Flash Cards

to accompany

A First Course in Linear Algebra
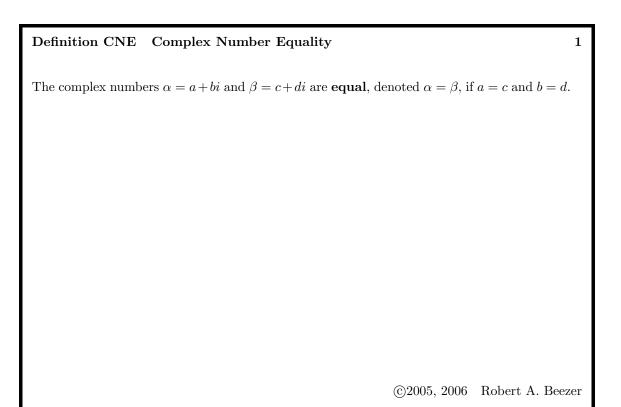
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Version 1.06

Definition CNE   Complex Number Equality

The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are **equal**, denoted $\alpha = \beta$, if $a = c$ and $b = d$.

Definition CNA   Complex Number Addition

The **sum** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + d)i$.

Definition CNM   Complex Number Multiplication   **3**

The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is $(ac - bd) + (ad + bc)i$.

Theorem PCNA   Properties of Complex Number Arithmetic   **4**
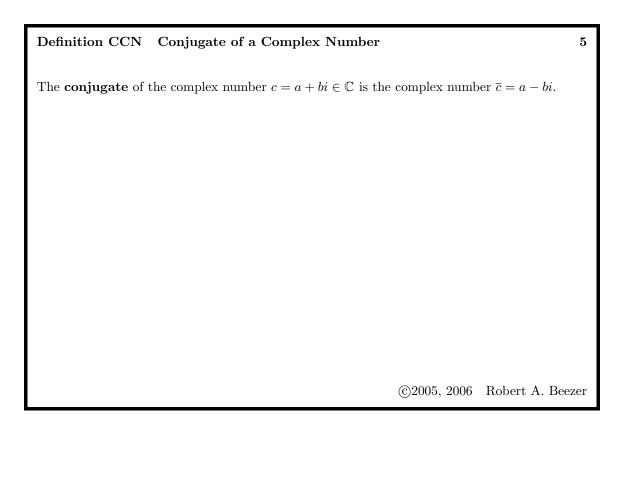
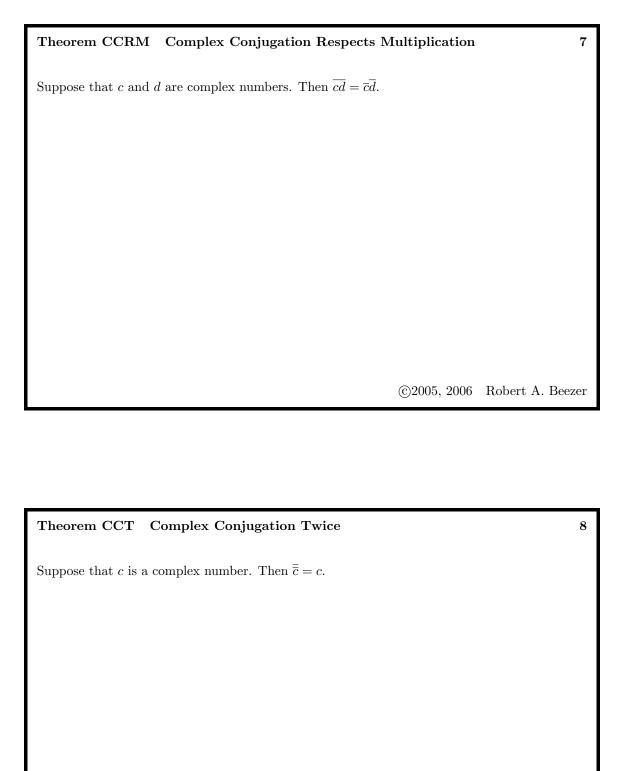The operations of addition and multiplication of complex numbers have the following properties.

- **ACCN   Additive Commutativity, Complex Numbers** For any α, $\beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.

- **MCCN   Multiplicative Commutativity, Complex Numbers** For any α, $\beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.

- **AACN   Additive Associativity, Complex Numbers** For any α, β, $\gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

- **MACN   Multiplicative Associativity, Complex Numbers** For any α, β, $\gamma \in \mathbb{C}$, $\alpha\left(\beta\gamma\right) = \left(\alpha\beta\right)\gamma$.

- **DCN   Distributivity, Complex Numbers** For any α, β, $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

- **ZCN   Zero, Complex Numbers** There is a complex number $0 = 0 + 0i$ so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.

- **OCN   One, Complex Numbers** There is a complex number $1 = 1 + 0i$ so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.

- **AICN   Additive Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.

- **MICN   Multiplicative Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

Suppose that c and d are complex numbers. Then $\overline{c + d} = \overline{c} + \overline{d}$.

Theorem CCRM   Complex Conjugation Respects Multiplication **7**

Suppose that c and d are complex numbers. Then $\overline{cd} = \overline{c}\,\overline{d}$.

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Theorem CCT   Complex Conjugation Twice **8**

Suppose that c is a complex number. Then $\overline{\overline{c}} = c$.

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The **modulus** of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|c| = \sqrt{c\bar{c}} = \sqrt{a^2 + b^2}.$$

A **set** is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its **elements**.

$$|c| = \sqrt{c\bar{c}} = \sqrt{a^2 + b^2}.$$

Definition SSET   Subset   **11**

If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Definition ES   Empty Set   **12**

The empty set is the set with no elements. Its is denoted by $\emptyset$.

Definition SE   Set Equality 13

Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write $S = T$.

Definition C   Cardinality 14

Suppose S is a finite set. Then the number of elements in S is called the **cardinality** or **size** of S, and is denoted $|S|$.

Definition SU   Set Union

Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

Definition SI   Set Intersection

Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted $\overline{S}$, is the set whose elements are those that are elements of U and not elements of S. More formally,

$$x \in \overline{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

A **system of linear equations** is a collection of m equations in the variable quantities $x_1,\ x_2,\ x_3, \ldots, x_n$ of the form,

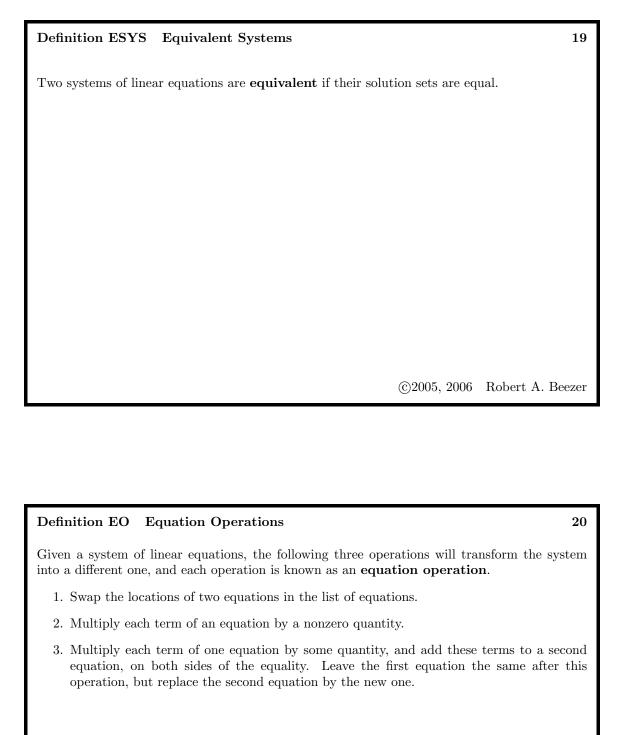$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
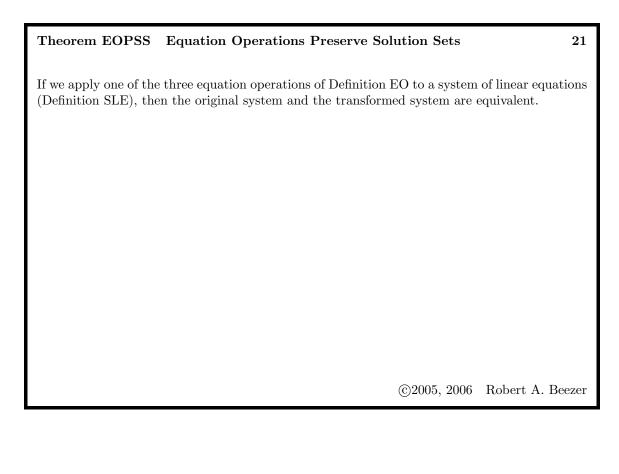$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Two systems of linear equations are **equivalent** if their solution sets are equal.

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. Swap the locations of two equations in the list of equations.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet $(A, B, C, \dots)$ to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Definition CV   Column Vector

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\underset{\sim}{u}$. To refer to the **entry** or **component** that is number i in the list that is the vector $\mathbf{v}$ we write $[\mathbf{v}]_i$.

Definition ZCV   Zero Column Vector

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \leq i \leq m$.

Definition CM Coefficient Matrix

For a system of linear equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

the **coefficient matrix** is the $m \times n$ matrix

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\
\vdots & & & & \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{bmatrix}
$$

Definition VOC Vector of Constants

For a system of linear equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

the **vector of constants** is the column vector of size m

$$
\mathbf{b} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_m
\end{bmatrix}
$$

Definition SV Solution Vector

For a system of linear equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

the **solution vector** is the column vector of size n

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}
$$

Definition LSMR   Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and $\mathbf{b}$ is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants $\mathbf{b}$. Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (number $n+1$) is the column vector $\mathbf{b}$. This matrix will be written as $[\,A\,|\,\mathbf{b}\,]$.
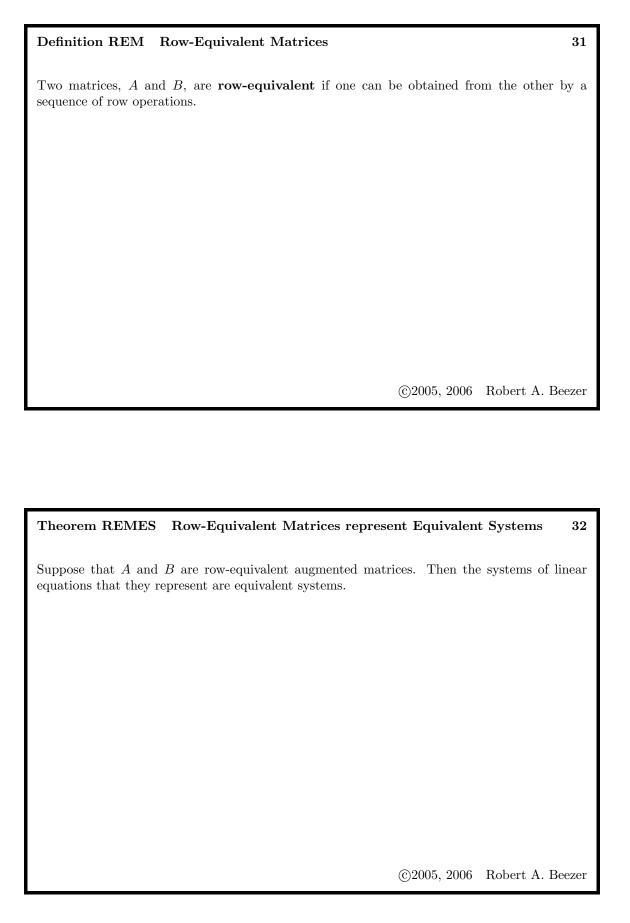
The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.

2. Multiply each entry of a single row by a nonzero quantity.

3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$:   Swap the location of rows i and j.

2. αR_i: Multiply row i by the nonzero scalar α.

3. $\alpha R_i + R_j$:   Multiply row i by the scalar α and add to row j.

