Flash Cards

to accompany

A First Course in Linear Algebra

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The most recent version of this work can always be found at http://linear.ups.edu.

Definition SLE System of Linear Equations

A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$:

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .

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 Definition ESYS
 Equivalent Systems
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 Two systems of linear equations are equivalent if their solution sets are equal.
 (2)

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Definition EO Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

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Definition M Matrix

An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

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Definition CV Column Vector

A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$. Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$.

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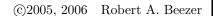
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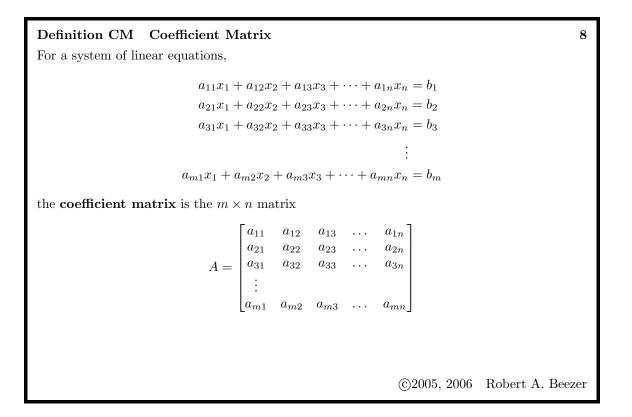
Definition ZCV Zero Column Vector

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0\\0\\0\\\vdots\\0 \end{bmatrix}$$

or more compactly, $\left[\mathbf{0}\right]_i=0$ for $1\leq i\leq m.$





Definition VOC Vector of Constants

For a system of linear equations,

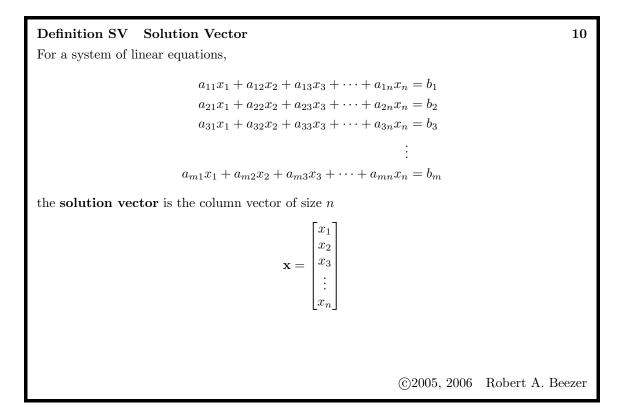
 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$:

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$





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Definition LSMR Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and **b** is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

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Definition AM Augmented Matrix

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants **b**. Then the **augmented matrix** of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (number n + 1) is the column vector **b**. This matrix will be written as $[A \mid \mathbf{b}]$.

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Definition RO Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows *i* and *j*.
- 2. αR_i : Multiply row *i* by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row *i* by the scalar α and add to row *j*.

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Definition REM Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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Theorem REMES Row-Equivalent Matrices represent Equivalent Systems 15 Suppose that *A* and *B* are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

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Definition RREF Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r. A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot colums will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Theorem REMEF Row-Equivalent Matrix in Echelon Form Suppose A is a matrix. Then there is a matrix B so that 1. A and B are row-equivalent. 2. B is in reduced row-echelon form. ©2005, 2006 Robert A. Beezer

Definition RR Row-Reducing

To row-reduce the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

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Definition CS Consistent System

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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Definition IDV Independent and Dependent Variables

Suppose A is the augmented matrix of a consistent system of linear equations and B is a rowequivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

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Theorem RCLS Recognizing Consistency of a Linear System

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n + 1 of B.

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Theorem ISRN Inconsistent Systems, r and n

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

Theorem CSRN Consistent Systems, r and n

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n - r free variables.

Theorem PSSLS Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

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Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions 26

Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions.

Definition HS Homogeneous System

A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

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Theorem HSC	Homogeneous Systems are Consistent	28
Suppose that a sy	stem of linear equations is homogeneous. Then the system is consistent.	
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Definition TSHSE	Trivial Solution to Homogeneous Systems	of Equations 29
	bus system of linear equations has n variables. e. $\mathbf{x} = 0$) is called the trivial solution .	The solution $x_1 = 0$,
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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 30

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

Definition NSM Null Space of a Matrix

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

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Definition	\mathbf{SQM}	Square	Matrix
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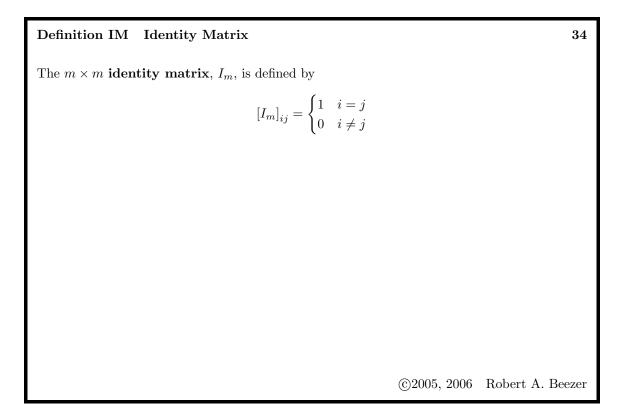
A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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Definition NM Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.



Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 35

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 3	6
	quare matrix. Then A is nonsingular if and only if the null space of A he zero vector, i.e. $\mathcal{N}(A) = \{0\}.$	1,
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Theorem NMUS Nonsingular Matrices and Unique Solutions

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} .

Theorem NME1	Nonsingular Matrix Equivalences, Round 1	38
Suppose that A is a	square matrix. The following are equivalent.	
1. A is nonsingul	ar.	
2. A row-reduces	to the identity matrix.	
3. The null space	of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear sys	tem $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible cl	hoice of \mathbf{b} .
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Definition VSCV	Vector Space of Column Vectors 39)
The vector space \mathbb{C}^m the set of complex n	is the set of all column vectors (Definition CV) of size m with entries from umbers, \mathbb{C} .	1
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Definition CVE	Column Vector E	quality		40
The vectors ${\bf u}$ and	\mathbf{v} are equal , written	$\mathbf{u} = \mathbf{v}$ provided the	hat	
	$\left[\mathbf{u} ight]_{i}=\left[\mathbf{v} ight]_{i}$		$1 \leq i \leq m$	
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Definition CVA	Column Vector Addition		41	
Given the vectors ${\bf u}$ and ${\bf v}$ the ${\bf sum}$ of ${\bf u}$ and ${\bf v}$ is the vector ${\bf u}+{\bf v}$ defined by				
	$\left[\mathbf{u}+\mathbf{v}\right]_i=\left[\mathbf{u}\right]_i+\left[\mathbf{v}\right]_i$	$1 \leq i \leq m$		
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Definition CVSM	Column Vector Scalar Multip	olication	42
Given the vector ${\bf u}$ an	nd the scalar $\alpha \in \mathbb{C}$, the scalar m	ultiple of u by α , α	\mathbf{u} is defined by
	$\left[\alpha \mathbf{u}\right]_{i}=\alpha\left[\mathbf{u}\right]_{i}$	$1 \leq i \leq m$	
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Theorem VSPCVVector Space Properties of Column Vectors43Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and
scalar multiplication as defined in Definition CVA and Definition CVSM. Then43

- ACC Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- ZC Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- OC One Column Vectors If $\mathbf{u} \in \mathbb{C}^m$ then $1\mathbf{u} = \mathbf{u}$

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Definition LCCV Linear Combination of Column Vectors

Given *n* vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and *n* scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear** combination is the vector

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n.$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n + 1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$ of size n by

$$\begin{aligned} \left[\mathbf{c} \right]_{i} &= \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, \, i = d_{k} \end{cases} \\ \left[\mathbf{u}_{j} \right]_{i} &= \begin{cases} 1 & \text{if } i \in F, \, i = f_{j} \\ 0 & \text{if } i \in F, \, i \neq f_{j} \\ -[B]_{k,f_{j}} & \text{if } i \in D, \, i = d_{k} \end{cases} \end{aligned}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \}$$

Theorem PSPHS Particular Solution Plus Homogeneous Solutions 47 Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then **y** is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Theorem RREFU Reduced Row-Echelon Form is Unique	48
Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are not to A and in reduced row-echelon form. Then $B = C$.	row-equivalent
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Definition SSCV Span of a Set of Column Vectors

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n - r vectors \mathbf{z}_j , $1 \le j \le n - r$ of size n as

$$\begin{bmatrix} \mathbf{z}_j \end{bmatrix}_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -\begin{bmatrix} B \end{bmatrix}_{k,f_i} & \text{if } i \in D, \ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \left\langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \right\rangle.$$

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Definition RLDCV Relation of Linear Dependence for Column Vectors

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the trivial relation of linear dependence on S.

