# Flash Cards

to accompany

# A First Course in Linear Algebra

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Version 1.00

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The most recent version of this work can always be found at http://linear.ups.edu.

## Definition SLE System of Linear Equations

1

A system of linear equations is a collection of m equations in the variable quantities  $x_1, x_2, x_3, \ldots, x_n$  of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of  $a_{ij}$ ,  $b_i$  and  $x_j$  are from the set of complex numbers,  $\mathbb{C}$ .

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## Definition ESYS Equivalent Systems

 $\mathbf{2}$ 

Two systems of linear equations are **equivalent** if their solution sets are equal.

## Definition EO Equation Operations

3

Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

- 1. Swap the locations of two equations in the list.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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#### Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M Matrix 5

An  $m \times n$  matrix is a rectangular layout of numbers from  $\mathbb{C}$  having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation  $[A]_{ij}$  will refer to the complex number in row i and column j of A.

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#### Definition CV Column Vector

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A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{z}$ . Some books like to write vectors with arrows, such as  $\vec{u}$ . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the **entry** or **component** that is number i in the list that is the vector  $\mathbf{v}$  we write  $[\mathbf{v}]_i$ .

#### Definition ZCV Zero Column Vector

7

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly,  $[\mathbf{0}]_i = 0$  for  $1 \le i \le m$ .

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#### Definition CM Coefficient Matrix

3

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

#### Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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#### Definition SV Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

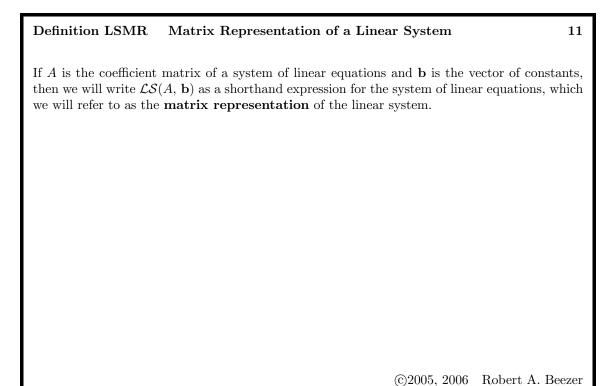
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



## Definition AM Augmented Matrix

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Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants  $\mathbf{b}$ . Then the **augmented matrix** of the system of equations is the  $m \times (n+1)$  matrix whose first n columns are the columns of A and whose last column (number n+1) is the column vector  $\mathbf{b}$ . This matrix will be written as  $[A \mid \mathbf{b}]$ .

## Definition RO Row Operations

**13** 

The following three operations will transform an  $m \times n$  matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

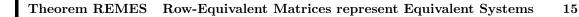
- 1.  $R_i \leftrightarrow R_j$ : Swap the location of rows i and j.
- 2.  $\alpha R_i$ : Multiply row i by the nonzero scalar  $\alpha$ .
- 3.  $\alpha R_i + R_j$ : Multiply row i by the scalar  $\alpha$  and add to row j.

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## Definition REM Row-Equivalent Matrices

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Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.



Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

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## Definition RREF Reduced Row-Echelon Form

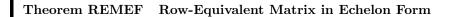
16

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  where  $d_1 < d_2 < d_3 < \cdots < d_r$ , while the columns that are not pivot columns will be denoted as  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  where  $f_1 < f_2 < f_3 < \cdots < f_{n-r}$ .



Suppose A is a matrix. Then there is a matrix B so that

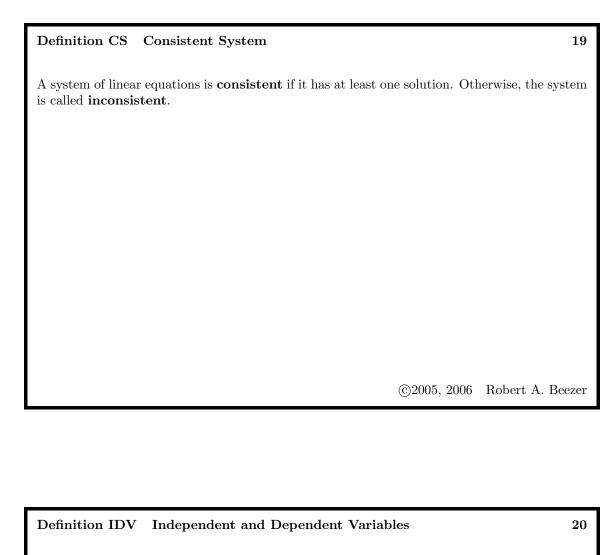
- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