Definition REM   Row-Equivalent Matrices   **31**

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Theorem REMES   Row-Equivalent Matrices represent Equivalent Systems   32

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

Definition RREF   Reduced Row-Echelon Form   **33**

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.
A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot colums will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

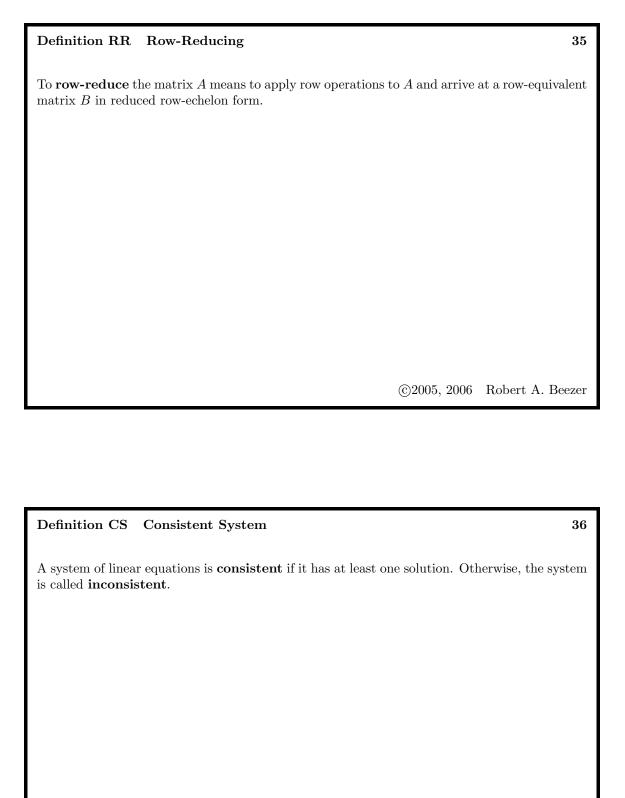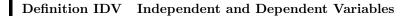**Theorem REMEF   Row-Equivalent Matrix in Echelon Form**   **34**

Suppose A is a matrix. Then there is a matrix B so that

1. A and B are row-equivalent.

2. B is in reduced row-echelon form.

Definition RR   Row-Reducing To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

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Definition CS   Consistent System A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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Definition IDV   Independent and Dependent Variables   **37**
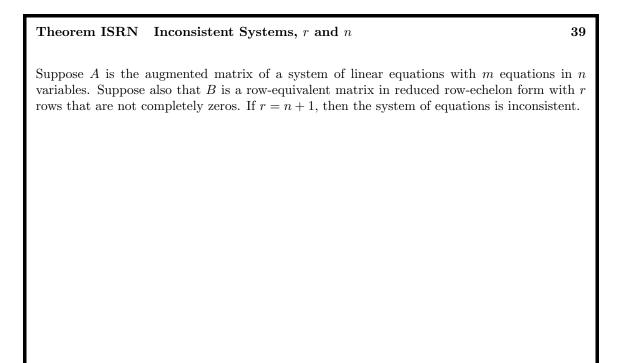
Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem RCLS   Recognizing Consistency of a Linear System   **38**

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B.
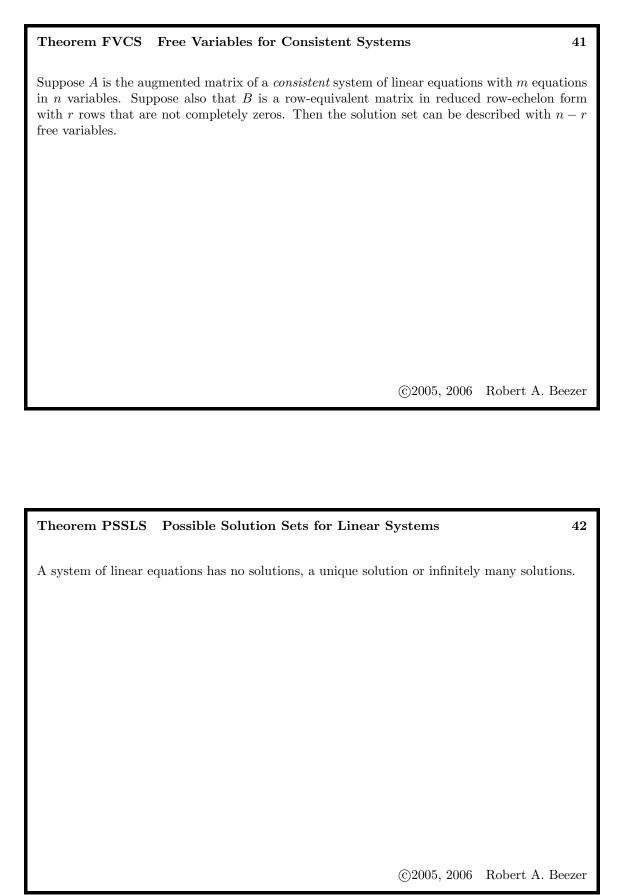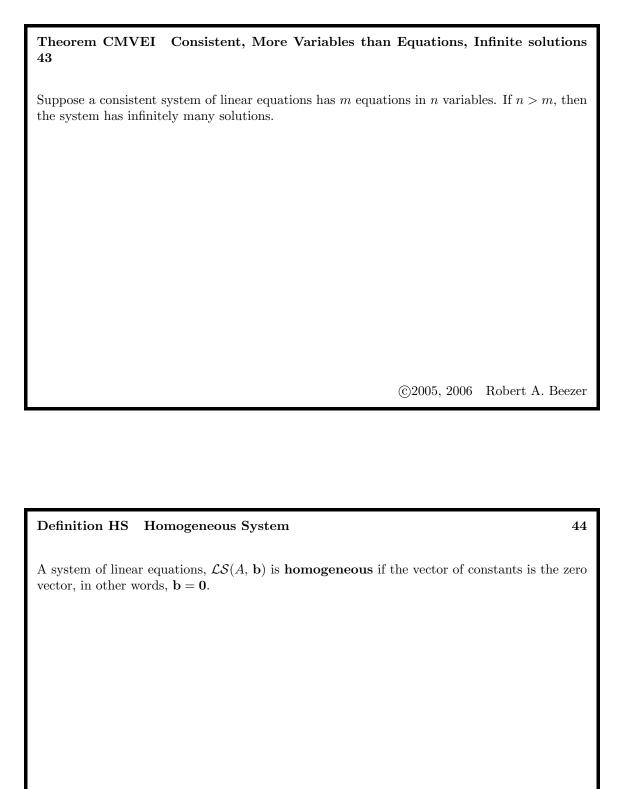
Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.
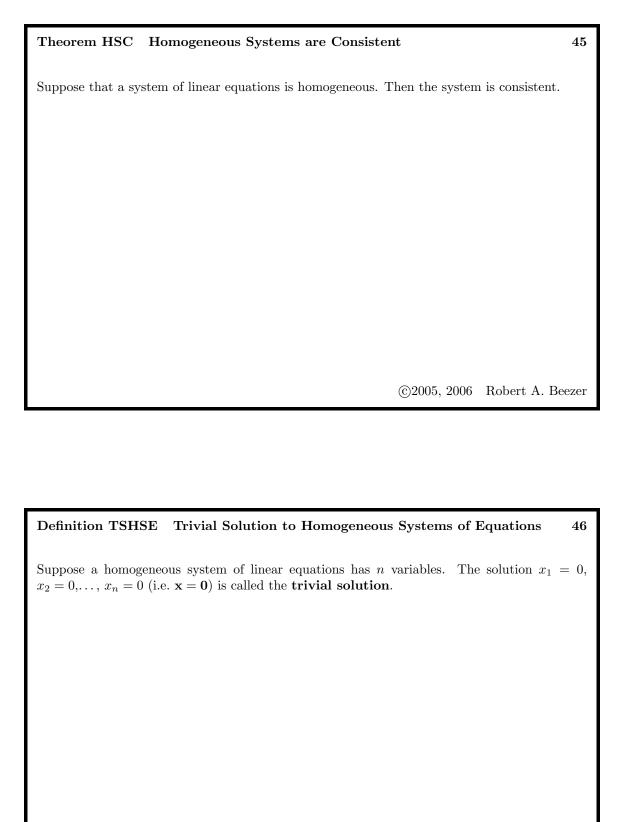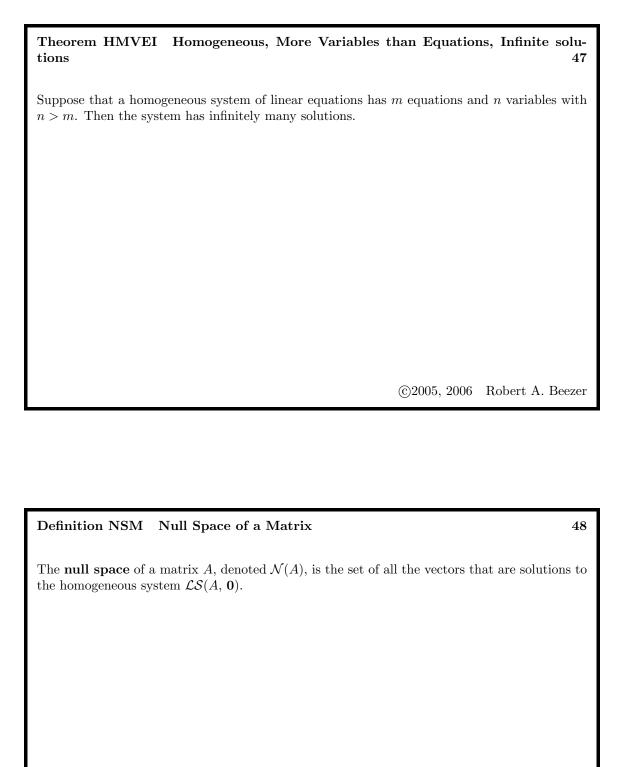
Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI   Consistent, More Variables than Equations, Infinite solutions 43

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Definition HS   Homogeneous System 44

A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

Theorem HSC   Homogeneous Systems are Consistent

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE   Trivial Solution to Homogeneous Systems of Equations

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition SQM   Square Matrix   **49**

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM   Nonsingular Matrix   **50**

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS   Nonsingular Matrices have Trivial Null Spaces   **53**

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Theorem NMUS   Nonsingular Matrices and Unique Solutions   **54**

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$. Then $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the **scalar multiple** of $\mathbf{u}$ by α is the vector $\alpha\mathbf{u}$ defined by

$$[\alpha\mathbf{u}]_i = \alpha\,[\mathbf{u}]_i \qquad\qquad 1 \le i \le m$$

Theorem VSPCV   Vector Space Properties of Column Vectors   **60**

Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC   Additive Closure, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- **SCC   Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.

- **CC   Commutativity, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AAC   Additive Associativity, Column Vectors** If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **ZC   Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.

- **AIC   Additive Inverses, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMAC   Scalar Multiplication Associativity, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVAC   Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSAC   Distributivity across Scalar Addition, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **OC   One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Definition LCCV   Linear Combination of Column Vectors   **61**

Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars α_1, α_2, α_3, $\ldots$, α_n, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Theorem SLSLC   Solutions to Linear Systems are Linear Combinations   **62**

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\ldots$, $\mathbf{A}_n$. Then $\mathbf{x}$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS   Vector Form of Solutions to Linear Systems 63

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}$, $\mathbf{u}_j$, $1 \le j \le n-r$ of size n by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, \ i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \ne f_j \ . \\ -[B]_{k,f_j} & \text{if } i \in D, \ i = d_k \end{cases}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r} \ \middle| \ x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \right\}$$
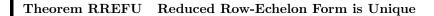
Theorem PSPHS   Particular Solution Plus Homogeneous Solutions 64

Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{\, \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_p \mathbf{u}_p \,|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p \,\}$$

$$= \left\{\, \sum_{i=1}^{p} \alpha_i \mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p \,\right\}$$

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -[B]_{k, f_j} & \text{if } i \in D, \ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\} \rangle.$$

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, a true statement of the form
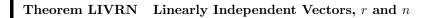
$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is the **trivial relation of linear dependence** on S.