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The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems 53

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

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Theorem LIVRN Linearly Independent Vectors, r and n

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

Theorem MVSLD More Vectors than Size implies Linear Dependence 5	55
Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that $n > m$. Then S a linearly dependent set.	is

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Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 56

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NME2 Nonsingular Matrix Equivalences, Round 2

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

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Theorem BNS Basis for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n - r vectors \mathbf{z}_j , $1 \le j \le n - r$ of size n as

$$\left[\mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k, f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Define the set $S = \{ \mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

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Theorem DLDS Dependency in Linearly Dependent Sets

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

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Theorem BS Basis of a Span

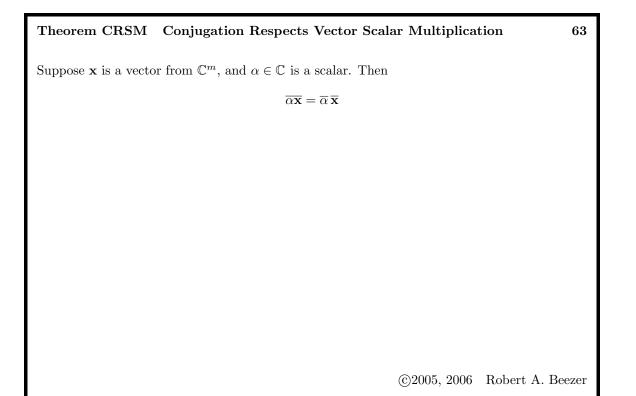
Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = {d_1, d_2, d_3, \dots, d_r}$ the set of column indices corresponding to the pivot columns of B. Then

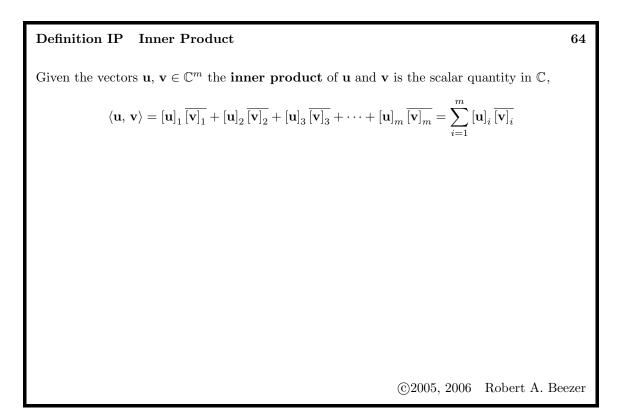
- 1. $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

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Definition CCCV Complex Conjugate of a Column Vector								
Suppose that u is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by								
	$\left[\overline{\mathbf{u}}\right]_i = \overline{\left[\mathbf{u}\right]_i} \qquad \qquad 1 \le i \le m$							
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Theorem CRVA	Conjugation Respects Vector Addition	62					
Suppose x and y are two vectors from \mathbb{C}^m . Then							
	$\overline{\mathbf{x}+\mathbf{y}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$						
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Theorem IPVA Inner Product and Vector Addition

Suppose $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

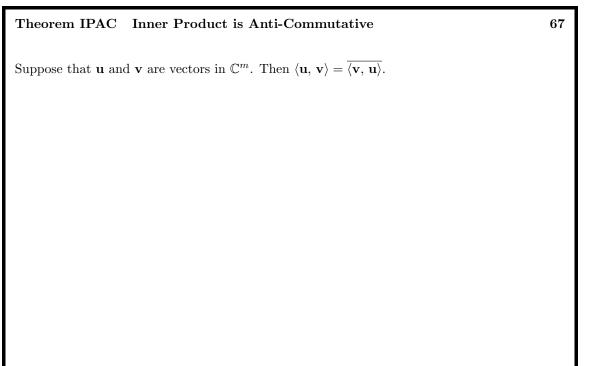
1.
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

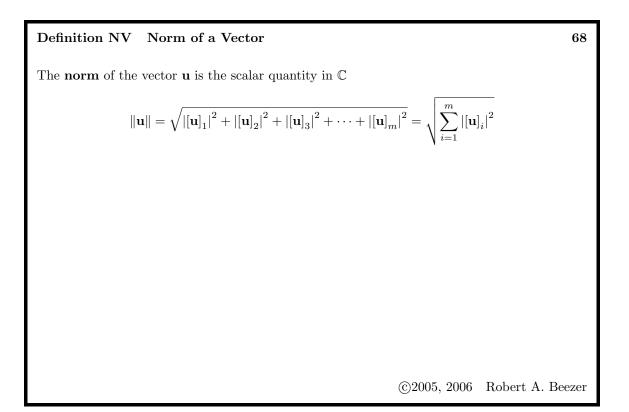
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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Theorem IPSM	rem IPSM Inner Product and Scalar Multiplication				
Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$	and $\alpha \in \mathbb{C}$. Then				
		$ \langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{u}, \alpha \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle $			
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Theorem IPN Inner Products and Norms	69
Suppose that u is a vector in \mathbb{C}^m . Then $\ \mathbf{u}\ ^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.	

Theorem PIP Positive Inner Products	70
Suppose that u is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ with equality if and only if $\mathbf{u} = 0$.	
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Definition OV Orthogonal Vectors

A pair of vectors, **u** and **v**, from \mathbb{C}^m are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

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Definition OSV Orthogonal Set of Vectors

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors from \mathbb{C}^m . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

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Theorem OSLI Orthogonal Sets are Linearly Independent

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

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Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \leq i \leq p$ by $\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$ Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$. ©2005, 2006 Robert A. Beezer

Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

Definition ONS OrthoNormal Set

Suppose $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is an orthogonal set of vectors such that $||\mathbf{u}_i|| = 1$ for all $1 \le i \le n$. Then S is an **orthonormal** set of vectors.

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Definition VSM	Vector Space of $m \times n$ Matrices
----------------	---------------------------------------

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

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Definition ME Matrix Equality

The $m \times n$ matrices A and B are equal, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

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Definition MA Matrix Addition

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A+B]_{ii} = [A]_{ii} + [B]_{ii} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Definition MSM Matrix Scalar Multiplication

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Theorem VSPM Vector Space Properties of Matrices

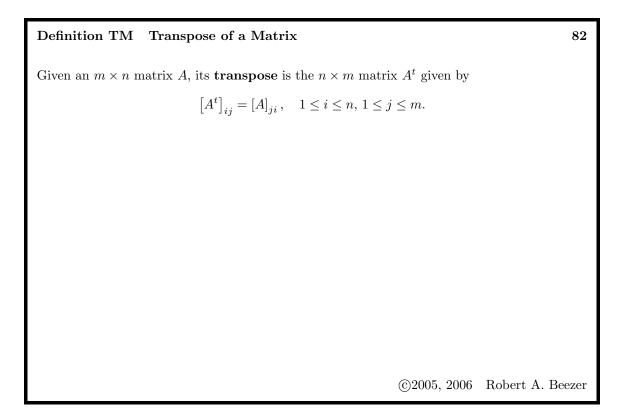
Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One Matrices If $A \in M$ then 1A A

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Definition ZM Zero Matrix

The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$.



Definition	\mathbf{SYM}	Symmetric	Matrix
------------	----------------	-----------	--------

The matrix A is symmetric if $A = A^t$.