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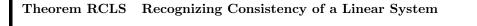
## Definition RR Row-Reducing

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To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.



Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable  $x_j$  is **dependent**. A variable that is not dependent is called **independent** or **free**.



 $\mathbf{21}$ 

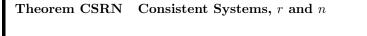
Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n+1 of B.

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## Theorem ISRN Inconsistent Systems, r and n

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Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.



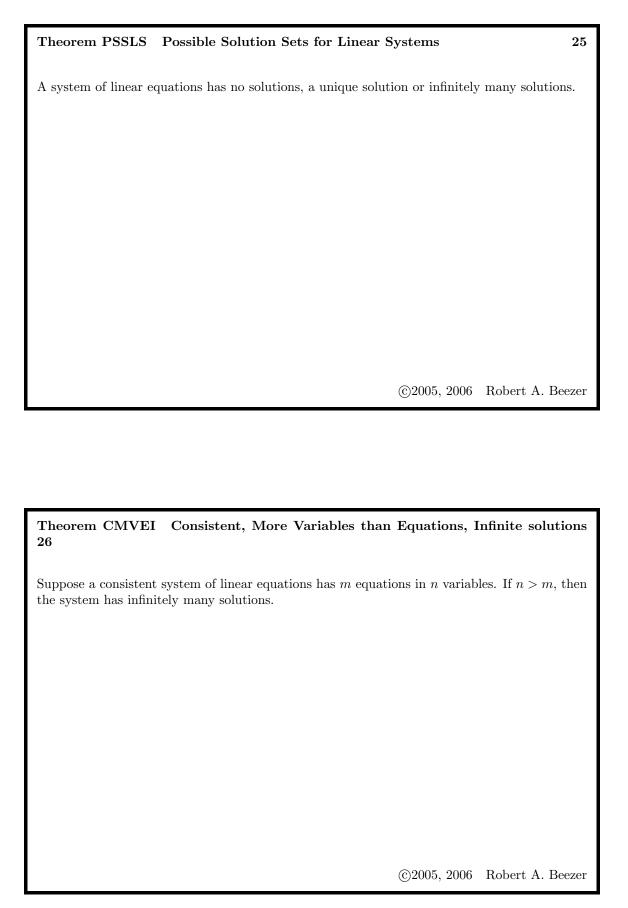
Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then  $r \leq n$ . If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