The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.
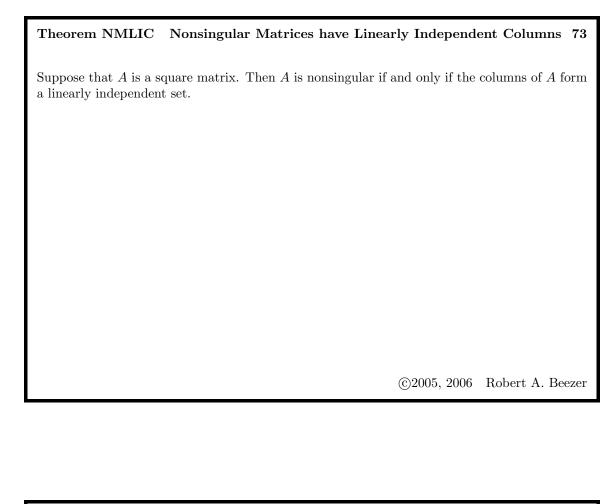
Theorem LIVRN   Linearly Independent Vectors, r and n   **71**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Theorem MVSLD   More Vectors than Size implies Linear Dependence   **72**

Suppose that $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Theorem NMLIC   Nonsingular Matrices have Linearly Independent Columns 73

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NME2   Nonsingular Matrix Equivalences, Round 2   74

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A form a linearly independent set.

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ and $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \le j \le n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \ne f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1,\, \mathbf{z}_2,\, \mathbf{z}_3,\, \ldots,\, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \le t \le n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_{t-1},\, \mathbf{u}_{t+1},\, \ldots,\, \mathbf{u}_n$.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \ldots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots \mathbf{v}_{d_r}\}$ is a linearly independent set.
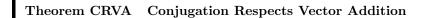
2. $W = \langle T \rangle$.

Suppose that $\mathbf{u}$ is a vector from $\mathbb{C}^m$. Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \qquad\qquad 1 \le i \le m$$

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Given the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\, \mathbf{v} \rangle = [\mathbf{u}]_1\, \overline{[\mathbf{v}]_1} + [\mathbf{u}]_2\, \overline{[\mathbf{v}]_2} + [\mathbf{u}]_3\, \overline{[\mathbf{v}]_3} + \cdots + [\mathbf{u}]_m\, \overline{[\mathbf{v}]_m} = \sum_{i=1}^{m} [\mathbf{u}]_i\, \overline{[\mathbf{v}]_i}$$

Suppose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^m$. Then

1. $\langle \mathbf{u} + \mathbf{v},\, \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{w} \rangle + \langle \mathbf{v},\, \mathbf{w} \rangle$
2. $\langle \mathbf{u},\, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle + \langle \mathbf{u},\, \mathbf{w} \rangle$

Theorem IPSM   Inner Product and Scalar Multiplication   **83**

Suppose $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

$$
\begin{aligned}
&1. \quad \langle \alpha\mathbf{u},\, \mathbf{v} \rangle = \alpha \langle \mathbf{u},\, \mathbf{v} \rangle \\
&2. \quad \langle \mathbf{u},\, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u},\, \mathbf{v} \rangle
\end{aligned}
$$

Theorem IPAC   Inner Product is Anti-Commutative   **84**

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{v} \rangle = \overline{\langle \mathbf{v},\, \mathbf{u} \rangle}$.

Definition NV   Norm of a Vector

The **norm** of the vector **u** is the scalar quantity in $\mathbb{C}$
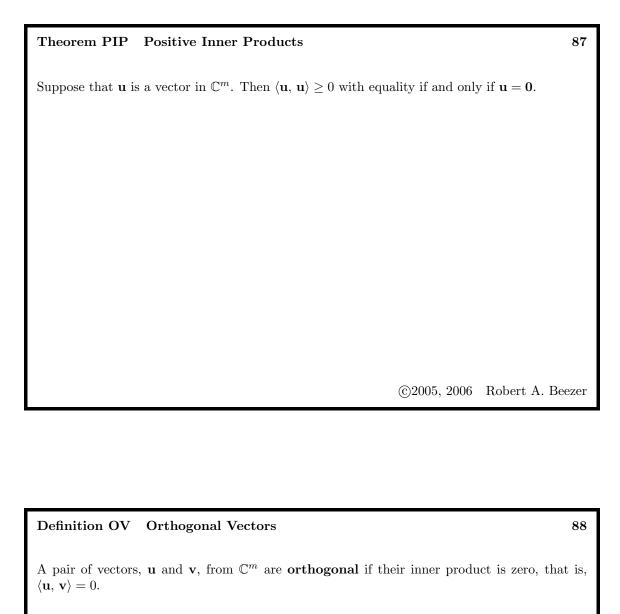
$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \cdots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^{m} |[\mathbf{u}]_i|^2}$$

Theorem IPN   Inner Products and Norms

Suppose that **u** is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u},\, \mathbf{u} \rangle$.

Theorem PIP   Positive Inner Products

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV   Orthogonal Vectors

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u},\, \mathbf{v} \rangle = 0$.

Definition OSV   Orthogonal Set of Vectors

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Theorem OSLI   Orthogonal Sets are Linearly Independent

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3,\, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1},\, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

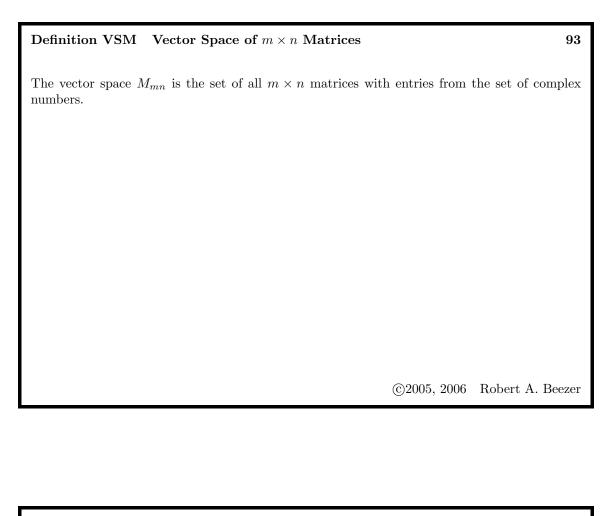Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an **orthonormal** set of vectors.

Definition VSM   Vector Space of $m \times n$ Matrices   **93**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME   Matrix Equality   **94**

The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition MA   Matrix Addition 95

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \qquad\qquad 1 \le i \le m,\, 1 \le j \le n$$

Definition MSM   Matrix Scalar Multiplication 96

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha\, [A]_{ij} \qquad\qquad 1 \le i \le m,\, 1 \le j \le n$$

Theorem VSPM   Vector Space Properties of Matrices   **97**

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then
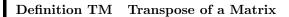
- **ACM   Additive Closure, Matrices** If A, $B \in M_{mn}$, then $A + B \in M_{mn}$.

- **SCM   Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

- **CM   Commutativity, Matrices** If A, $B \in M_{mn}$, then $A + B = B + A$.

- **AAM   Additive Associativity, Matrices** If A, B, $C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

- **ZM   Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.

- **AIM   Additive Inverses, Matrices** If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.

- **SMAM   Scalar Multiplication Associativity, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.

- **DMAM   Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and A, $B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

- **DSAM   Distributivity across Scalar Addition, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

- **OM   One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM   Zero Matrix   **98**

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM   Transpose of a Matrix   99

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \le i \le n,\, 1 \le j \le m.$$

Definition SYM   Symmetric Matrix   100

The matrix A is **symmetric** if $A = A^t$.

$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \le i \le n,\, 1 \le j \le m.$$

Theorem SMS   Symmetric Matrices are Square   101

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA   Transpose and Matrix Addition   102

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Theorem TMSM   Transpose and Matrix Scalar Multiplication   **103**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Theorem TT   Transpose of a Transpose   **104**

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Definition CCM   Complex Conjugate of a Matrix

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

Theorem CRMA   Conjugation Respects Matrix Addition

Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Theorem CRMSM   Conjugation Respects Matrix Scalar Multiplication

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Theorem CCM   Conjugate of the Conjugate of a Matrix

Suppose that A is an $m \times n$ matrix. Then $\overline{\left(\overline{A}\right)} = A$.

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.

If A is a square matrix, then its **adjoint** is $A^* = \left(\overline{A}\right)^t$.

Theorem AMA   Adjoint and Matrix Addition   **111**

Suppose A and B are matrices of the same size. Then $(A + B)^* = A^* + B^*$.

Theorem AMSM   Adjoint and Matrix Scalar Multiplication   **112**

Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$.

Theorem AA   Adjoint of an Adjoint   **113**

Suppose that A is a matrix. Then $(A^*)^* = A$

Definition MVP   Matrix-Vector Product   **114**

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \, \mathbf{A}_1 + [\mathbf{u}]_2 \, \mathbf{A}_2 + [\mathbf{u}]_3 \, \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \, \mathbf{A}_n$$

Theorem SLEMM   Systems of Linear Equations as Matrix Multiplication   115

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Theorem EMMVP   Equal Matrices and Matrix-Vector Products   116

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Definition MM   Matrix Multiplication 117

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, ..., $\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Theorem EMP   Entries of Matrix Products 118

Suppose A is an $m \times n$ matrix and $B =$ is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1}[B]_{1j} + [A]_{i2}[B]_{2j} + [A]_{i3}[B]_{3j} + \cdots + [A]_{in}[B]_{nj}$$
$$= \sum_{k=1}^{n}[A]_{ik}[B]_{kj}$$

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_mA = A$

Theorem MMDAA   Matrix Multiplication Distributes Across Addition   121

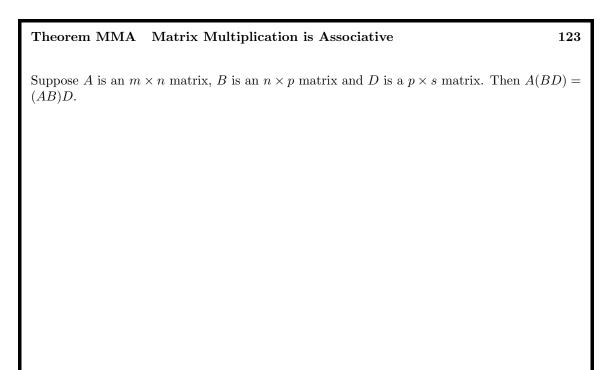Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Theorem MMSMM   Matrix Multiplication and Scalar Matrix Multiplication 122

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA   Matrix Multiplication is Associative   **123**

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.
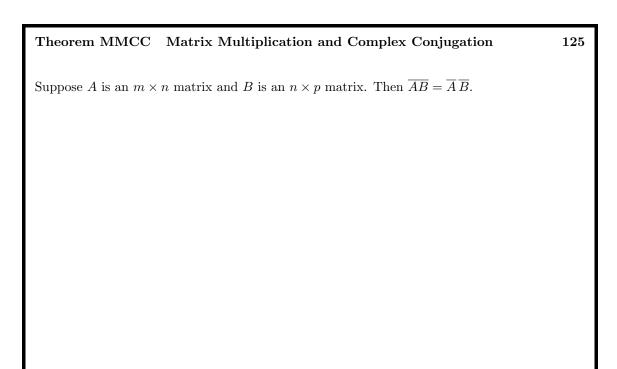
Theorem MMIP   Matrix Multiplication and Inner Products   **124**

If we consider the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Theorem MMCC   Matrix Multiplication and Complex Conjugation   **125**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.
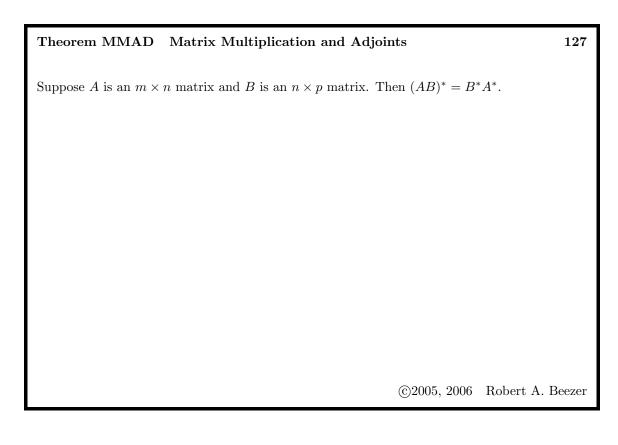
Theorem MMT   Matrix Multiplication and Transposes   **126**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Theorem MMAD   Matrix Multiplication and Adjoints   **127**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^* = B^*A^*$.