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 Theorem SMS
 Symmetric Matrices are Square
 84

 Suppose that A is a symmetric matrix. Then A is square.
 9

 Count of the second s

Theorem TMA Transpose and Matrix Addition

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

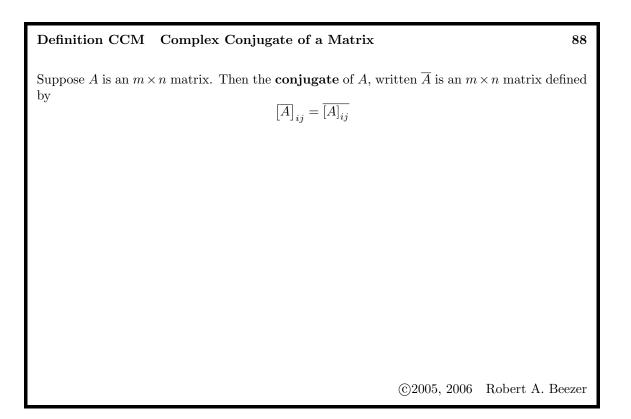
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Theorem TT Transpose of a Transpose

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

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Theorem CRMA Conjugation Respects Matrix Addition	89
Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.	
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Theorem CRMSM	Conjugation Respects Matrix Scalar Multiplication	90
Suppose that $\alpha \in \mathbb{C}$ and	and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.	
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Theorem MCT Matrix Conjugation and Transposes	91	
Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.		

Definition MVP Matrix-Vector Product

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

 $A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$

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Theorem SLEMM	Systems of Linear Equations as Matrix Multiplication	93
Solutions to the linear	system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for \mathbf{x} in the vector equation $A\mathbf{x} =$	b.
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Theorem EMMVP	Equal Matrices and Matrix-Vector Products	94
Suppose that A and B	are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A =$	В.
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Definition MM Matrix Multiplication

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

 $AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$

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Theorem EMPEntries of Matrix Products96Suppose A is an $m \times n$ matrix and B = is an $n \times p$ matrix. Then for $1 \le i \le m, 1 \le j \le p$, the
individual entries of AB are given by $[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$
 $= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$ $= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$ ©2005, 2006Robert A. Beezer

Theorem MMZM Matrix Multiplication and the Zero Matrix

1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$ 2. $\mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$

Theorem MMIM	Matrix Multiplication and Identity Matrix	98
Suppose A is an $m \times 1$. $AI_n = A$ 2 . $I_m A = A$	n matrix. Then	
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Theorem MMDAAMatrix Multiplication Distributes Across Addition99Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then1. A(B+C) = AB + AC2. (B+C)D = BD + CD

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 100

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

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Theorem MMA	Matrix Multiplication is Associative 10	01
Suppose A is an m (AB)D.	$\times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD)$	=
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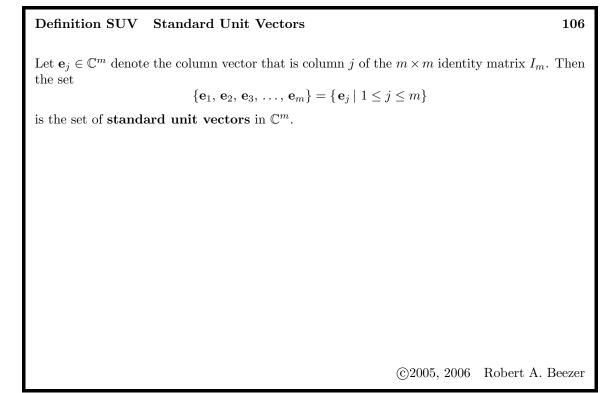
Theorem MMIP	Matrix Multiplication and Inner Products	102
If we consider the ve	ectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then	
	$\langle {f u},{f v} angle = {f u}^t \overline{f v}$	
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Theorem MMCC	Matrix Multiplication and Complex Conjugation 1	03
Suppose A is an $m \times$	<i>n</i> matrix and <i>B</i> is an $n \times p$ matrix. Then $\overline{AB} = \overline{A} \overline{B}$.	
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Theorem MMT	Matrix Multiplication and Transposes	104
Suppose A is an m	$\times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.	
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Definition MI Matrix Inverse

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.



Theorem TTMI Two-by-Two Matrix Inverse

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

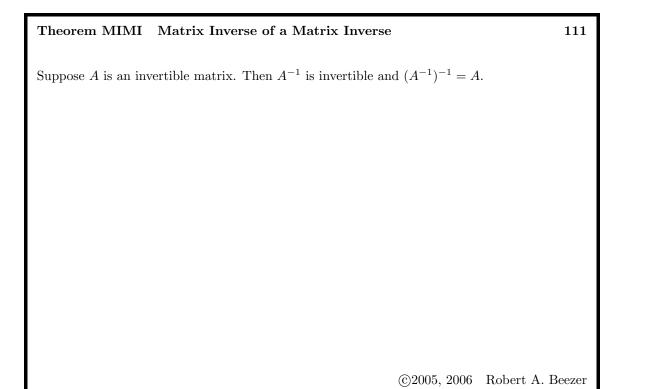
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

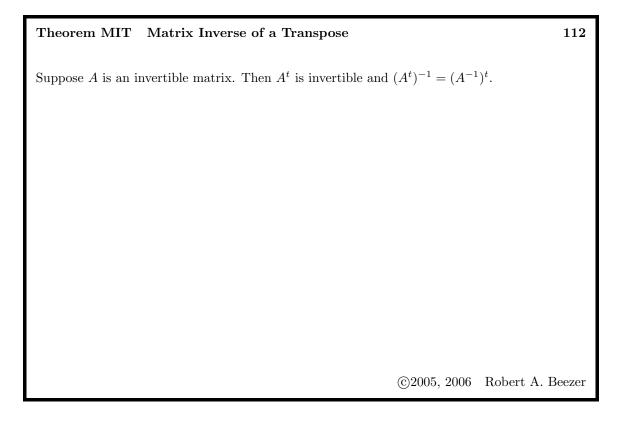
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Theorem CINM Computing the Inverse of a Nonsingular Matrix 108
Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.
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Theorem MIU Matrix Inverse is Unique	109
Suppose the square matrix A has an inverse. Then A^{-1} is unique.	
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Theorem SS Socks and Shoes	110
Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}$, invertible matrix.	A^{-1} and AB is an
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Theorem MISM Matrix Inverse of a Scalar Multiple 113 Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem NPNT	Nonsingular Product has Nonsingular Terms	114
Suppose that A and A and B are both not	B are square matrices of size n and the product AB is nonsingular. To onsingular.	Then
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Theorem OSIS	One-Sided Inverse is Sufficient 1	115
Suppose A and B	are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.	
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 Theorem NI Nonsingularity is Invertibility
 116

 Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.
 Image: Comparison of the second secon

Theorem NME3 Nonsingular Matrix Equivalences, Round 3
117
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system *LS*(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.

Solution with	Nonsingular Coefficient Matrix	118
nsingular. Then	the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is A^-	$^{1}\mathbf{b}.$
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		Solution with Nonsingular Coefficient Matrix nsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is A^- \bigcirc 2005, 2006

Definition UM Unitary Matrices

Suppose that Q is a square matrix of size n such that $(\overline{Q})^t Q = I_n$. Then we say Q is **unitary**.

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Theorem UMI Unitary Matrices are Invertible	120
Suppose that Q is a unitary matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.	
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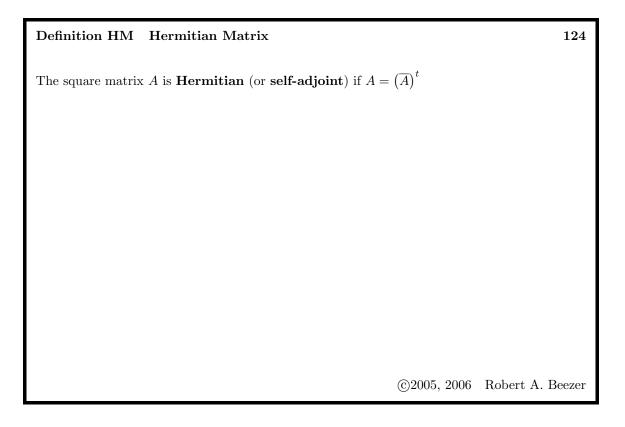
Theorem CUMOS Columns of Unitary Matrices are	e Orthonorm	al Sets 121
Suppose that A is a square matrix of size n with columns $S =$ is a unitary matrix if and only if S is an orthonormal set.	$= \{ \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, $	\ldots, \mathbf{A}_n }. Then A
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Theorem UMPIP Unitary Matrices Preserve Inner Products						122		
Suppose that (Q is a u	nitary mat	rix of size n	and \mathbf{u} and \mathbf{v}	v are two	vectors fr	com \mathbb{C}^n . Then	1
	$\langle Q \mathbf{u}, Q$	$ \mathbf{v} angle = \langle \mathbf{u}, \mathbf{v} angle$	·>	and		$\ Q\mathbf{v}\ =$	$\ \mathbf{v}\ $	
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Definition A Adjoint

If A is a square matrix, then its **adjoint** is $A^{H} = (\overline{A})^{t}$.