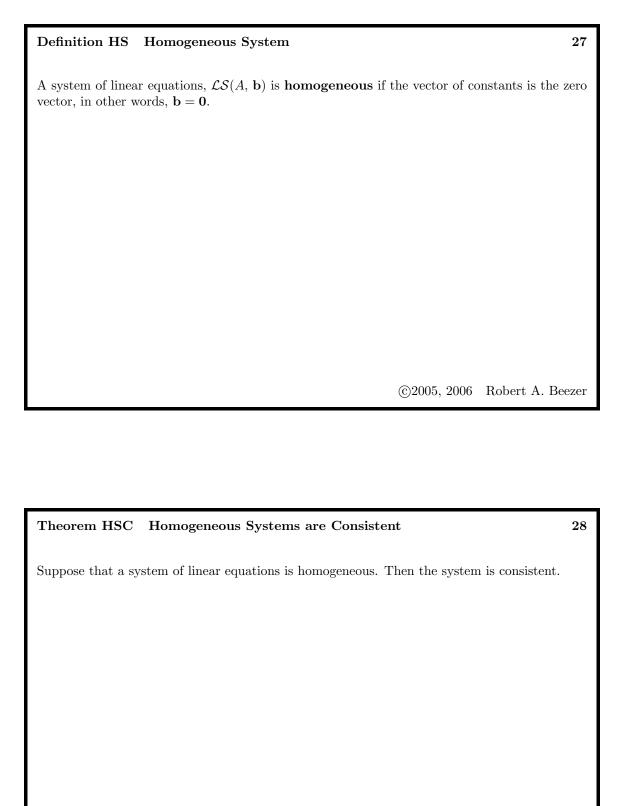
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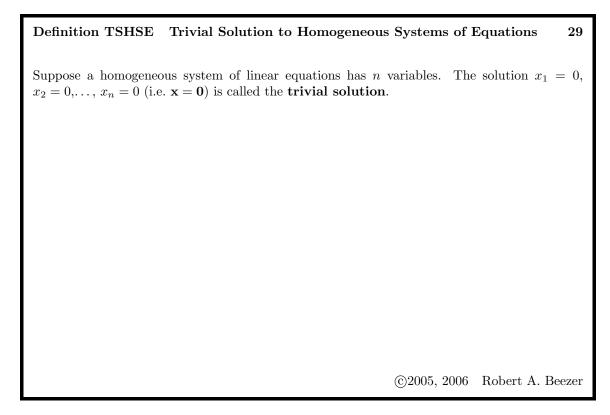
## Theorem FVCS Free Variables for Consistent Systems

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Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.

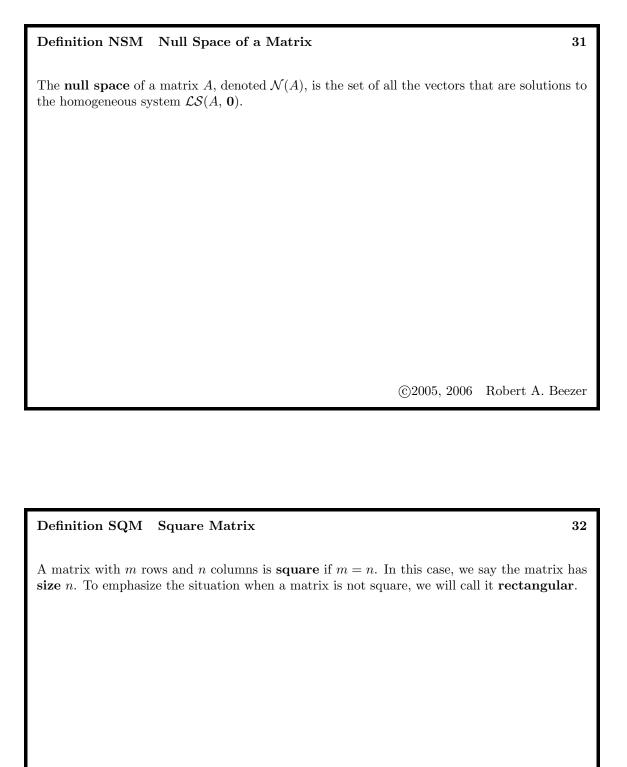


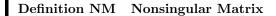




# Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions $\phantom{\Big|}30$

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.





Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations  $\mathcal{LS}(A, \mathbf{0})$  is  $\{\mathbf{0}\}$ , i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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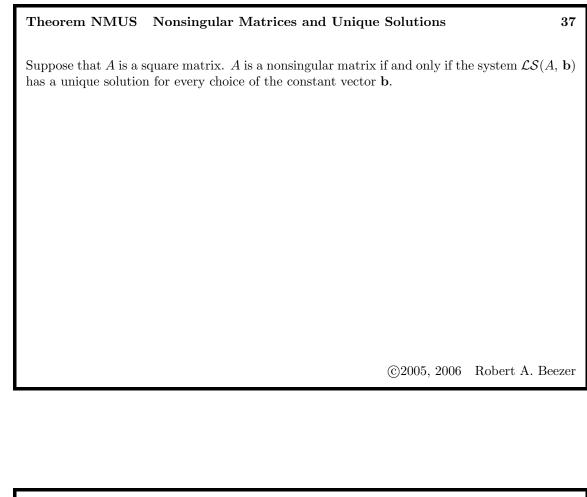
# Definition IM Identity Matrix

