations **216**

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.

2. If the row operation multiplies row i by α, then $B = E_i(\alpha)A$.

3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha)A$.

Theorem EMN Elementary Matrices are Nonsingular **217**

If E is an elementary matrix, then E is nonsingular.

Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices 218

Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \ldots, E_t$ so that $A = E_1 E_2 E_3 \ldots E_t$.

Definition SM SubMatrix **219**

Suppose that A is an $m \times n$ matrix. Then the **submatrix** $A(i|j)$ is the $(m-1)\times(n-1)$ matrix obtained from A by removing row i and column j.

Definition DM Determinant of a Matrix **220**

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\begin{aligned}\det(A) = [A]_{11}\det(A(1|1)) - [A]_{12}\det(A(1|2)) + [A]_{13}\det(A(1|3)) - \\ [A]_{14}\det(A(1|4)) + \cdots + (-1)^{n+1}[A]_{1n}\det(A(1|n))\end{aligned}$$

Theorem DMST Determinant of Matrices of Size Two **221**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Theorem DER Determinant Expansion about Rows **222**

Suppose that A is a square matrix of size n. Then

$$\begin{aligned} \det(A) &= (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2)) \\ &\quad + (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \qquad 1 \leq i \leq n \end{aligned}$$

which is known as **expansion** about row i.

Theorem DT Determinant of the Transpose 223

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Theorem DEC Determinant Expansion about Columns 224

Suppose that A is a square matrix of size n. Then

$$\begin{aligned}\det(A) &= (-1)^{1+j}\left[A\right]_{1j}\det\left(A\left(1|j\right)\right) + (-1)^{2+j}\left[A\right]_{2j}\det\left(A\left(2|j\right)\right)\\ &\quad + (-1)^{3+j}\left[A\right]_{3j}\det\left(A\left(3|j\right)\right) + \cdots + (-1)^{n+j}\left[A\right]_{nj}\det\left(A\left(n|j\right)\right) \qquad 1 \leq j \leq n\end{aligned}$$

which is known as **expansion** about column j.

Theorem DZRC Determinant with Zero Row or Column **225**

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

Theorem DRCS Determinant for Row or Column Swap **226**

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM Determinant for Row or Column Multiples **227**

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $\det(B) = \alpha\det(A)$.

Theorem DERC Determinant with Equal Rows or Columns **228**

Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Theorem DRCMA Determinant for Row or Column Multiples and Addition 229

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

Theorem DIM Determinant of the Identity Matrix 230

For every $n \geq 1$, $\det(I_n) = 1$.

Theorem DEM Determinants of Elementary Matrices **231**

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1. $\det(E_{i,j}) = -1$
2. $\det(E_i(\alpha)) = \alpha$
3. $\det(E_{i,j}(\alpha)) = 1$

Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 232

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det(EA) = \det(E)\det(A)$$

Theorem SMZD Singular Matrices have Zero Determinants 233

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NME7 Nonsingular Matrix Equivalences, Round 7 234

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.

Theorem DRMM Determinant Respects Matrix Multiplication **235**

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

Definition EEM Eigenvalues and Eigenvectors of a Matrix **236**

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in $\mathbb{C}^n$, and λ is a scalar in $\mathbb{C}$. Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

Theorem EMHE Every Matrix Has an Eigenvalue **237**

Suppose A is a square matrix. Then A has at least one eigenvalue.

Definition CP Characteristic Polynomial **238**

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A\left(x\right)$ defined by

$$p_A\left(x\right) = \det\left(A - xI_n\right)$$

Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials **239**

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM Eigenspace of a Matrix **240**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Theorem EMS Eigenspace for a Matrix is a Subspace **241**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $\mathcal{E}_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS Eigenspace of a Matrix is a Null Space **242**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

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Definition AME Algebraic Multiplicity of an Eigenvalue **243**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME Geometric Multiplicity of an Eigenvalue **244**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 245

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_3,\,\ldots,\,\mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1,\,\lambda_2,\,\lambda_3,\,\ldots,\,\lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE Singular Matrices have Zero Eigenvalues 246

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NME8 Nonsingular Matrix Equivalences, Round 8 **247**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix **248**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP Eigenvalues Of Matrix Powers **249**

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Theorem EPM Eigenvalues of the Polynomial of a Matrix **250**

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Theorem EIM Eigenvalues of the Inverse of a Matrix **251**

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM Eigenvalues of the Transpose of a Matrix **252**

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs **253**

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP Degree of the Characteristic Polynomial **254**

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A\left(x\right)$, has degree n.