Theorem AIP   Adjoint and Inner Product   **128**

Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$.

Definition HM   Hermitian Matrix   **129**

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = A^*$.

Theorem HMIP   Hermitian Matrices and Inner Products   **130**

Suppose that A is a square matrix of size n. Then A is Hermitian if and only if $\langle A\mathbf{x},\ \mathbf{y}\rangle = \langle \mathbf{x},\ A\mathbf{y}\rangle$ for all $\mathbf{x},\ \mathbf{y} \in \mathbb{C}^n$.

Definition MI   Matrix Inverse   **131**

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is
invertible and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Definition SUV   Standard Unit Vectors   **132**

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m. Then
the set
$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\, \mathbf{e}_j \mid 1 \le j \le m \,\}$$
is the set of **standard unit vectors** in $\mathbb{C}^m$.

Theorem TTMI   Two-by-Two Matrix Inverse   **133**

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have
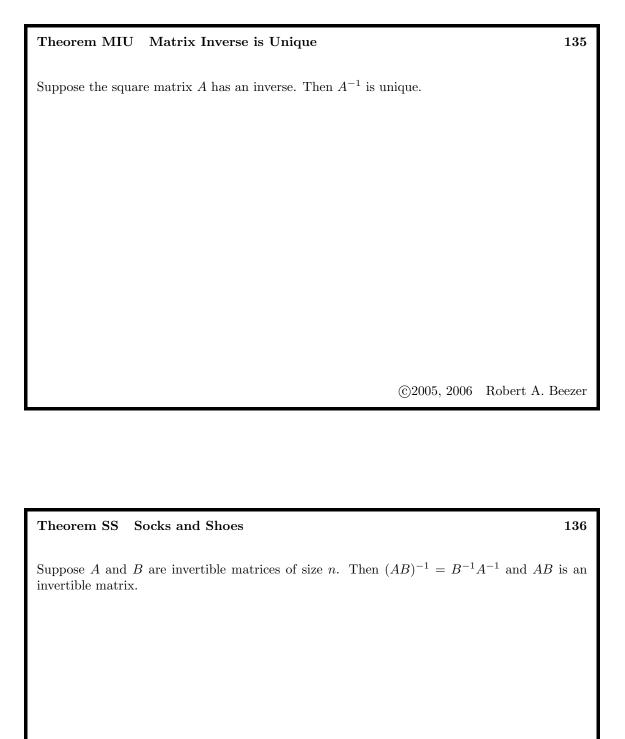
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem CINM   Computing the Inverse of a Nonsingular Matrix   **134**

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Theorem MIU   Matrix Inverse is Unique

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS   Socks and Shoes

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI   Matrix Inverse of a Matrix Inverse   137

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

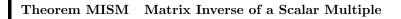**Theorem MIT   Matrix Inverse of a Transpose   138**

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Theorem MISM   Matrix Inverse of a Scalar Multiple   **139**
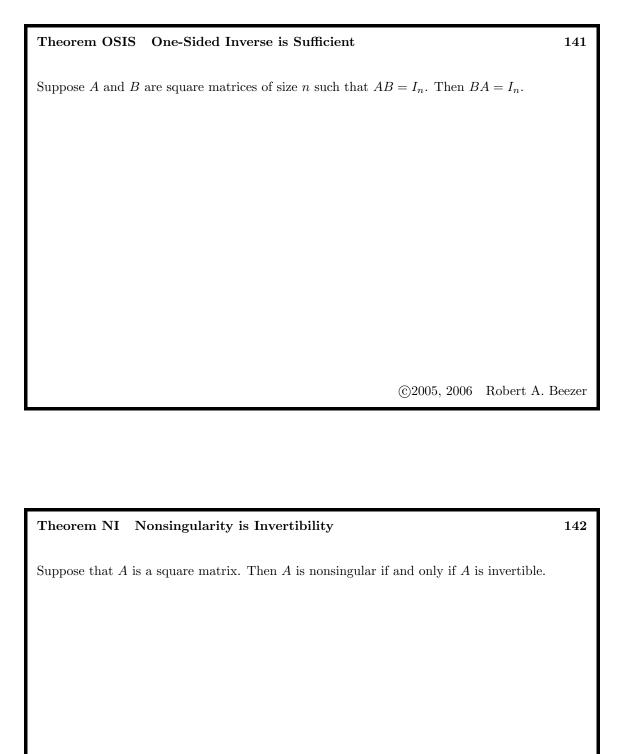
Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and αA is invertible.

Theorem NPNT   Nonsingular Product has Nonsingular Terms   **140**

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NME3   Nonsingular Matrix Equivalences, Round 3   **143**

Suppose that A is a square matrix of size n. The following are equivalent.
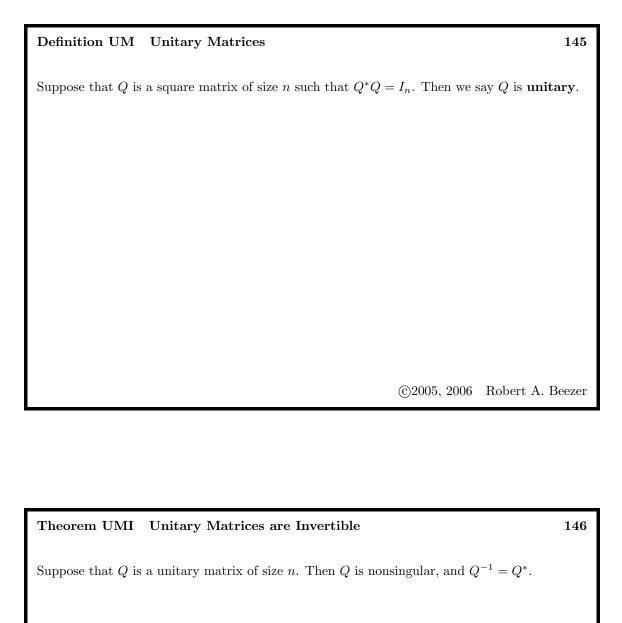
1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.
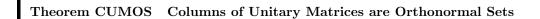
6. A is invertible.

Theorem SNCM   Solution with Nonsingular Coefficient Matrix   **144**

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Definition UM   Unitary Matrices   **145**

Suppose that Q is a square matrix of size n such that $Q^*Q = I_n$. Then we say Q is **unitary**.

Theorem UMI   Unitary Matrices are Invertible   **146**

Suppose that Q is a unitary matrix of size n. Then Q is nonsingular, and $Q^{-1} = Q^*$.

Theorem CUMOS   Columns of Unitary Matrices are Orthonormal Sets   147

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.
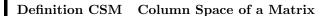
Theorem UMPIP   Unitary Matrices Preserve Inner Products   148

Suppose that Q is a unitary matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\, Q\mathbf{v} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle \qquad \text{and} \qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|$$

Definition CSM   Column Space of a Matrix   **149**

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\} \rangle$$

Theorem CSCS   Column Spaces and Consistent Systems   **150**

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.

2. $\mathcal{C}(A) = \langle T \rangle$.

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Theorem NME4   Nonsingular Matrix Equivalences, Round 4   **153**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
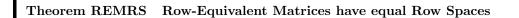

Definition RSM   Row Space of a Matrix   **154**

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Theorem REMRS   Row-Equivalent Matrices have equal Row Spaces   **155**

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Theorem BRS   Basis for the Row Space   **156**

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m - r) \times m$ matrix formed from the bottom $m - r$ rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array}\right]$$

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. J is nonsingular.

2. $B = JA$.

3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.

4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.

5. L is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m - r$ rows. Then

1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.

2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.

3. The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.

4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

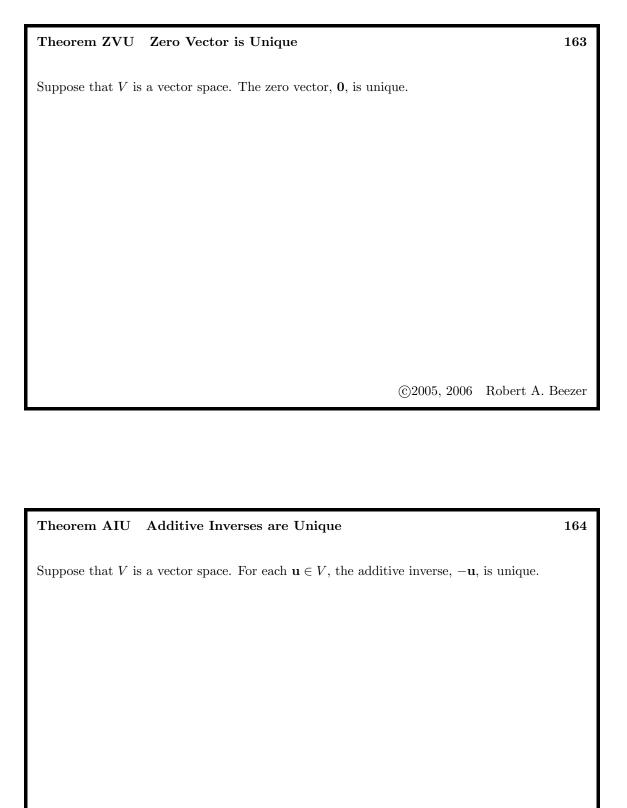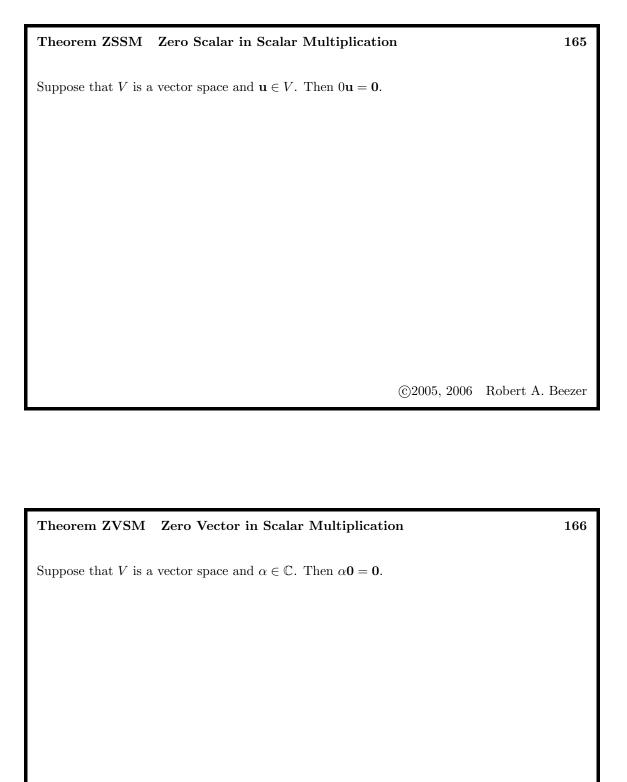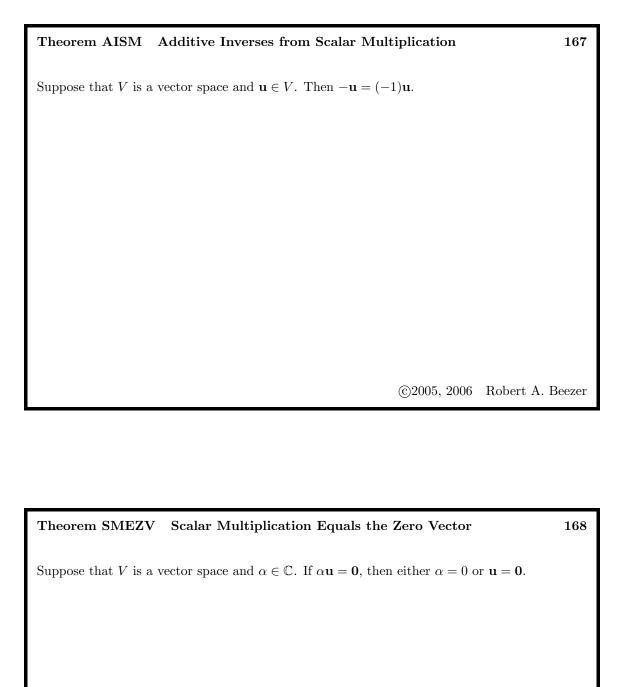
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.
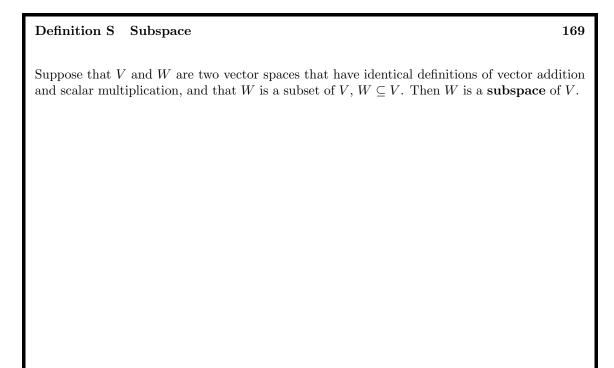
- **AC   Additive Closure** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

- **SC   Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

- **C   Commutativity** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AA   Additive Associativity** If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Z   Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

- **AI   Additive Inverses** If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMA   Scalar Multiplication Associativity** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVA   Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSA   Distributivity across Scalar Addition** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **O   One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU   Zero Vector is Unique   **163**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU   Additive Inverses are Unique   **164**

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM   Additive Inverses from Scalar Multiplication   **167**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Theorem SMEZV   Scalar Multiplication Equals the Zero Vector   **168**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Definition S   Subspace   **169**

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS   Testing Subsets for Subspaces   **170**

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.