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The  $m \times m$  identity matrix,  $I_m$ , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem NMRRI	Nonsingular Matrices Row Reduce to the Identity matrix 35
	square matrix and $B$ is a row-equivalent matrix in reduced row-echelon
form. Then A is nons	singular if and only if $B$ is the identity matrix.
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Theorem NMTNS	Nonsingular Matrices have Trivial Null Spaces 36
Suppose that $A$ is a s	Nonsingular Matrices have Trivial Null Spaces 36 square matrix. Then $A$ is nonsingular if and only if the null space of $A$ , the zero vector, i.e. $\mathcal{N}(A) = \{0\}.$
Suppose that $A$ is a s	square matrix. Then $A$ is nonsingular if and only if the null space of $A$ ,
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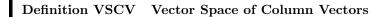


## Theorem NME1 Nonsingular Matrix Equivalences, Round 1

**38** 

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .



The vector space  $\mathbb{C}^m$  is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers,  $\mathbb{C}$ .

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# Definition CVE Column Vector Equality

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The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are **equal**, written  $\mathbf{u} = \mathbf{v}$  provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

# Definition CVA Column Vector Addition

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Given the vectors  $\mathbf{u}$  and  $\mathbf{v}$  the  $\mathbf{sum}$  of  $\mathbf{u}$  and  $\mathbf{v}$  is the vector  $\mathbf{u} + \mathbf{v}$  defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

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# Definition CVSM Column Vector Scalar Multiplication

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Given the vector  $\mathbf{u}$  and the scalar  $\alpha \in \mathbb{C}$ , the scalar multiple of  $\mathbf{u}$  by  $\alpha$ ,  $\alpha \mathbf{u}$  is defined by

$$[\alpha \mathbf{u}]_i = \alpha \left[ \mathbf{u} \right]_i$$

$$1 \leq i \leq m$$

#### Theorem VSPCV Vector Space Properties of Column Vectors

Suppose that  $\mathbb{C}^m$  is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ .
- SCC Scalar Closure, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha \mathbf{u} \in \mathbb{C}^m$ .
- CC Commutativity, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AAC Additive Associativity, Column Vectors If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- **ZC Zero Vector, Column Vectors** There is a vector, **0**, called the **zero vector**, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in \mathbb{C}^m$ .
- AIC Additive Inverses, Column Vectors If  $\mathbf{u} \in \mathbb{C}^m$ , then there exists a vector  $-\mathbf{u} \in \mathbb{C}^m$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMAC Scalar Multiplication Associativity, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVAC Distributivity across Vector Addition, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSAC Distributivity across Scalar Addition, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ .
- $\bullet$  OC One Column Vectors If  $u \in \mathbb{C}^m$  then 1u = u

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#### Definition LCCV Linear Combination of Column Vectors

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Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  from  $\mathbb{C}^m$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their **linear** combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

Denote the columns of the  $m \times n$  matrix A as the vectors  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ . Then  $\mathbf{x}$  is a solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$  if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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#### Theorem VFSLS Vector Form of Solutions to Linear Systems