Theorem NEM Number of Eigenvalues of a Matrix **255**

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A\left(\lambda_i\right) = n$$

Theorem ME Multiplicities of an Eigenvalue **256**

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A\left(\lambda\right) \leq \alpha_A\left(\lambda\right) \leq n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix 257

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Theorem HMRE Hermitian Matrices have Real Eigenvalues 258

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Theorem HMOE Hermitian Matrices have Orthogonal Eigenvectors **259**

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM Similar Matrices **260**

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER Similarity is an Equivalence Relation **261**

Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)
2. If A is similar to B, then B is similar to A. (Symmetric)
3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Theorem SMEE Similar Matrices have Equal Eigenvalues **262**

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $p_A(x) = p_B(x)$.

Definition DIM Diagonal Matrix 263

Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$.

Definition DZM Diagonalizable Matrix 264

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC Diagonalization Characterization

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMFE Diagonalizable Matrices have Full Eigenspaces

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED Distinct Eigenvalues implies Diagonalizable **267**

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Definition LT Linear Transformation **268**

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ Linear Transformations Take Zero to Zero 269

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Theorem MBLT Matrices Build Linear Transformations 270

Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV Matrix of a Linear Transformation, Column Vectors **271**

Suppose that $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations **272**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t$ are vectors from U and $a_1,\,a_2,\,a_3,\,\ldots,\,a_t$ are scalars from $\mathbb{C}$. Then

$$T\left(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t\right) = a_1T\left(\mathbf{u}_1\right) + a_2T\left(\mathbf{u}_2\right) + a_3T\left(\mathbf{u}_3\right) + \cdots + a_tT\left(\mathbf{u}_t\right)$$

Theorem LTDB Linear Transformation Defined on a Basis **273**

Suppose $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for the vector space U and $\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n$ is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation, $T\colon U \mapsto V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i$, $1 \leq i \leq n$.

Definition PI Pre-Image **274**

Suppose that $T\colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{\, \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\}$$

Definition LTA Linear Transformation Addition **275**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT Sum of Linear Transformations is a Linear Transformation **276**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Definition LTSM Linear Transformation Scalar Multiplication 277

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 278

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \mapsto V$ is a linear transformation.

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Theorem VSLT Vector Space of Linear Transformations **279**

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{L}T(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC Linear Transformation Composition **280**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT Composition of Linear Transformations is a Linear Transformation **281**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a linear transformation.

Definition ILT Injective Linear Transformation **282**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition KLT Kernel of a Linear Transformation 283

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{\, \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$$

Theorem KLTS Kernel of a Linear Transformation is a Subspace 284

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Theorem KPI Kernel and Pre-Image **285**

Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{\, \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$

Theorem KILT Kernel of an Injective Linear Transformation **286**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Theorem ILTLI Injective Linear Transformations and Linear Independence **287**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases **288**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

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Theorem ILTD Injective Linear Transformations and Dimension **289**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Theorem CILTI Composition of Injective Linear Transformations is Injective **290**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is an injective linear transformation.

Definition SLT Surjective Linear Transformation **291**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Definition RLT Range of a Linear Transformation **292**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{\, T(\mathbf{u}) \mid \mathbf{u} \in U\}$$

Theorem RLTS Range of a Linear Transformation is a Subspace **293**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT Range of a Surjective Linear Transformation **294**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT Spanning Set for Range of a Linear Transformation **295**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t\}$ spans U. Then

$$R = \{T\left(\mathbf{u}_1\right),\,T\left(\mathbf{u}_2\right),\,T\left(\mathbf{u}_3\right),\,\ldots,\,T\left(\mathbf{u}_t\right)\}$$

spans $\mathcal{R}(T)$.

Theorem RPI Range and Pre-Image **296**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}\left(\mathbf{v}\right) \neq \emptyset$$

Theorem SLTB Surjective Linear Transformations and Bases 297

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1),\,T(\mathbf{u}_2),\,T(\mathbf{u}_3),\,\ldots,\,T(\mathbf{u}_m)\}$ is a spanning set for V.