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Definition CSM Column Space of a Matrix

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS	Column Spaces and Consistent Systems 126	i
Suppose A is an m $\mathcal{LS}(A, \mathbf{b})$ is consistent	$n \times n$ matrix and b is a vector of size m . Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if ent.	a
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Theorem BCS Basis of the Column Space

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $\mathcal{C}(A) = \langle T \rangle$.

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Theorem NME4 Nonsingular Matrix Equivalences, Round 4 129
Suppose that A is a square matrix of size n. The following are equivalent.
1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, N(A) = {0}.
4. The linear system LS(A, b) has a unique solution for every possible choice of b.
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is Cⁿ, C(A) = Cⁿ.

Definition RSM Row Space of a Matrix

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

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Theorem REMRS	Row-Equivalent Matrices have equal Row Sp	baces 131
Suppose A and B are	row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.	
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Theorem BRS Basis for the Row Space	132		
Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then			
1. $\mathcal{R}(A) = \langle S \rangle.$			
2. S is a linearly independent set.			
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Theorem CSRST	Column Space, Row Space, Transpose	133
Suppose A is a matrix	ix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.	

Definition LNS Left Null Space 134 Suppose A is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

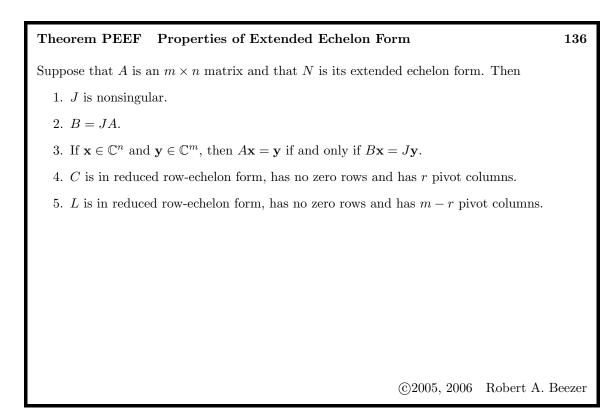
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Definition EEF Extended Echelon Form

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m - r) \times m$ matrix formed from the bottom m - r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$



Theorem FS Four Subsets

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m - r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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Definition VS Vector Space

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space if the following ten properties hold.

- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- Z Zero Vector There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One If $\mathbf{u} \in V$ then $1\mathbf{u} = \mathbf{u}$

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

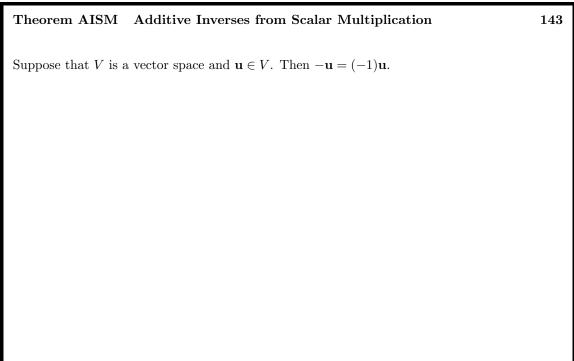
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Theorem ZVU Zero Vector is Unique	139	
Suppose that V is a vector space. The zero vector, 0 , is unique.		
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Theorem AIU Additive Inverses are Unique	140
Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.	
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Theorem ZSSM	Zero Scalar in Scalar Multiplicatio	n	141
Suppose that V is a	vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = 0$.		
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Theorem ZVSM	Zero Vector in Scalar Multiplication	142
Suppose that V is a	vector space and $\alpha \in \mathbb{C}$. Then $\alpha 0 = 0$.	
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Theorem SMEZV	Scalar Multiplication Equals the Zero Vector	144
Suppose that V is a v	vector space and $\alpha \in \mathbb{C}$. If $\alpha \mathbf{u} = 0$, then either $\alpha = 0$ or $\mathbf{u} = 0$.	
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Theorem VAC Vector Addition Cancellation	145
Suppose that V is a vector space, and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.	
Suppose that v is a vector space, and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{v}$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.	
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Theorem CSSM	Canceling Scalars in Scalar Multiplication	146
Suppose V is a vec $\mathbf{u} = \mathbf{v}$.	ctor space, $\mathbf{u}, \mathbf{v} \in V$ and α is a nonzero scalar from \mathbb{C} . If $\alpha \mathbf{u} = \alpha \mathbf{v}$,	then
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Theorem CVSM Ca	anceling Vectors in Scalar Mult	iplication	147
Suppose V is a vector space of V is a vector space.	pace, $\mathbf{u} \neq 0$ is a vector in V and α ,	$\beta \in \mathbb{C}$. If $\alpha \mathbf{u} = \beta$	$\beta \mathbf{u}$, then $\alpha = \beta$.
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Definition S	Subspace	148
	and W are two vector spaces that have identical definitions iplication, and that W is a subset of $V, W \subseteq V$. Then W is a	
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Theorem TSS Testing Subsets for Subspaces

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

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Definition TS	Trivial Subspaces	150
Given the vector	space V, the subspaces V and $\{0\}$ are each called a trivial subspace .	
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Theorem NSMS Null Space of a Matrix is a Subspace Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of \mathbb{C}^n . ©2005, 2006 Robert A. Beezer

Definition LC Linear Combination 152Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$ ©2005, 2006 Robert A. Beezer

Definition SS Span of a Set

Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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Theorem SSS Span of a Set is a Subspace	154
Suppose V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t} \subseteq V$, their $\langle S \rangle$, is a subspace.	span,
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Theorem CSMS	Column Space of a Matrix is a Subspace	155
Suppose that A is a	on $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .	
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Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .	Theorem RSMS	Row Space of	a Matrix is a	a Subspac	e		156
	Suppose that A is a	n $m \times n$ matrix.	Then $\mathcal{R}(A)$ is	a subspace	of \mathbb{C}^n .		
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Theorem LNSMS Left Null Space of a Matrix is a Subspace	157
Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .	
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Definition RLD Relation of Linear Dependence 158 Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, ..., \mathbf{u}_n}$, an equation of the form $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$ is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is a trivial relation of linear dependence on S. If this equation of linear dependence on S. ©2005, 2006 Robert A. Beezer Column 1 Column 2

Definition LI Linear Independence

Suppose that V is a vector space. The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

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Theorem VRRB Vector Representation Relative to a Basis

Suppose that V is a vector space and $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ is a linearly independent set that spans V. Let **w** be any vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \dots, a_m$ such that

 $\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$

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Definition B	Basis					162
Suppose V is a and spans V .	vector space.	Then a subse	et $S \subseteq V$ is	a basis	of V if it is lin	nearly independent
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Theorem SUVB Standard Unit Vectors are a Basis	163
The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m \}$ $\{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .	,} =
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Theorem CNMB	Columns of Nonsing	ular Matrix a	re a Basis	164
Suppose that A is a solution only if A is nonsingular	quare matrix of size m . ar.	Then the colum	ans of A are a	basis of \mathbb{C}^m if and
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Theorem NME5 Nonsingular Matrix Equivalences, Round 5 165Suppose that A is a square matrix of size n. The following are equivalent. 1. A is nonsingular. 2. A row-reduces to the identity matrix. 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$ 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} . 5. The columns of A are a linearly independent set. 6. A is invertible. 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$. 8. The columns of A are a basis for \mathbb{C}^n . ©2005, 2006 Robert A. Beezer

Theorem COB Coordinates and Orthonormal Bases 166Suppose that $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$, $\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$

Definition D Dimension

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

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Theorem SSLD	Spanning Sets and Linear Dependence	168
	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans t + 1 or more vectors from V is linearly dependent.	he vector space V .
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Theorem BIS Bases have Identical Sizes	169
Suppose that V is a vector space with a finite basis B and a second basis C have the same size.	. Then B and C
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Theorem DCM Dimension of \mathbb{C}^m		170
The dimension of \mathbb{C}^m (Example VSCV) is m .		
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Theorem DP Dimension of P_n	171
The dimension of P_n (Example VSP) is $n + 1$.	
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Definition NOM Nullity Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

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Definition ROM Rank Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim (\mathcal{C}(A))$.