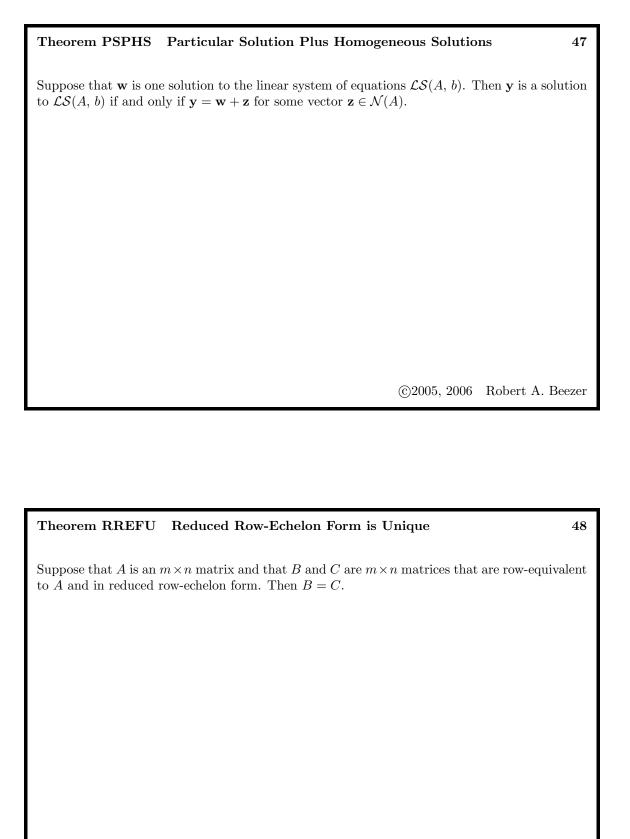
46

Suppose that  $[A \mid \mathbf{b}]$  is the augmented matrix for a consistent linear system  $\mathcal{LS}(A, \mathbf{b})$  of m equations in n variables. Let B be a row-equivalent  $m \times (n+1)$  matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$ , and columns with leading 1's (pivot columns) having indices  $D = \{d_1, d_2, d_3, \ldots, d_r\}$ . Define vectors  $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$  of size n by

$$\begin{aligned} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} . \end{aligned}$$

Then the set of solutions to the system of equations  $\mathcal{LS}(A, \mathbf{b})$  is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \right\}$$



Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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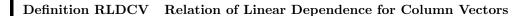
#### Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the column indices where B has leading 1's (pivot columns) and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the set of column indices where B does not have leading 1's. Construct the n-r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n-r$  of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle$$
.



Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ , a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

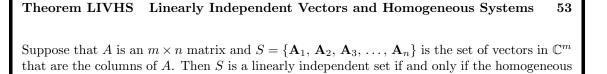
is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is the **trivial relation of linear dependence** on S.

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## Definition LICV Linear Independence of Column Vectors

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The set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.



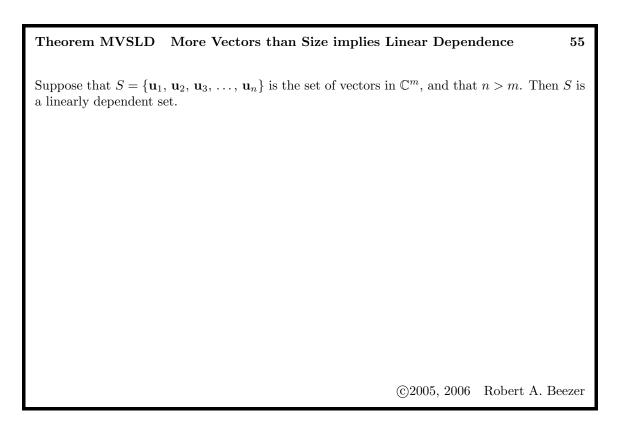
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## Theorem LIVRN Linearly Independent Vectors, r and n

system  $\mathcal{LS}(A, \mathbf{0})$  has a unique solution.

**54** 

Suppose that A is an  $m \times n$  matrix and  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.



## Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 56

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A form a linearly independent set.

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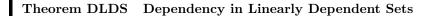
#### Theorem BNS Basis for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n-r$  of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set  $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$ . Then

- 1.  $\mathcal{N}(A) = \langle S \rangle$ .
- 2. S is a linearly independent set.



Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors. Then S is a linearly dependent set if and only if there is an index  $t, 1 \le t \le n$  such that  $\mathbf{u_t}$  is a linear combination of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$ .

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## Theorem BS Basis of a Span

60

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  is a set of column vectors. Define  $W = \langle S \rangle$  and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with  $D = \{d_1, d_2, d_3, \dots, d_r\}$  the set of column indices corresponding to the pivot columns of B. Then

- 1.  $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$  is a linearly independent set.
- 2.  $W = \langle T \rangle$ .