Theorem SLTD Surjective Linear Transformations and Dimension 298

Suppose that $T\colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective **299**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation **300**

The **identity linear transformation** on the vector space W is defined as

$$I_W\colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

Definition IVLT Invertible Linear Transformations **301**

Suppose that $T\colon U \mapsto V$ is a linear transformation. If there is a function $S\colon V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 302

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}\colon V \mapsto U$ is a linear transformation.

Theorem IILT Inverse of an Invertible Linear Transformation **303**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Theorem ILTIS Invertible Linear Transformations are Injective and Surjective **304**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT Composition of Invertible Linear Transformations 305

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T) : U \mapsto W$ is an invertible linear transformation.

Theorem ICLT Inverse of a Composition of Linear Transformations 306

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS Isomorphic Vector Spaces **307**

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension **308**

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Definition ROLT Rank Of a Linear Transformation **309**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,

$$r(T) = \dim(\mathcal{R}(T))$$

Definition NOLT Nullity Of a Linear Transformation **310**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n(T)$, is the dimension of the kernel of T,

$$n(T) = \dim(\mathcal{K}(T))$$

Theorem ROSLT Rank Of a Surjective Linear Transformation **311**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT Nullity Of an Injective Linear Transformation **312**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Theorem RPNDD Rank Plus Nullity is Domain Dimension **313**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$r\left(T\right) + n\left(T\right) = \dim\left(U\right)$$

Definition VR Vector Representation **314**

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\}$. Define a function $\rho_B\colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B\left(\mathbf{w}\right) \in \mathbb{C}^n$ by

$$\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \cdots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$$

Theorem VRLT Vector Representation is a Linear Transformation **315**

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI Vector Representation is Injective **316**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS Vector Representation is Surjective **317**

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT Vector Representation is an Invertible Linear Transformation 318

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

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Theorem CLI Coordinatization and Linear Independence **321**

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B\left(\mathbf{u}_1\right),\, \rho_B\left(\mathbf{u}_2\right),\, \rho_B\left(\mathbf{u}_3\right),\, \ldots,\, \rho_B\left(\mathbf{u}_k\right)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS Coordinatization and Spanning Sets **322**

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}\rangle$ if and only if $\rho_B\left(\mathbf{u}\right) \in \langle\{\rho_B\left(\mathbf{u}_1\right),\, \rho_B\left(\mathbf{u}_2\right),\, \rho_B\left(\mathbf{u}_3\right),\, \ldots,\, \rho_B\left(\mathbf{u}_k\right)\}\rangle$.

Definition MR Matrix Representation **323**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M^T_{B,C} = \left[\rho_C\left(T\left(\mathbf{u}_1\right)\right)\middle|\,\rho_C\left(T\left(\mathbf{u}_2\right)\right)\middle|\,\rho_C\left(T\left(\mathbf{u}_3\right)\right)\middle|\ldots\middle|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\right]$$

Theorem FTMR Fundamental Theorem of Matrix Representation **324**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 325

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M^{T+S}_{B,C} = M^{T}_{B,C} + M^{S}_{B,C}$$

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 326

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M^{\alpha T}_{B,C} = \alpha M^{T}_{B,C}$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations **327**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M^{S\circ T}_{B,D} = M^{S}_{C,D} M^{T}_{B,C}$$

Theorem KNSI Kernel and Null Space Isomorphism **328**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M^{T}_{B,C}$,

$$\mathcal{K}(T) \cong \mathcal{N}\left(M^{T}_{B,C}\right)$$

Theorem RCSI Range and Column Space Isomorphism **329**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M^{T}_{B,C}$,

$$\mathcal{R}(T) \cong \mathcal{C}\left(M^{T}_{B,C}\right)$$

Theorem IMR Invertible Matrix Representations **330**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M^{T}_{B,C}$ is an invertible matrix, and

$$M^{T^{-1}}_{C,B} = \left(M^{T}_{B,C}\right)^{-1}$$

Theorem IMILT Invertible Matrices, Invertible Linear Transformation **331**

Suppose that A is a square matrix of size n and $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Theorem NME9 Nonsingular Matrix Equivalences, Round 9 **332**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of A.
13. The linear transformation $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Definition EELT Eigenvalue and Eigenvector of a Linear Transformation **333**