2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.

3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Definition TS   Trivial Subspaces   **171**

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS   Null Space of a Matrix is a Subspace   **172**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC   Linear Combination   **173**

Suppose that V is a vector space.   Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_n$ and n scalars α_1, α_2, α_3, ..., α_n, their **linear combination** is the vector

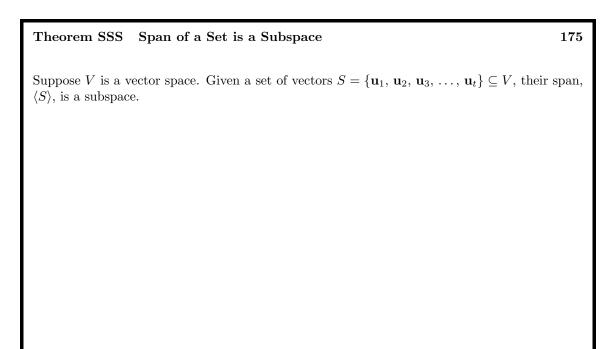$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Definition SS   Span of a Set   **174**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{\, \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\ 1 \le i \le t \right\}$$
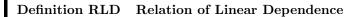
Theorem SSS   Span of a Set is a Subspace   **175**

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Theorem CSMS   Column Space of a Matrix is a Subspace   **176**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Theorem RSMS   Row Space of a Matrix is a Subspace   **177**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Theorem LNSMS   Left Null Space of a Matrix is a Subspace   **178**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$.

Definition RLD   Relation of Linear Dependence 179

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$
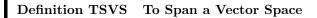
is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LI   Linear Independence 180

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let $\mathbf{w}$ be any vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Definition B   Basis   **183**

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem SUVB   Standard Unit Vectors are a Basis   **184**

The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Suppose that A is a square matrix of size m. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
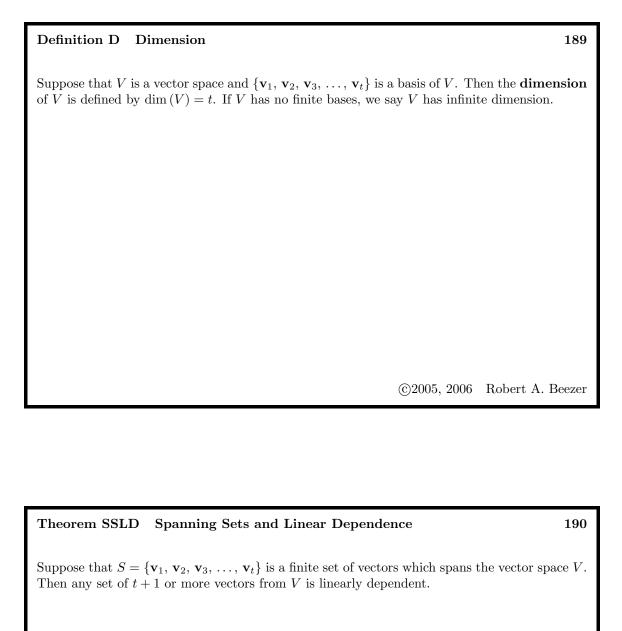
8. The columns of A are a basis for $\mathbb{C}^n$.

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Suppose that U is a unitary matrix of sixe n and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\}$ is an orthonormal basis of $\mathbb{C}^n$. Then the set $C = \{U\mathbf{x}_1, U\mathbf{x}_2, U\mathbf{x}_3, \ldots, U\mathbf{x}_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

Definition D   Dimension   **189**

Suppose that V is a vector space and $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD   Spanning Sets and Linear Dependence   **190**

Suppose that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.
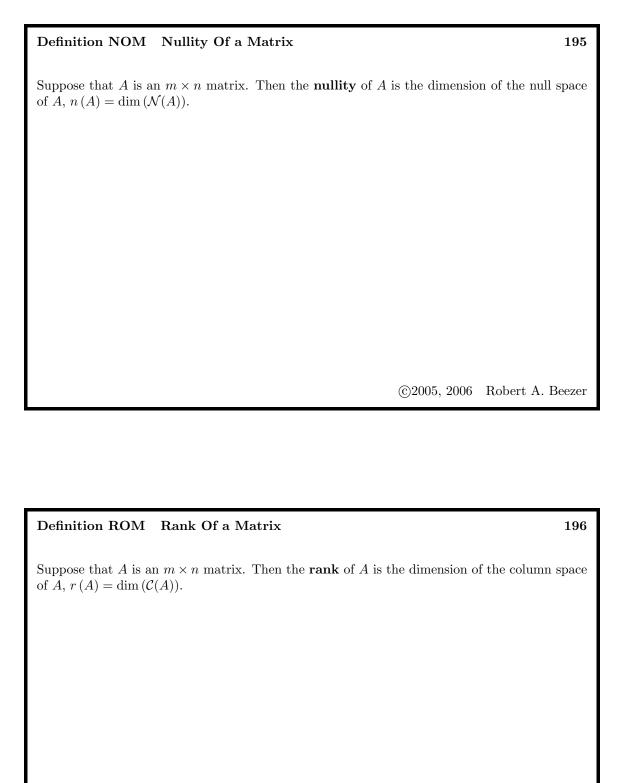
Theorem BIS   Bases have Identical Sizes   191

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM   Dimension of $\mathbb{C}^m$ 192

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP   Dimension of P_n   **193**

The dimension of P_n (Example VSP) is $n + 1$.

Theorem DM   Dimension of M_{mn}   **194**

The dimension of M_{mn} (Example VSM) is mn.

Definition NOM   Nullity Of a Matrix   **195**

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM   Rank Of a Matrix   **196**

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Theorem CRN   Computing Rank and Nullity   **197**

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Theorem RPNC   Rank Plus Nullity is Columns   **198**

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. The rank of A is n, $r(A) = n$.
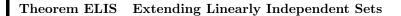
3. The nullity of A is zero, $n(A) = 0$.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

Theorem ELIS   Extending Linearly Independent Sets   **201**

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.
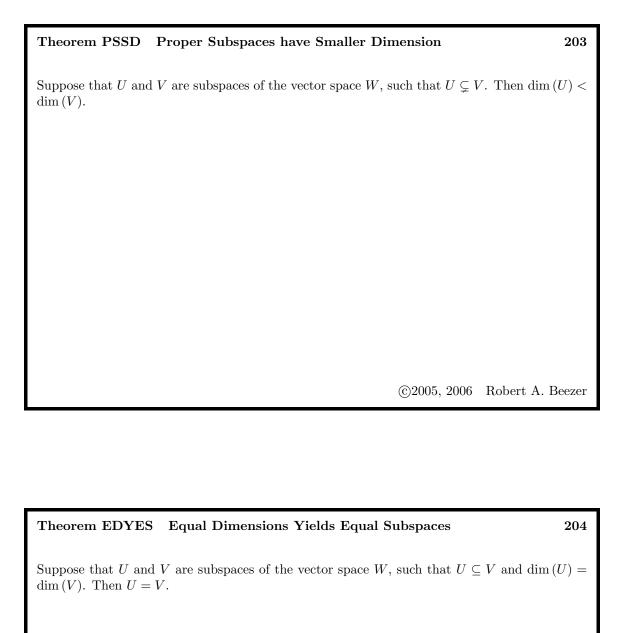
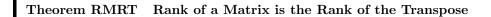---

Theorem G   Goldilocks   **202**

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.

2. If $m < t$, then S does not span V.

3. If $m = t$ and S is linearly independent, then S spans V.

4. If $m = t$ and S spans V, then S is linearly independent.

Theorem PSSD   Proper Subspaces have Smaller Dimension   **203**

Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.

Theorem EDYES   Equal Dimensions Yields Equal Subspaces   **204**

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Suppose A is an $m \times n$ matrix. Then $r\left(A\right) = r\left(A^t\right)$.

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim\left(\mathcal{N}(A)\right) = n - r$

2. $\dim\left(\mathcal{C}(A)\right) = r$

3. $\dim\left(\mathcal{R}(A)\right) = r$

4. $\dim\left(\mathcal{L}(A)\right) = m - r$

Definition DS   Direct Sum   **207**

Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$

2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U$, $\mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

Then V is the **direct sum** of U and W and we write $V = U \oplus W$.

Theorem DSFB   Direct Sum From a Basis   **208**

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define

$$U = \langle \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}\rangle \qquad W = \langle \{\mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \mathbf{v}_{m+3}, \ldots, \mathbf{v}_n\}\rangle$$

Then $V = U \oplus W$.

Theorem DSFOS   Direct Sum From One Subspace   **209**

Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.

Theorem DSZV   Direct Sums and Zero Vectors   **210**

Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

2. Whenever $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$, $\mathbf{w} \in W$ then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

Theorem DSZI   Direct Sums and Zero Intersection   **211**

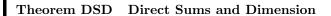Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

2. $U \cap W = \{\mathbf{0}\}$.

Theorem DSLI   Direct Sums and Linear Independence   **212**

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose that R is a linearly independent subset of U and S is a linearly independent subset of W. Then $R \cup S$ is a linearly independent subset of V.

Theorem DSD   Direct Sums and Dimension   **213**

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then $\dim(V) = \dim(U) + \dim(W)$.

Theorem RDS   Repeated Direct Sums   **214**

Suppose V is a vector space with subspaces U and W with $V = U \oplus W$. Suppose that X and Y are subspaces of W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

Definition ELEM   Elementary Matrices 215

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$
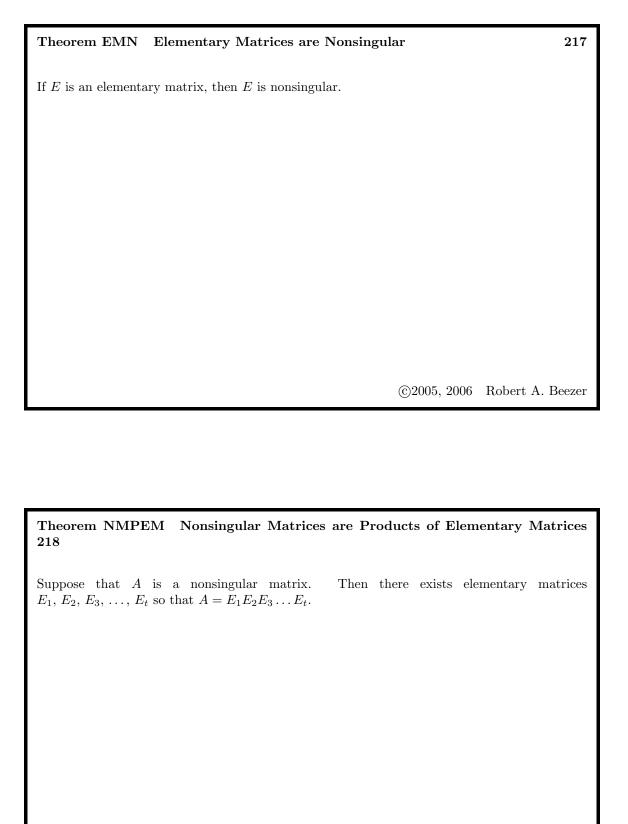
Theorem EMDRO   Elementary Matrices Do Row Operations 216

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.