# Definition CCCV Complex Conjugate of a Column Vector

61

Suppose that **u** is a vector from  $\mathbb{C}^m$ . Then the conjugate of the vector,  $\overline{\mathbf{u}}$ , is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

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# Theorem CRVA Conjugation Respects Vector Addition

62

Suppose  $\mathbf{x}$  and  $\mathbf{y}$  are two vectors from  $\mathbb{C}^m$ . Then

$$\overline{\mathbf{x}+\mathbf{y}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$$



Suppose **x** is a vector from  $\mathbb{C}^m$ , and  $\alpha \in \mathbb{C}$  is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

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## Definition IP Inner Product

64

Given the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  the **inner product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity in  $\mathbb{C}$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[ \mathbf{u} \right]_1 \overline{\left[ \mathbf{v} \right]_1} + \left[ \mathbf{u} \right]_2 \overline{\left[ \mathbf{v} \right]_2} + \left[ \mathbf{u} \right]_3 \overline{\left[ \mathbf{v} \right]_3} + \dots + \left[ \mathbf{u} \right]_m \overline{\left[ \mathbf{v} \right]_m} = \sum_{i=1}^m \left[ \mathbf{u} \right]_i \overline{\left[ \mathbf{v} \right]_i}$$

## Theorem IPVA Inner Product and Vector Addition

65

Suppose  $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ . Then

- 1.  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- 2.  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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## 

66

Suppose  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ . Then

- 1.  $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- 2.  $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$

		_	_	_				
Theorem IP.	$\Delta C$	Inner	Pro	duct	is /	∆nti_	Comn	nutative

Suppose that **u** and **v** are vectors in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ .

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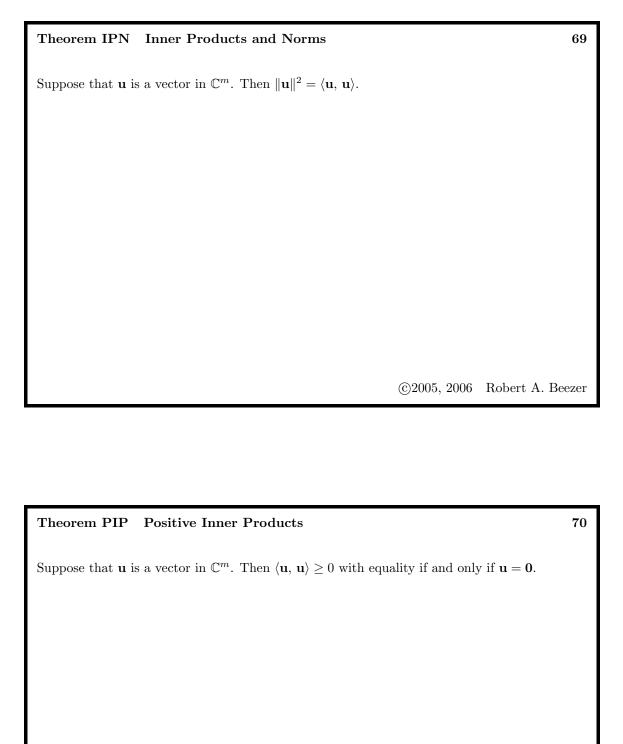
## Definition NV Norm of a Vector

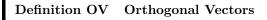
**68** 

67

The  $\bf{norm}$  of the vector  $\bf{u}$  is the scalar quantity in  $\mathbb C$ 

$$\|\mathbf{u}\| = \sqrt{\left|\left[\mathbf{u}\right]_{1}\right|^{2} + \left|\left[\mathbf{u}\right]_{2}\right|^{2} + \left|\left[\mathbf{u}\right]_{3}\right|^{2} + \dots + \left|\left[\mathbf{u}\right]_{m}\right|^{2}} = \sqrt{\sum_{i=1}^{m} \left|\left[\mathbf{u}\right]_{i}\right|^{2}}$$





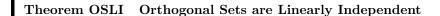
A pair of vectors, **u** and **v**, from  $\mathbb{C}^m$  are **orthogonal** if their inner product is zero, that is,  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .