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM Change-of-Basis Matrix **334**

Suppose that V is a vector space, and $I_V\colon V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$\begin{aligned}
C_{B,C} &= M^{I_V}_{B,C} \\
&= \left[\rho_C\left(I_V\left(\mathbf{v}_1\right)\right)\middle|\, \rho_C\left(I_V\left(\mathbf{v}_2\right)\right)\middle|\, \rho_C\left(I_V\left(\mathbf{v}_3\right)\right)\middle| \ldots \middle|\rho_C\left(I_V\left(\mathbf{v}_n\right)\right)\right] \\
&= \left[\rho_C\left(\mathbf{v}_1\right)\middle|\, \rho_C\left(\mathbf{v}_2\right)\middle|\, \rho_C\left(\mathbf{v}_3\right)\middle| \ldots \middle|\rho_C\left(\mathbf{v}_n\right)\right]
\end{aligned}$$

Theorem CB Change-of-Basis **335**

Suppose that $\mathbf{v}$ is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C(\mathbf{v}) = C_{B,C}\,\rho_B(\mathbf{v})$$

Theorem ICBM Inverse of Change-of-Basis Matrix **336**

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB Matrix Representation and Change of Basis **337**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M^T_{B,D} = C_{E,D} M^T_{C,E} C_{B,C}$$

Theorem SCB Similarity and Change of Basis **338**

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M^T_{B,B} = C^{-1}_{B,C} M^T_{C,C} C_{B,C}$$

Theorem EER Eigenvalues, Eigenvectors, Representations **339**

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ.

Definition NLT Nilpotent Linear Transformation **340**

Suppose that $T\colon V \mapsto V$ is a linear transformation such that there is an integer $p > 0$ such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.

Definition JB Jordan Block **341**

Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$[J_n(\lambda)]_{ij} = \begin{cases} \lambda & i = j \\ 1 & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem NJB Nilpotent Jordan Blocks **342**

The Jordan block $J_n(0)$ is nilpotent of index n.

Theorem ENLT Eigenvalues of Nilpotent Linear Transformations **343**

Suppose that $T\colon V \mapsto V$ is a nilpotent linear transformation and λ is an eigenvalue of T. Then $\lambda = 0$.

Theorem DNLT Diagonalizable Nilpotent Linear Transformations **344**

Suppose the linear transformation $T\colon V \mapsto V$ is nilpotent. Then T is diagonalizable if and only T is the zero linear transformation.

Theorem KPLT Kernels of Powers of Linear Transformations

Suppose $T\colon V \mapsto V$ is a linear transformation, where $\dim(V) = n$. Then there is an integer m, $0 \leq m \leq n$, such that

$$\{\mathbf{0}\} = \mathcal{K}\left(T^0\right) \subsetneq \mathcal{K}\left(T^1\right) \subsetneq \mathcal{K}\left(T^2\right) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}\left(T^{m+1}\right) = \mathcal{K}\left(T^{m+2}\right) = \cdots$$

Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations

Suppose $T\colon V \mapsto V$ is a nilpotent linear transformation with index p and $\dim(V) = n$. Then $0 \leq p \leq n$ and

$$\{\mathbf{0}\} = \mathcal{K}\left(T^0\right) \subsetneq \mathcal{K}\left(T^1\right) \subsetneq \mathcal{K}\left(T^2\right) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}\left(T^{p+1}\right) = \cdots = V$$

Theorem CFNLT Canonical Form for Nilpotent Linear Transformations **347**

Suppose that $T\colon V \mapsto V$ is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation, $M^T_{B,B}$, is block diagonal with each block being a Jordan block, $J_n(0)$. The size of the largest block is the index p, and the total number of blocks is the nullity of T, $n(T)$.

Definition IS Invariant Subspace **348**

Suppose that $T\colon V \mapsto V$ is a linear transformation and W is a subspace of V. Suppose further that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$. Then W is an **invariant subspace** of V relative to T.

Theorem EIS Eigenspaces are Invariant Subspaces **349**

Suppose that $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ and associated eigenspace $\mathcal{E}_T(\lambda)$. Let W be any subspace of $\mathcal{E}_T(\lambda)$. Then W is an invariant subspace of V relative to T.