Theorem CRN Computing Rank and Nullity

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

Theorem RPNC	Rank Plus Nullity is Columns	176
Suppose that A is an	$m m \times n$ matrix. Then $r(A) + n(A) = n$.	
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Theorem RNNM Rank and Nullity of a Nonsingular Matrix

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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Theorem NME6 Nonsingular Matrix Equivalences, Round 6 Suppose that A is a square matrix of size n . The following are equivalent.	178
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of A are a basis for \mathbb{C}^n .	
9. The rank of A is $n, r(A) = n$.	
10. The nullity of A is zero, $n(A) = 0$.	
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Theorem ELIS Extending Linearly Independent Sets

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Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks 180
Suppose that V is a vector space of dimension t. Let $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ be a set of vectors from V. Then
1. If $m > t$, then S is linearly dependent.
2. If $m < t$, then S does not span V.
3. If $m = t$ and S is linearly independent, then S spans V.
4. If $m = t$ and S spans V, then S is linearly independent.
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Theorem PSSD Proper Subspaces have Smaller Dimension 181 Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.

Theorem EDYES Equal Dimensions Yields Equal Subspaces	182
Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and dim $(U \dim (V))$. Then $U = V$.	U) =
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Theorem RMRT Rank of a Matrix is the Rank of the Transpose

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

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Theorem DFS Dimensions of Four Subspaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

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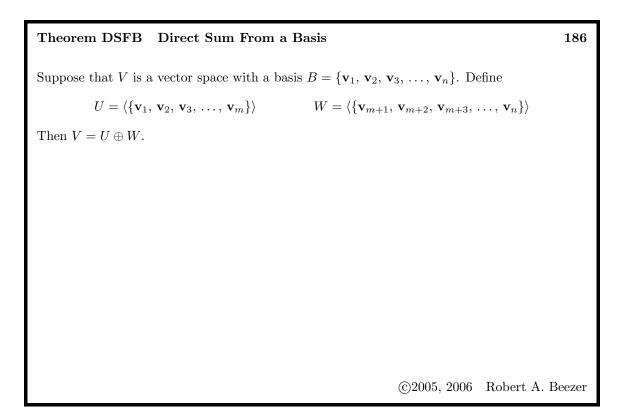
183

Definition DS Direct Sum

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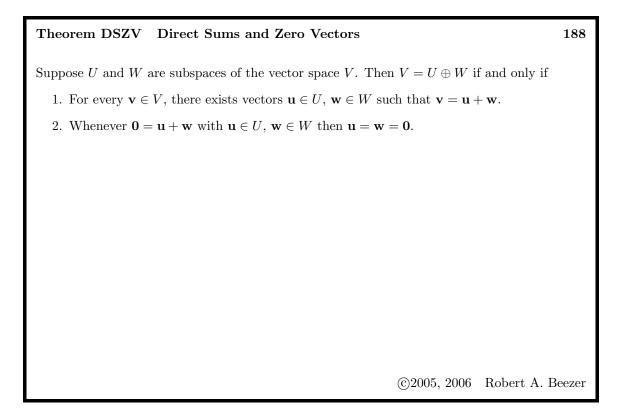
Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

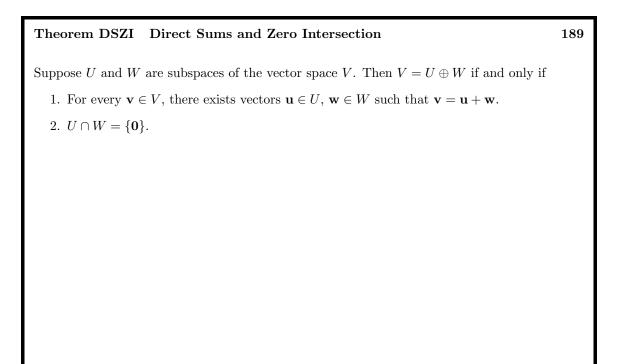
- 1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.
- Then V is the **direct sum** of U and W and we write $V = U \oplus W$.



Theorem DSFOS Direct Sum From One Subspace

Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.





Theorem DSLI Direct Sums and Linear Independence	190
Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose that R linearly independent subset of U and S is a linearly independent subset of W. Then $R \cup S$ linearly independent subset of V.	
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Theorem DSD Direct Sums and Dimension

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then dim (V) = $\dim\left(U\right) + \dim\left(W\right).$

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Theorem RDS	Repeated Direct Sums 19	92
	ector space with subspaces U and W with $V = U \oplus W$. Suppose that X and f W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.	nd
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Definition ELEM Elementary Matrices

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq \\ 1 & k \neq i, k \neq j, \ell = \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

 $k \\ k$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$\left[E_{i}\left(\alpha\right)\right]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

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Theorem EMDRO Elementary Matrices Do Row Operations

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row *i* by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.

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Theorem EMN	Elementary Matrices are Nonsingular	195
If E is an elementa	ry matrix, then E is nonsingular.	

Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices

Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \ldots, E_t$ so that $A = E_1 E_2 E_3 \ldots E_t$.

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Definition SM SubMatrix

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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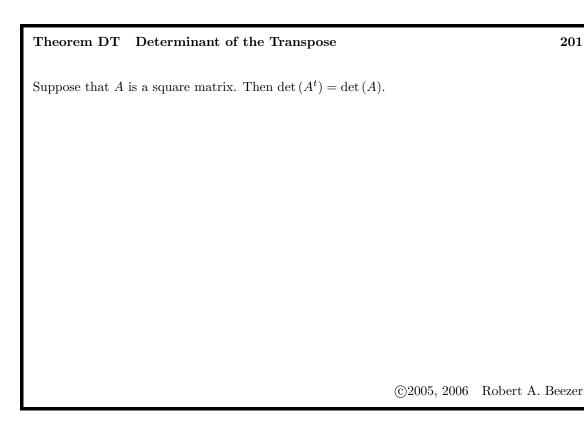
Definition DMDeterminant of a Matrix198Suppose A is a square matrix. Then its determinant, det (A) = |A|, is an element of C defined
recursively by:If A is a 1×1 matrix, then det $(A) = [A]_{11}$.If A is a matrix of size n with $n \ge 2$, thendet $(A) = [A]_{11} \det (A(1|1)) - [A]_{12} \det (A(1|2)) + [A]_{13} \det (A(1|3)) - [A]_{14} \det (A(1|4)) + \dots + (-1)^{n+1} [A]_{1n} \det (A(1|n))$ $[A]_{14} \det (A(1|4)) + \dots + (-1)^{n+1} [A]_{1n} \det (A(1|n))$ ©2005, 2006Robert A. Beezer

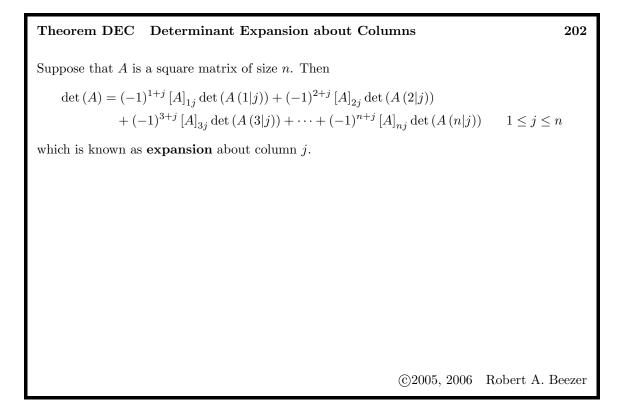
Theorem DMST Determinant of Matrices of Size Two

Suppose that
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Then det $(A) = ad - bc$

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Theorem DER Determinant Expansion about Rows	200
Suppose that A is a square matrix of size n . Then	
$\det (A) = (-1)^{i+1} [A]_{i1} \det (A(i 1)) + (-1)^{i+2} [A]_{i2} \det (A(i 2)) + (-1)^{i+3} [A]_{i3} \det (A(i 3)) + \dots + (-1)^{i+n} [A]_{in} \det (A(i n)) \qquad 1 \le i \le n$	n
which is known as expansion about row i .	
@2005_2006_Dabart A_I	Dooron
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Theorem DZRC Determinant with Zero Row or Column	203
Suppose that A is a square matrix with a row where every entry is zero, or a column every entry is zero. Then $\det(A) = 0$.	where

Theorem DRCS Determinant for Row or Column Swap

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

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Theorem DRCM Determinant for Row or Column Multiples

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Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α , or by multiplying a single column by the scalar α . Then det $(B) = \alpha \det(A)$.