2. If the row operation multiplies row i by α, then $B = E_i(\alpha) A$.

3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.

Theorem EMN   Elementary Matrices are Nonsingular   **217**

If E is an elementary matrix, then E is nonsingular.

Theorem NMPEM   Nonsingular Matrices are Products of Elementary Matrices 218

Suppose that A is a nonsingular matrix.   Then there exists elementary matrices E_1, E_2, E_3, ..., E_t so that $A = E_1 E_2 E_3 \ldots E_t$.

Definition SM   SubMatrix   219

Suppose that A is an $m \times n$ matrix. Then the **submatrix** $A\,(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM   Determinant of a Matrix   220

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11}\det(A\,(1|1)) - [A]_{12}\det(A\,(1|2)) + [A]_{13}\det(A\,(1|3)) - $$
$$[A]_{14}\det(A\,(1|4)) + \cdots + (-1)^{n+1}[A]_{1n}\det(A\,(1|n))$$

Theorem DMST   Determinant of Matrices of Size Two   **221**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det (A) = ad - bc$

Theorem DER   Determinant Expansion about Rows   **222**
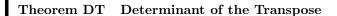
Suppose that A is a square matrix of size n. Then

$$\det (A) = (-1)^{i+1} [A]_{i1} \det (A (i|1)) + (-1)^{i+2} [A]_{i2} \det (A (i|2))$$
$$+ (-1)^{i+3} [A]_{i3} \det (A (i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det (A (i|n)) \qquad 1 \le i \le n$$

which is known as **expansion** about row i.

Theorem DT   Determinant of the Transpose   **223**

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

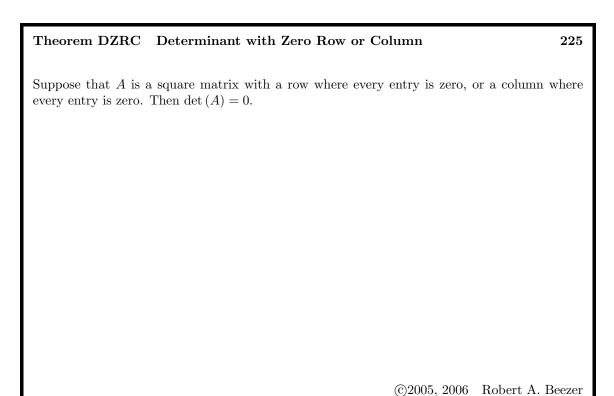**Theorem DEC   Determinant Expansion about Columns**   **224**

Suppose that A is a square matrix of size n. Then

$$\det(A) = (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j))$$
$$+ (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j)) \qquad 1 \le j \le n$$
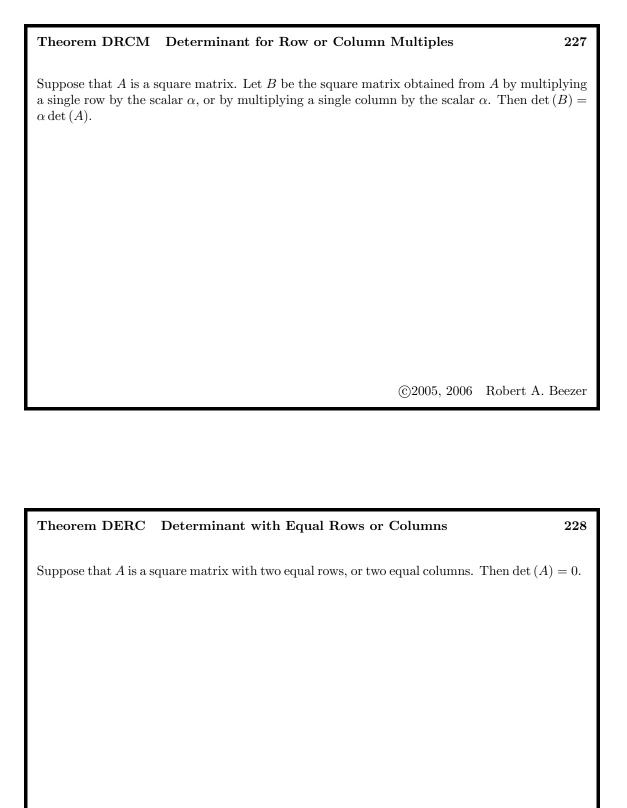
which is known as **expansion** about column j.

Theorem DZRC   Determinant with Zero Row or Column   225

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.
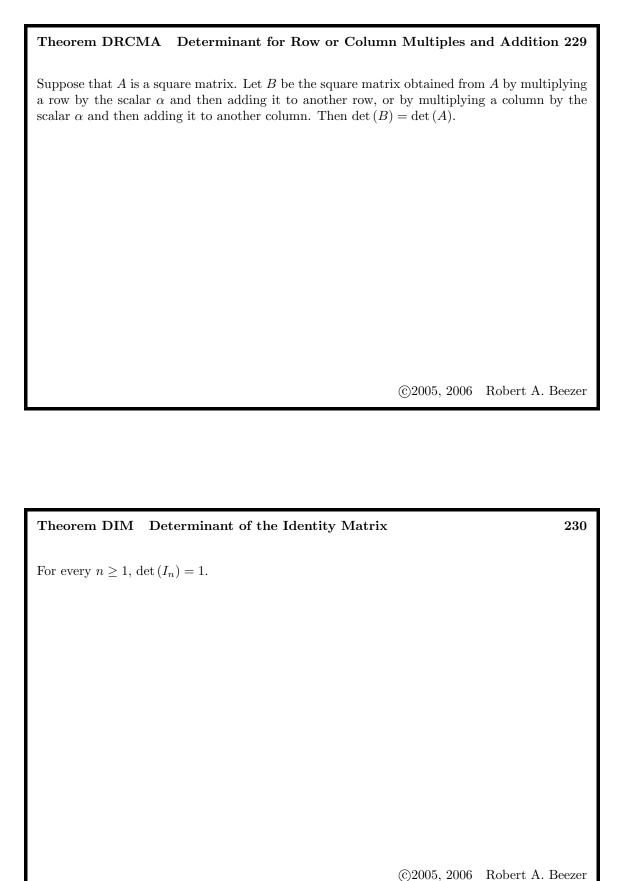
Theorem DRCS   Determinant for Row or Column Swap   226

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM   Determinant for Row or Column Multiples   **227**

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $\det(B) = \alpha \det(A)$.

Theorem DERC   Determinant with Equal Rows or Columns   **228**

Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Theorem DRCMA   Determinant for Row or Column Multiples and Addition 229

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

Theorem DIM   Determinant of the Identity Matrix   **230**

For every $n \geq 1$, $\det(I_n) = 1$.

Theorem DEM   Determinants of Elementary Matrices   231

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1. $\det(E_{i,j}) = -1$

2. $\det(E_i(\alpha)) = \alpha$

3. $\det(E_{i,j}(\alpha)) = 1$

Theorem DEMMM   Determinants, Elementary Matrices, Matrix Multiplication 232

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then
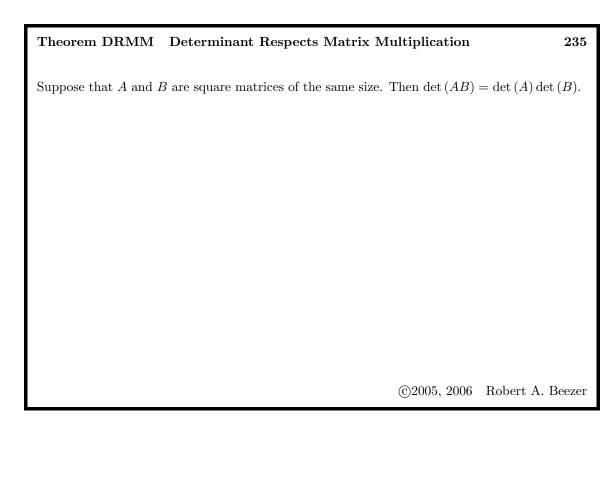
$$\det(EA) = \det(E)\det(A)$$

Theorem SMZD   Singular Matrices have Zero Determinants 233

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NME7   Nonsingular Matrix Equivalences, Round 7 234

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

Theorem DRMM   Determinant Respects Matrix Multiplication   **235**

Suppose that A and B are square matrices of the same size. Then $\det (AB) = \det (A) \det (B)$.
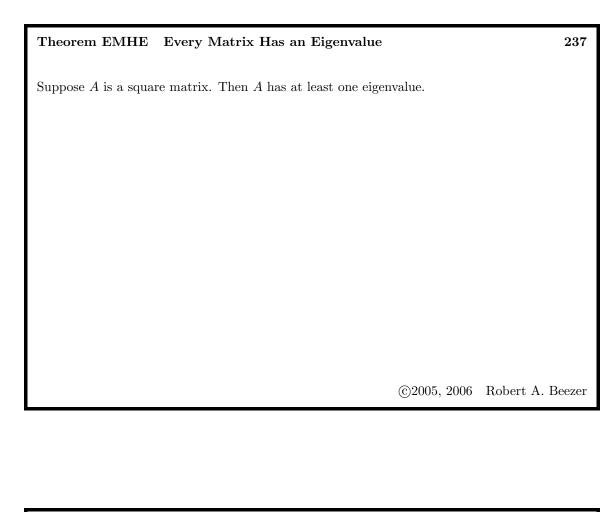

Definition EEM   Eigenvalues and Eigenvectors of a Matrix   **236**

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in $\mathbb{C}^n$, and λ is a scalar in $\mathbb{C}$. Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ if

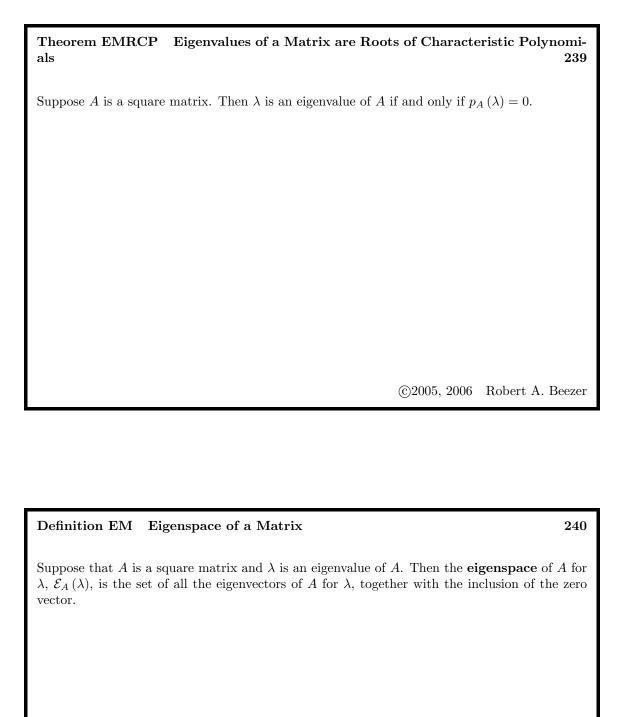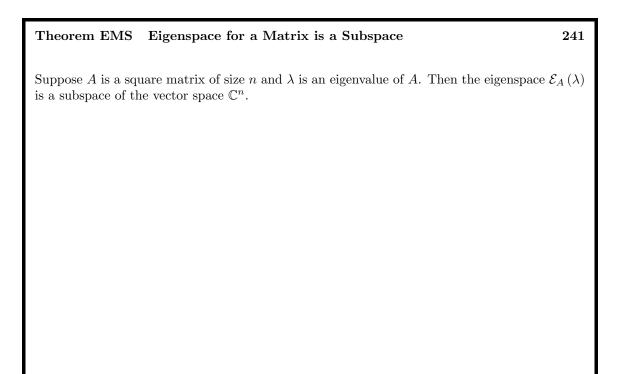$$A\mathbf{x} = \lambda\mathbf{x}$$

Suppose A is a square matrix. Then A has at least one eigenvalue.