Theorem KPIS Kernels of Powers are Invariant Subspaces **350**

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then $\mathcal{K}(T^k)$ is an invariant subspace of V.

Definition GEV Generalized Eigenvector 351

Suppose that $T\colon V \mapsto V$ is a linear transformation. Suppose further that for $\mathbf{x} \neq \mathbf{0}$, $(T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0}$ for some $k > 0$. Then $\mathbf{x}$ is a **generalized eigenvector** of T with eigenvalue λ.

Definition GES Generalized Eigenspace 352

Suppose that $T\colon V \mapsto V$ is a linear transformation. Define the **generalized eigenspace** of T for λ as

$$\mathcal{G}_T(\lambda) = \left\{ \mathbf{x} \mid (T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0} \text{ for some } k \geq 0 \right\}$$

Theorem GESIS Generalized Eigenspace is an Invariant Subspace **353**

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then the generalized eigenspace $\mathcal{G}_T(\lambda)$ is an invariant subspace of V relative to T.

Theorem GEK Generalized Eigenspace as a Kernel **354**

Suppose that $T\colon V \mapsto V$ is a linear transformation, $\dim(V) = n$, and λ is an eigenvalue of T. Then $\mathcal{G}_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n)$.

Definition LTR **Linear Transformation Restriction** **355**

Suppose that $T\colon V \mapsto V$ is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by

$$T|_U\colon U \mapsto U \qquad\qquad T|_U(\mathbf{u}) = T(\mathbf{u})$$

Theorem RGEN **Restriction to Generalized Eigenspace is Nilpotent** **356**

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is nilpotent.

Definition IE Index of an Eigenvalue **357**

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the **index** of λ, $\iota_T(\lambda)$, is the index of the nilpotent linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$.

Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace **358**

Suppose that $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then there is a basis of the the generalized eigenspace $\mathcal{G}_T(\lambda)$ such that the restriction $T|_{\mathcal{G}_T(\lambda)}\colon \mathcal{G}_T(\lambda) \mapsto \mathcal{G}_T(\lambda)$ has a matrix representation that is block diagonal where each block is a Jordan block of the form $J_n(\lambda)$.

Theorem GESD Generalized Eigenspace Decomposition **359**

Suppose that $T(V)\,V$ is a linear transformation with distinct eigenvalues λ_1, λ_2, λ_3, …, λ_m. Then

$$V = \mathcal{G}_T\left(\lambda_1\right) \oplus \mathcal{G}_T\left(\lambda_2\right) \oplus \mathcal{G}_T\left(\lambda_3\right) \oplus \cdots \oplus \mathcal{G}_T\left(\lambda_m\right)$$

Theorem DGES Dimension of Generalized Eigenspaces **360**

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the dimension of the generalized eigenspace for λ is the algebraic multiplicity of λ, $\dim\left(\mathcal{G}_T\left(\lambda_i\right)\right) = \alpha_T\left(\lambda_i\right)$.

Definition JCF   Jordan Canonical Form **361**

A square matrix is in **Jordan canonical form** if it meets the following requirements:

1. The matrix is block diagonal.
2. Each block is a Jordan block.
3. If $\rho < \lambda$ then the block $J_k(\rho)$ occupies rows with indices greater than the indices of the rows occupied by $J_\ell(\lambda)$.
4. If $\rho = \lambda$ and $\ell < k$, then the block $J_\ell(\lambda)$ occupies rows with indices greater than the indices of the rows occupied by $J_k(\lambda)$.

Theorem JCFLT   Jordan Canonical Form for a Linear Transformation **362**

Suppose $T\colon V \mapsto V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

1. The matrix representation is in Jordan canonical form.
2. If $J_k(\lambda)$ is one of the Jordan blocks, then λ is an eigenvalue of T.
3. For a fixed value of λ, the largest block of the form $J_k(\lambda)$ has size equal to the index of λ, $\iota_T(\lambda)$.
4. For a fixed value of λ, the number of blocks of the form $J_k(\lambda)$ is the geometric multiplicity of λ, $\gamma_T(\lambda)$.
5. For a fixed value of λ, the number of rows occupied by blocks of the form $J_k(\lambda)$ is the algebraic multiplicity of λ, $\alpha_T(\lambda)$.

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Theorem CHT Cayley-Hamilton Theorem

Suppose A is a square matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = \mathcal{O}$.