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## Definition OSV Orthogonal Set of Vectors

72

Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors from  $\mathbb{C}^m$ . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  whenever  $i \neq j$ .



Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is an orthogonal set of nonzero vectors. Then S is linearly independent.

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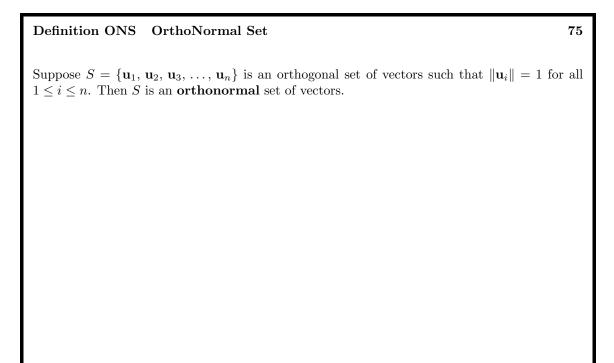
#### Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

74

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$  is a linearly independent set of vectors in  $\mathbb{C}^m$ . Define the vectors  $\mathbf{u}_i$ ,  $1 \le i \le p$  by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if  $T = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_p\}$ , then T is an orthogonal set of non-zero vectors, and  $\langle T \rangle = \langle S \rangle$ .

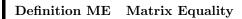


## Definition VSM Vector Space of $m \times n$ Matrices

**76** 

The vector space  $M_{mn}$  is the set of all  $m \times n$  matrices with entries from the set of complex numbers.

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The  $m \times n$  matrices A and B are **equal**, written A = B provided  $[A]_{ij} = [B]_{ij}$  for all  $1 \le i \le m$ ,  $1 \le j \le n$ .

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#### Definition MA Matrix Addition

**78** 

Given the  $m \times n$  matrices A and B, define the **sum** of A and B as an  $m \times n$  matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

Given the  $m \times n$  matrix A and the scalar  $\alpha \in \mathbb{C}$ , the **scalar multiple** of A is an  $m \times n$  matrix, written  $\alpha A$  and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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#### Theorem VSPM Vector Space Properties of Matrices

80

Suppose that  $M_{mn}$  is the set of all  $m \times n$  matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If  $A, B \in M_{mn}$ , then  $A + B \in M_{mn}$ .
- SCM Scalar Closure, Matrices If  $\alpha \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha A \in M_{mn}$ .
- CM Commutativity, Matrices If  $A, B \in M_{mn}$ , then A + B = B + A.
- AAM Additive Associativity, Matrices If  $A, B, C \in M_{mn}$ , then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix,  $\mathcal{O}$ , called the zero matrix, such that  $A + \mathcal{O} = A$  for all  $A \in M_{mn}$ .
- AIM Additive Inverses, Matrices If  $A \in M_{mn}$ , then there exists a matrix  $-A \in M_{mn}$  so that  $A + (-A) = \mathcal{O}$ .
- SMAM Scalar Multiplication Associativity, Matrices If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha(\beta A) = (\alpha \beta)A$ .
- DMAM Distributivity across Matrix Addition, Matrices If  $\alpha \in \mathbb{C}$  and  $A, B \in M_{mn}$ , then  $\alpha(A+B) = \alpha A + \alpha B$ .
- DSAM Distributivity across Scalar Addition, Matrices If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $(\alpha + \beta)A = \alpha A + \beta A$ .
- OM One Matrices If  $A \in M_{min}$  then 1A = A



The  $m \times n$  zero matrix is written as  $\mathcal{O} = \mathcal{O}_{m \times n}$  and defined by  $[\mathcal{O}]_{ij} = 0$ , for all  $1 \leq i \leq m$ ,  $1 \leq i \leq n$ .