Theorem DERC Determinant with Equal Rows or Columns	206
Suppose that A is a square matrix with two equal rows, or two equal columns.	Then $\det(A) = 0.$
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Theorem DRCMA Determinant for Row or Column Multiples and Addition 207

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then det $(B) = \det(A)$.

Theorem DIM	Determinant of the Identity Matrix	208
For every $n \ge 1$, d	$\det\left(I_n\right) = 1.$	
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Theorem DEM Determinants of Elementary Matrices

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. det $(E_{i,j}) = -1$
- 2. det $(E_i(\alpha)) = \alpha$
- 3. det $(E_{i,j}(\alpha)) = 1$

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Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 210

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

 $\det (EA) = \det (E) \det (A)$

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Theorem SMZD Singular Matrices have Zero Determinants

Let A be a square matrix. Then A is singular if and only if det(A) = 0.

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Theorem NME7 Nonsingular Matrix Equivalences, Round 7 Suppose that A is a square matrix of size n . The following are equivalent.	212
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of A are a basis for \mathbb{C}^n .	
9. The rank of A is $n, r(A) = n$.	
10. The nullity of A is zero, $n(A) = 0$.	
11. The determinant of A is nonzero, $\det(A) \neq 0$.	
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Theorem DRMM Determinant Respects Matrix Multiplication

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A) \det(B)$.

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Definition EEM Eigenvalues and Eigenvectors of a Matrix	214
Suppose that A is a square matrix of size $n, \mathbf{x} \neq 0$ is a vector in \mathbb{C}^n , and λ is a scalar in Then we say \mathbf{x} is an eigenvector of A with eigenvalue λ if	n C.
$A\mathbf{x} = \lambda \mathbf{x}$	
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Theorem EMHE Every Matrix Has an Eigenvalue	215
Suppose A is a square matrix. Then A has at least one eigenvalue.	
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Definition CPCharacteristic Polynomial216Suppose that A is a square matrix of size n. Then the characteristic polynomial of A is the
polynomial $p_A(x)$ defined by $p_A(x) = \det(A - xI_n)$

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Theorem EMRCP	Eigenvalues of a Matrix are Roots of Characteristic Polynomi-
als	217

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

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Definition EM Eigenspace of a Matrix

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.

Theorem EMS	Eigenspace for a Matrix is a Subspace 21	19
	hare matrix of size n and λ is an eigenvalue of A . Then the eigenspace $\mathcal{E}_A(\lambda)$ we vector space \mathbb{C}^n .	$\lambda)$
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Theorem EMNS	Eigenspace of a Matrix is a Null Space	220
Suppose A is a squa	re matrix of size n and λ is an eigenvalue of A . Then	
	$\mathcal{E}_{A}\left(\lambda\right) = \mathcal{N}(A - \lambda I_{n})$	
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Definition AME Algebraic Multiplicity of an Eigenvalue

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Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME	Geometric Multiplicity of an Eigenvalue 22	22
	square matrix and λ is an eigenvalue of A . Then the geometric mult is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.	i-
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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 223

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE	Singular Matrices have Zero Eigenvalues	224
Suppose A is a squa	are matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of	А.
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Theorem NME8Nonsingular Matrix Equivalences, Round 8Suppose that A is a square matrix of size n . The following are equivalent.	225
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	
8. The columns of A are a basis for \mathbb{C}^n .	
9. The rank of A is $n, r(A) = n$.	
10. The nullity of A is zero, $n(A) = 0$.	
11. The determinant of A is nonzero, $\det(A) \neq 0$.	
12. $\lambda = 0$ is not an eigenvalue of A.	
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Theorem ESMM	Eigenvalues of a Scalar Multiple of a Matrix 2	26
Suppose A is a square	re matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of αA .	
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Theorem EOMP	Eigenvalues Of Matrix Powers	227
Suppose A is a squa eigenvalue of A^s .	re matrix, λ is an eigenvalue of A , and $s \ge 0$ is an integr	er. Then λ^s is an
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Suppose A is a square matrix and λ is an eigenvalue of A. variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.	Let $q(x)$ be a	polynomial in the
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Theorem EPM Eigenvalues of the Polynomial of a Matrix

Theorem EIM Eigenvalues of the Inverse of a Matrix 229 Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1} . 1

Theorem ETM	Eigenvalues of the Transpose of a Matrix 230	0
Suppose A is a squ A^t .	are matrix and λ is an eigenvalue of A . Then λ is an eigenvalue of the matrix	x
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Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs 231 Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue λ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP	Degree of the Charac	teristic Polynomial		232
Suppose that A is has degree n .	a square matrix of size n	. Then the characteri	stic polyno	omial of A , $p_A(x)$,
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Theorem NEM Number of Eigenvalues of a Matrix

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Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) =$$

n

Theorem ME Multiplicities of an Eigenvalue	234
Suppose that A is a square matrix of size n and λ is an eigenvalue. Then	
$1 \le \gamma_A\left(\lambda\right) \le \alpha_A\left(\lambda\right) \le n$	
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Theorem HMOE	Hermitian Matrices have Orthogonal Ei	genvect	tors	237
	a Hermitian matrix and \mathbf{x} and \mathbf{y} are two eigen and \mathbf{y} are orthogonal vectors.	nvectors	of A for	different
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Definition SIM	Similar	Matrices
----------------	---------	----------

Suppose A and B are two square matrices of size n. Then A and B are similar if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

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Theorem SER Similarity is an Equivalence Relation 239 Suppose A, B and C are square matrices of size n. Then 1. A is similar to A. (Reflexive) 2. If A is similar to B, then B is similar to A. (Symmetric) 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Theorem SMEE Similar Matrices	have	Equal Eigenvalues		240
Suppose A and B are similar matrices. equal, that is $p_A(x) = p_B(x)$.	Then	the characteristic po	lynomia	ls of A and B are
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Definition DIM	Diagonal Matrix	241
Suppose that A is a	a square matrix. Then A is a diagonal matrix if $[A]_{ij} = 0$ whenever i	$\neq j.$
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Definition DZM	Diagonalizable Matrix	242
Suppose A is a squa	are matrix. Then A is diagonalizable if A is similar to a diagonal matrix	rix.
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Diagonalization Characterization	243
have matrix of size n . Then A is diagonalizable if and only not set S that contains n eigenvectors of A .	y if there exists a
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	hare matrix of size n . Then A is diagonalizable if and only

Theorem DMFE	Diagonalizable Mat	rices have Full Eigenspaces	244
Suppose A is a square eigenvalue λ of A .	e matrix. Then A is dia	gonalizable if and only if $\gamma_{A}(\lambda)$	$= \alpha_A(\lambda)$ for every
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Theorem DED Distinct Eigenvalues implies Diagonalizable

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

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Definition LT Linear Transformation

A linear transformation, $T: U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_{1} + \mathbf{u}_{2}) = T(\mathbf{u}_{1}) + T(\mathbf{u}_{2})$ for all $\mathbf{u}_{1}, \mathbf{u}_{2} \in U$

2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ	Linear Transformations Take Zero to Zero	248
Suppose $T \colon U \mapsto V$	is a linear transformation. Then $T(0) = 0$.	
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Theorem MBLT Matrices Build Linear Transformations 249 Suppose that *A* is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then *T* is a linear transformation.