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by
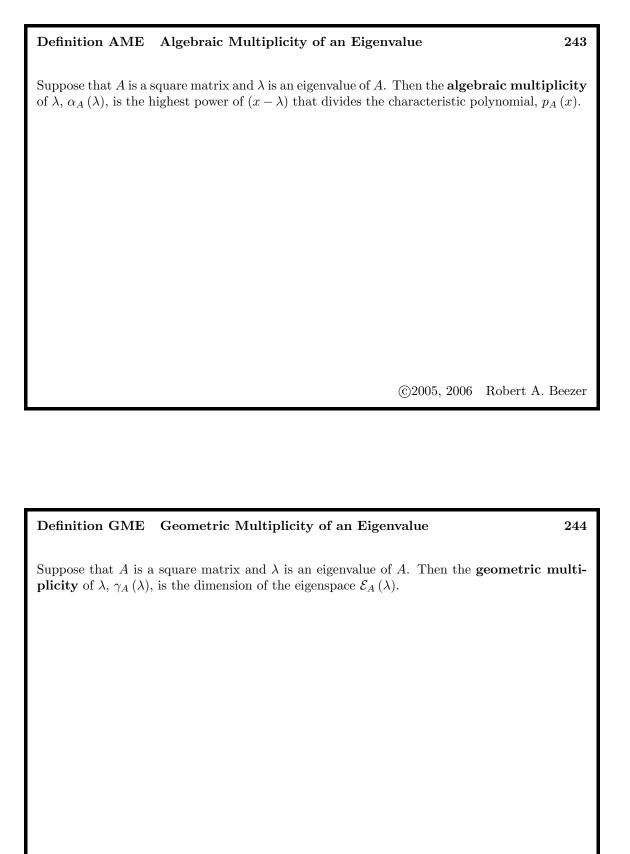$$p_A(x) = \det(A - xI_n)$$

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Theorem EMS   Eigenspace for a Matrix is a Subspace   **241**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $\mathcal{E}_A\left(\lambda\right)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS   Eigenspace of a Matrix is a Null Space   **242**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A\left(\lambda\right) = \mathcal{N}(A - \lambda I_n)$$

Definition AME   Algebraic Multiplicity of an Eigenvalue   **243**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME   Geometric Multiplicity of an Eigenvalue   **244**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.

Theorem EDELI   Eigenvectors with Distinct Eigenvalues are Linearly Independent
245

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE   Singular Matrices have Zero Eigenvalues   **246**

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NME8   Nonsingular Matrix Equivalences, Round 8   **247**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

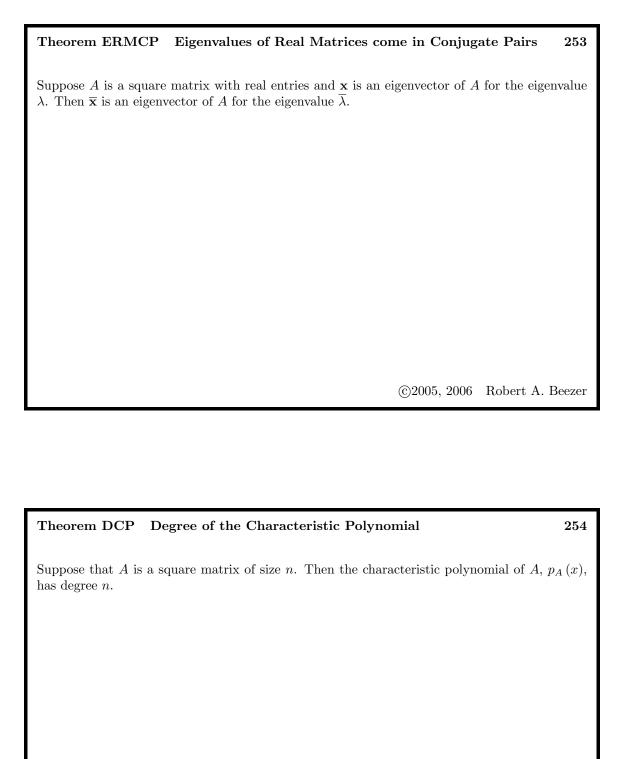**Theorem ESMM   Eigenvalues of a Scalar Multiple of a Matrix**   **248**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP   Eigenvalues Of Matrix Powers   249

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Theorem EPM   Eigenvalues of the Polynomial of a Matrix   250

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Theorem EIM   Eigenvalues of the Inverse of a Matrix   **251**

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM   Eigenvalues of the Transpose of a Matrix   **252**

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP   Eigenvalues of Real Matrices come in Conjugate Pairs   253

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP   Degree of the Characteristic Polynomial   254

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Theorem NEM   Number of Eigenvalues of a Matrix   **255**

Suppose that A is a square matrix of size n with distinct eigenvalues λ_1, λ_2, λ_3, ..., λ_k. Then

$$\sum_{i=1}^{k} \alpha_A\left(\lambda_i\right) = n$$

Theorem ME   Multiplicities of an Eigenvalue   **256**

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A\left(\lambda\right) \leq \alpha_A\left(\lambda\right) \leq n$$

Theorem MNEM   Maximum Number of Eigenvalues of a Matrix   **257**

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Theorem HMRE   Hermitian Matrices have Real Eigenvalues   **258**

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Theorem HMOE Hermitian Matrices have Orthogonal Eigenvectors   259

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM   Similar Matrices   260

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER   Similarity is an Equivalence Relation   **261**

Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)

2. If A is similar to B, then B is similar to A. (Symmetric)

3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Theorem SMEE   Similar Matrices have Equal Eigenvalues   **262**

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.

Definition DIM   Diagonal Matrix   **263**

Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$.

Definition DZM   Diagonalizable Matrix   **264**

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC   Diagonalization Characterization   **265**

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMFE   Diagonalizable Matrices have Full Eigenspaces   **266**

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T\left(\mathbf{u}_1 + \mathbf{u}_2\right) = T\left(\mathbf{u}_1\right) + T\left(\mathbf{u}_2\right)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$

2. $T\left(\alpha \mathbf{u}\right) = \alpha T\left(\mathbf{u}\right)$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T\left(\mathbf{0}\right) = \mathbf{0}$.

Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T\left(\mathbf{x}\right) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV   Matrix of a Linear Transformation, Column Vectors   271

Suppose that $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC   Linear Transformations and Linear Combinations   272

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, ..., a_t are scalars from $\mathbb{C}$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \cdots + a_t T(\mathbf{u}_t)$$

Theorem LTDB   Linear Transformation Defined on a Basis   **273**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let $a_1, a_2, a_3, \ldots, a_n$ be the scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \cdots + a_n T(\mathbf{u}_n)$$

Definition PI   Pre-Image   **274**

Suppose that $T\colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\}$$

Definition LTA   Linear Transformation Addition

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT   Sum of Linear Transformations is a Linear Transformation 276

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Definition LTSM   Linear Transformation Scalar Multiplication   277

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)\,(\mathbf{u}) = \alpha T\,(\mathbf{u})$$

Theorem MLTLT   Multiple of a Linear Transformation is a Linear Transformation 278

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \mapsto V$ is a linear transformation.

Theorem VSLT   Vector Space of Linear Transformations   279

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\text{LT}\,(U,\,V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC   Linear Transformation Composition   280

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)\,(\mathbf{u}) = S\,(T\,(\mathbf{u}))$$

Theorem CLTLT   Composition of Linear Transformations is a Linear Transformation   281

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a linear transformation.

Definition ILT   Injective Linear Transformation   282

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition KLT   Kernel of a Linear Transformation   **283**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{\,\mathbf{u} \in U \mid T\,(\mathbf{u}) = \mathbf{0}\,\}$$

Theorem KLTS   Kernel of a Linear Transformation is a Subspace   **284**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Theorem KPI   Kernel and Pre-Image **285**

Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then
$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$$

Theorem KILT   Kernel of an Injective Linear Transformation **286**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Theorem ILTLI   Injective Linear Transformations and Linear Independence   287

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

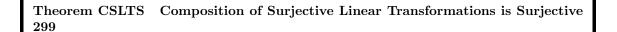**Theorem ILTB   Injective Linear Transformations and Bases   288**

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Theorem ILTD   Injective Linear Transformations and Dimension   **289**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Theorem CILTI   Composition of Injective Linear Transformations is Injective 290

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S\circ T)\colon U \mapsto W$ is an injective linear transformation.

Definition SLT   Surjective Linear Transformation

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Definition RLT   Range of a Linear Transformation

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{\, T(\mathbf{u}) \mid \mathbf{u} \in U \,\}$$

Theorem RLTS   Range of a Linear Transformation is a Subspace   **293**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT   Range of a Surjective Linear Transformation   **294**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT   Spanning Set for Range of a Linear Transformation   295

Suppose that $T\colon U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ spans U. Then

$$R = \{T\left(\mathbf{u}_1\right),\, T\left(\mathbf{u}_2\right),\, T\left(\mathbf{u}_3\right),\, \ldots,\, T\left(\mathbf{u}_t\right)\}$$

spans $\mathcal{R}(T)$.

Theorem RPI   Range and Pre-Image   296

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}\left(\mathbf{v}\right) \neq \emptyset$$

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a spanning set for V.

Suppose that $T \colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS   Composition of Surjective Linear Transformations is Surjective
299

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT   Identity Linear Transformation   **300**

The **identity linear transformation** on the vector space W is defined as

$$I_W\colon W \mapsto W, \qquad I_W\left(\mathbf{w}\right) = \mathbf{w}$$

Definition IVLT   Invertible Linear Transformations   **301**

Suppose that $T : U \mapsto V$ is a linear transformation. If there is a function $S : V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT   Inverse of a Linear Transformation is a Linear Transformation
302

Suppose that $T : U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1} : V \mapsto U$ is a linear transformation.

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

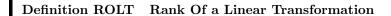**Theorem CIVLT   Composition of Invertible Linear Transformations**   **305**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T)\colon U \mapsto W$ is an invertible linear transformation.

Theorem ICLT   Inverse of a Composition of Linear Transformations   **306**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS   Isomorphic Vector Spaces   307

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED   Isomorphic Vector Spaces have Equal Dimension   308

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **rank** of T, $r\,(T)$, is the dimension of the range of T,

$$r\,(T) = \dim\left(\mathcal{R}(T)\right)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n\,(T)$, is the dimension of the kernel of T,

$$n\,(T) = \dim\left(\mathcal{K}(T)\right)$$

Theorem ROSLT   Rank Of a Surjective Linear Transformation   **311**

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT   Nullity Of an Injective Linear Transformation   **312**

Suppose that $T \colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Theorem RPNDD **Rank Plus Nullity is Domain Dimension** 313

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$r\left(T\right) + n\left(T\right) = \dim\left(U\right)$$

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Definition VR **Vector Representation** 314

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$. Define a function $\rho_B\colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i \qquad\qquad\qquad 1 \le i \le n$$

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Theorem VRLT   Vector Representation is a Linear Transformation   **315**

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI   Vector Representation is Injective   **316**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS   Vector Representation is Surjective   317

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT   Vector Representation is an Invertible Linear Transformation
318

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS   Characterization of Finite Dimensional Vector Spaces   319

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS   Isomorphism of Finite Dimensional Vector Spaces   320

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI   Coordinatization and Linear Independence   321

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B\left(\mathbf{u}_1\right), \rho_B\left(\mathbf{u}_2\right), \rho_B\left(\mathbf{u}_3\right), \ldots, \rho_B\left(\mathbf{u}_k\right)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS   Coordinatization and Spanning Sets   322

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}\rangle$ if and only if $\rho_B\left(\mathbf{u}\right) \in \langle\{\rho_B\left(\mathbf{u}_1\right), \rho_B\left(\mathbf{u}_2\right), \rho_B\left(\mathbf{u}_3\right), \ldots, \rho_B\left(\mathbf{u}_k\right)\}\rangle$.

Definition MR   Matrix Representation   **323**

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = [\, \rho_C \left(T \left(\mathbf{u}_1 \right) \right) | \ \rho_C \left(T \left(\mathbf{u}_2 \right) \right) | \ \rho_C \left(T \left(\mathbf{u}_3 \right) \right) | \ldots | \rho_C \left(T \left(\mathbf{u}_n \right) \right) \,]$$

Theorem FTMR   Fundamental Theorem of Matrix Representation   **324**

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C \left(T \left(\mathbf{u} \right) \right) = M_{B,C}^T \left(\rho_B \left(\mathbf{u} \right) \right)$$

or equivalently

$$T \left(\mathbf{u} \right) = \rho_C^{-1} \left(M_{B,C}^T \left(\rho_B \left(\mathbf{u} \right) \right) \right)$$

Theorem MRSLT   Matrix Representation of a Sum of Linear Transformations 325

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^{T} + M_{B,C}^{S}$$

Theorem MRMLT   Matrix Representation of a Multiple of a Linear Transformation 326

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^{T}$$

Theorem MRCLT   Matrix Representation of a Composition of Linear Transformations **327**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^{S} M_{B,C}^{T}$$

Theorem KNSI   Kernel and Null Space Isomorphism **328**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^{T}$,

$$\mathcal{K}(T) \cong \mathcal{N}\left(M_{B,C}^{T}\right)$$

Theorem RCSI   Range and Column Space Isomorphism   **329**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M^T_{B,C}$,

$$\mathcal{R}(T) \cong \mathcal{C}\big(M^T_{B,C}\big)$$

Theorem IMR   Invertible Matrix Representations   **330**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M^T_{B,C}$ is an invertible matrix, and

$$M^{T^{-1}}_{C,B} = \big(M^T_{B,C}\big)^{-1}$$

Theorem IMILT   Invertible Matrices, Invertible Linear Transformation   **331**

Suppose that A is a square matrix of size n and $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Theorem NME9   Nonsingular Matrix Equivalences, Round 9   **332**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

13. The linear transformation $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Definition EELT   Eigenvalue and Eigenvector of a Linear Transformation   333

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM   Change-of-Basis Matrix   334

Suppose that V is a vector space, and $I_V\colon V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M_{B,C}^{I_V} \\
&= \left[\,\rho_C\left(I_V\left(\mathbf{v}_1\right)\right)\big|\ \rho_C\left(I_V\left(\mathbf{v}_2\right)\right)\big|\ \rho_C\left(I_V\left(\mathbf{v}_3\right)\right)\big|\ldots\big|\rho_C\left(I_V\left(\mathbf{v}_n\right)\right)\,\right] \\
&= \left[\,\rho_C\left(\mathbf{v}_1\right)\big|\ \rho_C\left(\mathbf{v}_2\right)\big|\ \rho_C\left(\mathbf{v}_3\right)\big|\ldots\big|\rho_C\left(\mathbf{v}_n\right)\,\right]
\end{aligned}
$$

Theorem CB   Change-of-Basis   **335**

Suppose that $\mathbf{v}$ is a vector in the vector space V and B and C are bases of V. Then

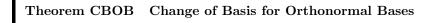$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

Theorem ICBM   Inverse of Change-of-Basis Matrix   **336**

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem CBOB   Change of Basis for Orthonormal Bases   **337**

Suppose that B and C are orthonormal bases for $\mathbb{C}^n$. Then the change-of-basis matrix, $C_{B,C}$, is a unitary matrix.

Theorem MRCB   Matrix Representation and Change of Basis   **338**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^{T} = C_{E,D} M_{C,E}^{T} C_{B,C}$$

Theorem SCB   Similarity and Change of Basis

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

Theorem EER   Eigenvalues, Eigenvectors, Representations

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ.

Suppose that $T \colon V \mapsto V$ is a linear transformation such that there is an integer $p > 0$ such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.
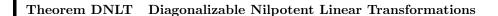
Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$[J_n(\lambda)]_{ij} = \begin{cases} \lambda & i = j \\ 1 & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem NJB   Nilpotent Jordan Blocks   **343**

The Jordan block $J_n(0)$ is nilpotent of index n.

Theorem ENLT   Eigenvalues of Nilpotent Linear Transformations   **344**

Suppose that $T\colon V \mapsto V$ is a linear transformation and λ is an eigenvalue of T. Then $\lambda = 0$.

Theorem DNLT   Diagonalizable Nilpotent Linear Transformations   345

Suppose the linear transformation $T\colon V \mapsto V$ is nilpotent. Then T is diagonalizable if and only T is the zero linear transformation.

Theorem KPLT   Kernels of Powers of Linear Transformations   346

Suppose $T\colon V \mapsto V$ is a linear transformation, where $\dim(V) = n$. Then there is an integer m, $0 \leq m \leq n$, such that
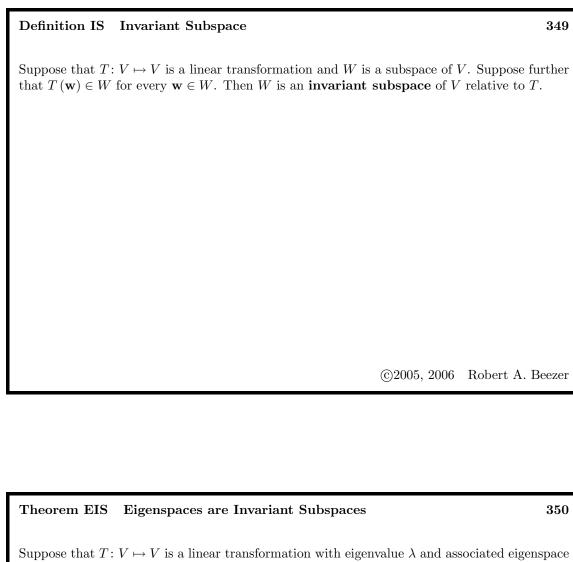
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations   347

Suppose $T \colon V \mapsto V$ is a nilpotent linear transformation with index p and $\dim(V) = n$. Then $0 \le p \le n$ and

$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$

Theorem CFNLT Canonical Form for Nilpotent Linear Transformations   348

Suppose that $T \colon V \mapsto V$ is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation, $M_{B,B}^T$, is block diagonal with each block being a Jordan block, $J_n(0)$. The size of the largest block is the index p, and the total number of blocks is the nullity of T, $n(T)$.

Definition IS   Invariant Subspace 349

Suppose that $T\colon V \mapsto V$ is a linear transformation and W is a subspace of V. Suppose further that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$. Then W is an **invariant subspace** of V relative to T.

Theorem EIS   Eigenspaces are Invariant Subspaces 350

Suppose that $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ and associated eigenspace $\mathcal{E}_T(\lambda)$. Let W be any subspace of $\mathcal{E}_T(\lambda)$. Then W is an invariant subspace of V relative to T.

Theorem KPIS   Kernels of Powers are Invariant Subspaces   **351**

Suppose that $T \colon V \mapsto V$ is a linear transformation. Then $\mathcal{K}\left(T^{k}\right)$ is an invariant subspace of V.

Definition GEV   Generalized Eigenvector   **352**

Suppose that $T \colon V \mapsto V$ is a linear transformation. Suppose further that for $\mathbf{x} \neq \mathbf{0}$, $\left(T - \lambda I_{V}\right)^{k}(\mathbf{x}) = \mathbf{0}$ for some $k > 0$. Then $\mathbf{x}$ is a **generalized eigenvector** of T with eigenvalue λ.

Definition GES   Generalized Eigenspace

Suppose that $T\colon V \mapsto V$ is a linear transformation. Define the **generalized eigenspace** of T for λ as

$$\mathcal{G}_T\left(\lambda\right) = \left\{\, \mathbf{x} \mid \left(T - \lambda I_V\right)^k\left(\mathbf{x}\right) = \mathbf{0} \text{ for some } k \geq 0 \right\}$$

Theorem GESIS   Generalized Eigenspace is an Invariant Subspace

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then the generalized eigenspace $\mathcal{G}_T\left(\lambda\right)$ is an invariant subspace of V relative to T.

Suppose that $T\colon V \mapsto V$ is a linear transformation, $\dim(V) = n$, and λ is an eigenvalue of T. Then $\mathcal{G}_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n)$.
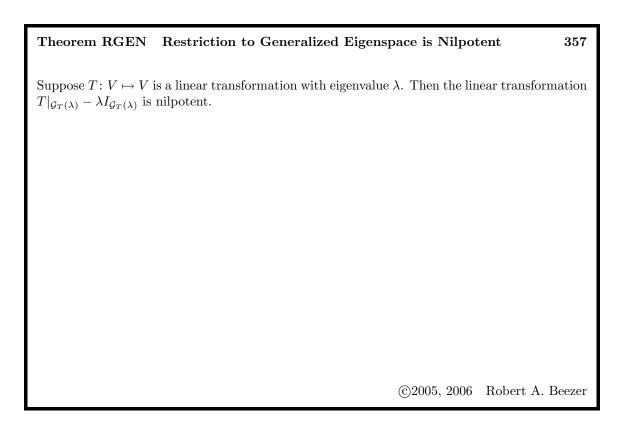
Suppose that $T\colon V \mapsto V$ is a linear transformation, and U is an invariant subspace of V relative to T. Define the **restriction** of T to U by
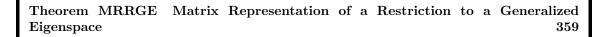
$$T|_U \colon U \mapsto U \qquad\qquad T|_U(\mathbf{u}) = T(\mathbf{u})$$

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is nilpotent.

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the **index** of λ, $\iota_T(\lambda)$, is the index of the nilpotent linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$.

Theorem MRRGE   Matrix Representation of a Restriction to a Generalized Eigenspace

Suppose that $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then there is a basis of the the generalized eigenspace $\mathcal{G}_T\left(\lambda\right)$ such that the restriction $T|_{\mathcal{G}_T(\lambda)}\colon \mathcal{G}_T\left(\lambda\right) \mapsto \mathcal{G}_T\left(\lambda\right)$ has a matrix representation that is block diagonal where each block is a Jordan block of the form $J_n\left(\lambda\right)$.
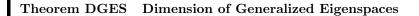

Theorem GESD   Generalized Eigenspace Decomposition

Suppose that $T\left(V\right)V$ is a linear transformation with distinct eigenvalues λ_1, λ_2, λ_3, ..., λ_m. Then

$$V = \mathcal{G}_T\left(\lambda_1\right) \oplus \mathcal{G}_T\left(\lambda_2\right) \oplus \mathcal{G}_T\left(\lambda_3\right) \oplus \cdots \oplus \mathcal{G}_T\left(\lambda_m\right)$$

Suppose $T\colon V \mapsto V$ is a linear transformation with eigenvalue λ. Then the dimension of the generalized eigenspace for λ is the algebraic multiplicity of λ, $\dim\left(\mathcal{G}_T\left(\lambda_i\right)\right) = \alpha_T\left(\lambda_i\right)$.

A square matrix is in **Jordan canonical form** if it meets the following requirements:

1. The matrix is block diagonal.

2. Each block is a Jordan block.

3. If $\rho < \lambda$ then the block $J_k\left(\rho\right)$ occupies rows with indices greater than the indices of the rows occupied by $J_\ell\left(\lambda\right)$.

4. If $\rho = \lambda$ and $\ell < k$, then the block $J_\ell\left(\lambda\right)$ occupies rows with indices greater than the indices of the rows occupied by $J_k\left(\lambda\right)$.

Theorem JCFLT   Jordan Canonical Form for a Linear Transformation   363

Suppose $T\colon V \mapsto V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

1. The matrix representation is in Jordan canonical form.

2. If $J_k(\lambda)$ is one of the Jordan blocks, then λ is an eigenvalue of T.

3. For a fixed value of λ, the largest block of the form $J_k(\lambda)$ has size equal to the index of λ, $\iota_T(\lambda)$.

4. For a fixed value of λ, the number of blocks of the form $J_k(\lambda)$ is the geometric multiplicity of λ, $\gamma_T(\lambda)$.

5. For a fixed value of λ, the number of rows occupied by blocks of the form $J_k(\lambda)$ is the algebraic multiplicity of λ, $\alpha_T(\lambda)$.

Theorem CHT   Cayley-Hamilton Theorem   364

Suppose A is a square matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = \mathcal{O}$.