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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors 250

Suppose that $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

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Theorem LTLC Linear Transformations and Linear Combinations

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Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

 $T(a_{1}\mathbf{u}_{1} + a_{2}\mathbf{u}_{2} + a_{3}\mathbf{u}_{3} + \dots + a_{t}\mathbf{u}_{t}) = a_{1}T(\mathbf{u}_{1}) + a_{2}T(\mathbf{u}_{2}) + a_{3}T(\mathbf{u}_{3}) + \dots + a_{t}T(\mathbf{u}_{t})$

Theorem LTDB Linear Transformation Defined on a Basis	252
Suppose that $T: U \mapsto V$ is a linear transformation, $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a basis for and w is a vector from U. Let $a_1, a_2, a_3, \dots, a_n$ be the scalars from \mathbb{C} such that	or U
$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_n\mathbf{u}_n$	
Then $T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$	
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Definition PI Pre-Image

Suppose that $T: U \mapsto V$ is a linear transformation. For each **v**, define the **pre-image** of **v** to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v} \}$$

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Definition LTA Linear Transformation Addition

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S: U \mapsto V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 255

Suppose that $T \colon U \mapsto V$ and $S \colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S \colon U \mapsto V$ is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication 256
Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the scalar multiple is the function $\alpha T: U \mapsto V$ whose outputs are defined by
$\left(\alpha T\right)\left(\mathbf{u}\right)=\alpha T\left(\mathbf{u}\right)$
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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 257

Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T): U \mapsto V$ is a linear transformation.

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Theorem VSLT Vector Space of Linear Transformations

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, LT(U, V) is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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Definition LTC Linear Transformation Composition

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Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T): U \mapsto W$ whose outputs are defined by

$$\left(S\circ T\right)\left(\mathbf{u}\right)=S\left(T\left(\mathbf{u}\right)\right)$$

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Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 260

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then $(S \circ T): U \mapsto W$ is a linear transformation.

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Definition KLT	Kernel of a Linear Transformation	262
Suppose $T \colon U \mapsto V$	' is a linear transformation. Then the kernel of T is the set	
	$\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = 0 \}$	
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Theorem KLTS Kernel of a Linear Transformation is a Subspace

Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

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Theorem KPI Kernel and Pre-Image

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

 $T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$

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Theorem KILT Kernel of an Injective Linear Transformation

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}.$

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Theorem ILTLI Injective Linear Transformations and Linear Independence 266

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

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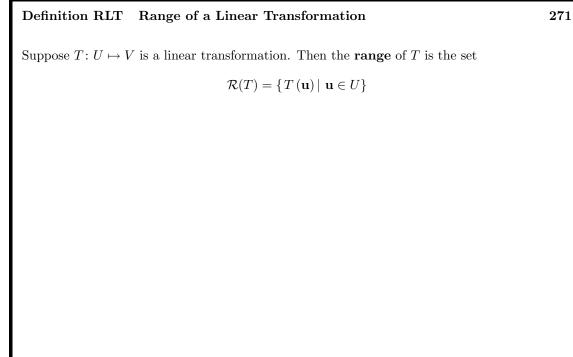
Theorem ILTB Injective Linear Transformations and Bases

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Theorem ILTD	Injective Linear Transformations and Dimension	268
Suppose that $T: U$	$U \mapsto V$ is an injective linear transformation. Then dim $(U) \leq \dim (V)$.	
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Theorem CILTI Composition of Injective Linear Transformations is Injective 269 Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are injective linear transformations. Then $(S \circ T): U \mapsto W$ is an injective linear transformation.

Definition SLT Surjective Linear Transformation	270
Suppose $T: U \mapsto V$ is a linear transformation. Then T is surjective if for every $\mathbf{v} \in V$ exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.	there
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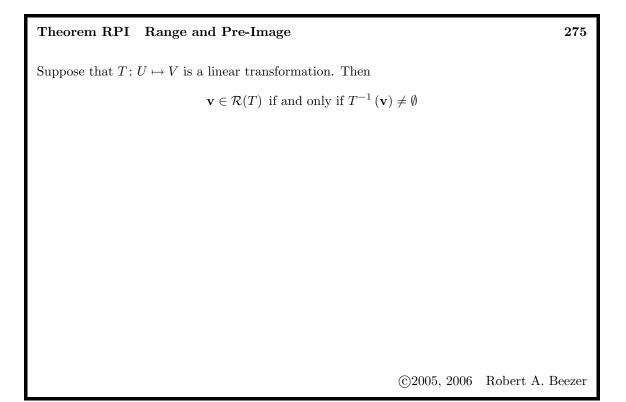
Theorem RLTSRange of a Linear Transformation is a Subspace272
Suppose that $T: U \mapsto V$ is a linear transformation. Then the range of T , $\mathcal{R}(T)$, is a subspace of V .
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Theorem RSLT Range of a Surjective Linear Transformation

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

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Theorem SSRLTSpanning Set for Range of a Linear Transformation274Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, ..., \mathbf{u}_t\}$ spans U.
Then
 $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), ..., T(\mathbf{u}_t)\}$
spans $\mathcal{R}(T)$.Spans $\mathcal{R}(T)$.



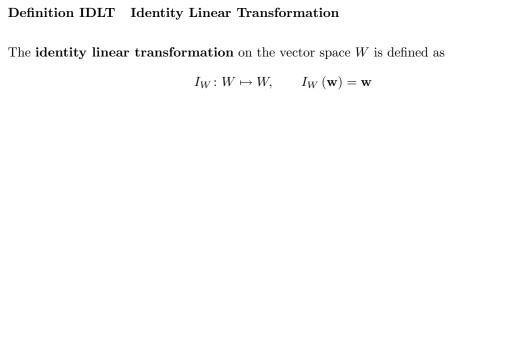
Theorem SLTB	Surjective Linear Transformations and Bases 27	76
	$V \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis exclive if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning	
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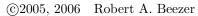
Theorem SLTD Surjective Linear Transformations and Dimension	277
Suppose that $T: U \mapsto V$ is a surjective linear transformation. Then dim $(U) \ge \dim (V)$.	

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 278

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are surjective linear transformations. Then $(S \circ T): U \mapsto W$ is a surjective linear transformation.

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Definition IVLT	Invertible Linear Transformation	ons	280
Suppose that $T: U$	$\mapsto V$ is a linear transformation. If the	ere is a function $S \colon V \mapsto U$ suc	h that
	$S \circ T = I_U$	$T \circ S = I_V$	
then T is invertible	e . In this case, we call S the inverse	of T and write $S = T^{-1}$.	
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Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 281
Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}: V \mapsto U$ is a linear transformation.

Theorem IILT	Inverse of an Invertible Linea	ar Transformation	282
Suppose that $T: U$ transformation and	$\mapsto V$ is an invertible linear transfer $(T^{-1})^{-1} = T.$	ormation. Then T^{-1}	is an invertible linear
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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective283

Suppose $T: U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT	Composition of Invertible Linear Transformations	284	
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. composition, $(S \circ T): U \mapsto W$ is an invertible linear transformation.			
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Theorem ICLT Inverse of a Composition of Linear Transforma	ations 285
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformation invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.	ations. Then $S \circ T$ is

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Definition IVS Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain $V, T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

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Theorem IVSED	Isomorphic Vector Spaces have Equal Dimension 28	37
Suppose U and V are	re isomorphic vector spaces. Then $\dim(U) = \dim(V)$.	
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Definition ROLT	Rank Of a Linear Transformation 2	288
Suppose that $T: U \vdash$ of the range of T ,	→ V is a linear transformation. Then the rank of T, $r(T)$, is the dimension $r(T) = \dim(\mathcal{R}(T))$	ion
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Definition NOLT Nullity Of a	a Linear Transform	mation	289
Suppose that $T: U \mapsto V$ is a line dimension of the kernel of T ,	ear transformation. $n\left(T\right) = \dim\left(\mathcal{K}(T)\right)$		of T , $n(T)$, is the
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Theorem ROSLT Rank Of a Surjective Linear Transformation	290
Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the $r(T) = \dim(V)$, if and only if T is surjective.	ne dimension of V ,
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Theorem NOILTNullity Of an Injective Linear Transformation291Suppose that $T: U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero,
n(T) = 0, if and only if T is injective.

Theorem RPNDD Rank Plus Nullity is Domain Dimension	292
Suppose that $T: U \mapsto V$ is a linear transformation. Then	
$r(T) + n(T) = \dim(U)$	
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Definition VR Vector Representation

Suppose that V is a vector space with a basis $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$. Define a function $\rho_B \colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \dots, a_n$ so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i \qquad \qquad 1 \le i \le n$$

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Theorem VRLT	Vector Representation is a Linear Transformat	ion 294
The function ρ_B (D	Definition VR) is a linear transformation.	
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Theorem VRI Vector Representation is Injective	295
The function ρ_B (Definition VR) is an injective linear transformation.	
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Theorem VRS Vector Representation is Surjective	296
The function ρ_B (Definition VR) is a surjective linear transformation.	
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Theorem VRILTVector Representation is an Invertible Linear Transformation297The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS	Characterization of Finite Dimensional Vector Spaces	298
Suppose that V is a x	vector space with dimension n . Then V is isomorphic to \mathbb{C}^n .	
Suppose that V is a v	vector space with dimension n . Then V is isomorphic to \mathbb{C} .	
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Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces

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Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

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Theorem CLI	Coordinatization and Linear Independence 3	800
linearly independ	is a vector space with a basis <i>B</i> of size <i>n</i> . Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is dent subset of <i>U</i> if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}$ spendent subset of \mathbb{C}^n .	s a k
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Theorem CSS Coordinatization and Spanning Sets

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Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\} \rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\} \rangle$.

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Definition MR Matrix Representation

Suppose that $T: U \mapsto V$ is a linear transformation, $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

 $M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$

Theorem FTMR Fundamental Theorem of Matrix Representation

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

 $T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations304

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

 $M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 305

Suppose that $T: U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 306

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

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Theorem KNSI Kernel and Null Space Isomorphism

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism	308
Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n , and C for V of size m . Then the range of T is isomorphic to the column space of $M_{B,C}^T$,	C is a basis
$\mathcal{R}(T) \cong \mathcal{C}\left(M_{B,C}^T\right)$	
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Theorem IMR Invertible Matrix Representations

Suppose that $T: U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^{T}$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^-$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation	310
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Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

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Theorem NME9 Nonsingular Matrix Equivalences, Round 9 Suppose that A is a square matrix of size n . The following are equivalent.	311	
1. A is nonsingular.		
2. A row-reduces to the identity matrix.		
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$		
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .		
5. The columns of A are a linearly independent set.		
6. A is invertible.		
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.		
8. The columns of A are a basis for \mathbb{C}^n .		
9. The rank of A is $n, r(A) = n$.		
10. The nullity of A is zero, $n(A) = 0$.		
11. The determinant of A is nonzero, $\det(A) \neq 0$.		
12. $\lambda = 0$ is not an eigenvalue of A.		
13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.		
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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation 312

Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

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Definition CBM Change-of-Basis Matrix

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

= $[\rho_C (I_V (\mathbf{v}_1)) | \rho_C (I_V (\mathbf{v}_2)) | \rho_C (I_V (\mathbf{v}_3)) | \dots | \rho_C (I_V (\mathbf{v}_n))]$
= $[\rho_C (\mathbf{v}_1) | \rho_C (\mathbf{v}_2) | \rho_C (\mathbf{v}_3) | \dots | \rho_C (\mathbf{v}_n)]$

Theorem CB Change-of-Basis	314
Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then	
$\rho_{C}\left(\mathbf{v}\right) = C_{B,C}\rho_{B}\left(\mathbf{v}\right)$	
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Theorem ICBM Inverse of Change-of-Basis Matrix	315
Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis m $C_{B,C}$ is nonsingular and $C_{B,C}^{-1} = C_{C,B}$	atrix
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Theorem MRCB	Matrix Representation and Change of Basis 316
	→ V is a linear transformation, B and C are bases for U, and D and E are $M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$
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Theorem SCB Similarity and Change of Basis Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then $M_{B,B}^{T} = C_{B,C}^{-1} M_{C,C}^{T} C_{B,C}$ ©2005, 2006 Robert A. Beezer

Theorem EER Eigenvalues, Eigenvectors, Representations

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Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V . Then $\mathbf{v} \in V$ i eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for eigenvalue λ .	
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Definition NLT Nilpotent Linear Transformation

Suppose that $T: V \mapsto V$ is a linear transformation such that there is an integer p > 0 such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.

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 Definition JB Jordan Block
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 Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by
 $[J_n(\lambda)]_{ij} = \begin{cases} \lambda & i = j \\ 1 & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$

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Theorem NJB Nilpotent Jordan Blocks	321
The Jordan block $J_{n}(0)$ is nilpotent of index n .	

Theorem ENLT	Eigenvalues of Nilpotent Linear Transformations 322	2
Suppose that $T: V$	$\mapsto V$ is a linear transformation and λ is an eigenvalue of T . Then $\lambda = 0$.	
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Theorem DNLT	Diagonalizable Nilpotent Linear Transformations	323
Suppose the linear tr T is the zero linear t	transformation $T: V \mapsto V$ is nilpotent. Then T is diagonalizable if and transformation.	only

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Theorem KPLT Kernels of Powers of Linear Transformations 324 Suppose $T: V \mapsto V$ is a linear transformation, where dim (V) = n. Then there is an integer m, $0 \le m \le n$, such that $\{0\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$ $\{0\} = \mathcal{K}(T^0) \backsim \mathcal{K}(T^1) \backsim \mathcal{K}(T^2) \backsim \mathcal{K}(T^2) \lor \mathcal{K}(T^m) = \mathcal{K}(T^m$

Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 325

Suppose $T: V \mapsto V$ is a nilpotent linear transformation with index p and dim (V) = n. Then $0 \le p \le n$ and

 $\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$

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Theorem CFNLT Canonical Form for Nilpotent Linear Transformations 326

Suppose that $T: V \mapsto V$ is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation, $M_{B,B}^T$, is block diagonal with each block being a Jordan block, $J_n(0)$. The size of the largest block is the index p, and the total number of blocks is the nullity of T, n(T).

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Definition CNE	Complex Number Equality	327
The complex numb	ers $\alpha = a + bi$ and $\beta = c + di$ are equal , denoted $\alpha = \beta$, if $a = c$ and $b = c$	d.
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Definition CNA	Complex Number Addition	328
The \mathbf{sum} of the com	applex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + c)$	l)i.
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Definition CNM Complex Number Multiplication

The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is (ac - bd) + (ad + bc)i.

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Theorem PCNAProperties of Complex Number Arithmetic330The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Commutativity, Complex Numbers For any $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$.
- MCCN Multiplicative Commutativity, Complex Numbers For any $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}, \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers For any α , β , $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}, 0 + \alpha = \alpha$.
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha(\frac{1}{\alpha}) = 1$.

Definition CCN	Conjugate of a Complex Number	331
The conjugate of t	the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.	
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Theorem CCRA	Complex Conjugation Respects Addition	332
Suppose that c and	d are complex numbers. Then $\overline{c+d} = \overline{c} + \overline{d}$.	
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Theorem CCRM	Complex Conjugation Respects	Multiplication	333
Suppose that c and	d are complex numbers. Then $\overline{cd} = \overline{cd}$	Ī.	
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Theorem CCT Complex Conjugation Twice	334
Suppose that c is a complex number. Then $\overline{\overline{c}} = c$.	
Suppose that c is a complex number. Then c – c.	
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Definition MCN Modulus of a Complex Number The modulus of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number $|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}$.

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Definition SET	Set	336
	red collection of objects. If S is a set and x is an object t x is not in S, then we write $x \notin S$. We refer to the obj	
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Definition SSET	Subset		337
If S and T are two s	ets, then S is a subset of T, written $S \subseteq \mathcal{I}$	T if whenever $x \in$	S then $x \in T$.
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Definition ES	Empty Set	338		
The empty set is	The empty set is the set with no elements. Its is denoted by \emptyset .			
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Definition SESet Equality339Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write S = T. $(C_2005, 2006)$ Robert A. Beezer

Definition C	Cardinality	340
	finite set. Then the number of elements in S is called the cardinality or s	
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Definition SU Set Union

Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$

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Definition SI Set Intersection	342	
Suppose S and T are sets. Then the intersection of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,		
$x \in S \cap T$ if and only if $x \in S$ and $x \in T$		
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Definition SC Set Complement

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$