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## Definition TM Transpose of a Matrix

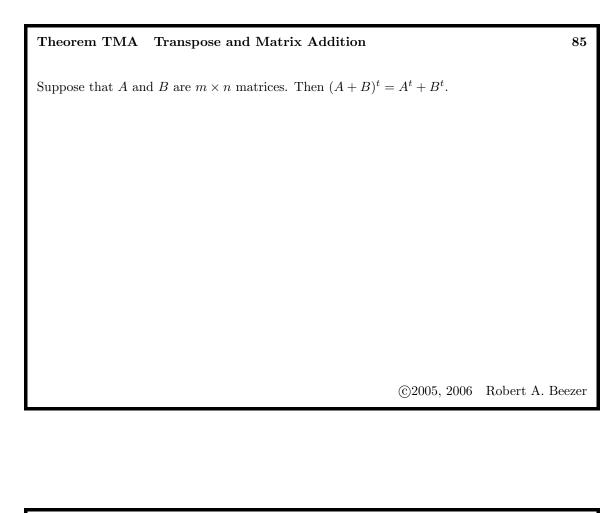
82

Given an  $m \times n$  matrix A, its **transpose** is the  $n \times m$  matrix  $A^t$  given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM	Symmetric Matrix		83
The matrix $A$ is sy	vmmetric if $A = A^t$ .		
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Theorem SMS	Symmetric Matrices are Square		84

Suppose that A is a symmetric matrix. Then A is square.



## Theorem TMSM Transpose and Matrix Scalar Multiplication

86

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $(\alpha A)^t = \alpha A^t$ .



Suppose that A is an  $m \times n$  matrix. Then  $(A^t)^t = A$ .

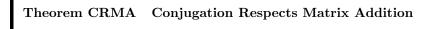
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## Definition CCM Complex Conjugate of a Matrix

88

Suppose A is an  $m \times n$  matrix. Then the **conjugate** of A, written  $\overline{A}$  is an  $m \times n$  matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$



Suppose that A and B are  $m \times n$  matrices. Then  $\overline{A+B} = \overline{A} + \overline{B}$ .

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## Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

90

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $\overline{\alpha A} = \overline{\alpha} \overline{A}$ .



Suppose that A is an  $m \times n$  matrix. Then  $\overline{(A^t)} = \left(\overline{A}\right)^t$ .

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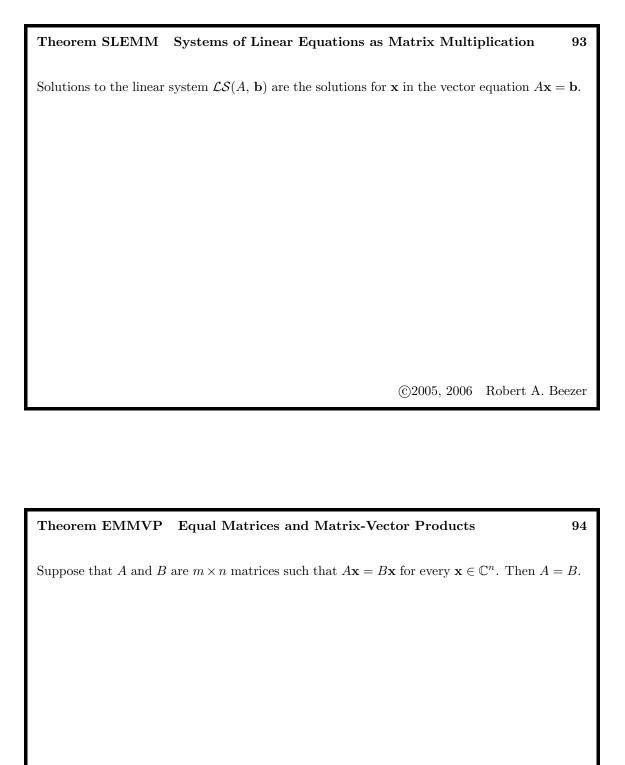
#### Definition MVP Matrix-Vector Product

**92** 

91

Suppose A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$  and  $\mathbf{u}$  is a vector of size n. Then the **matrix-vector product** of A with  $\mathbf{u}$  is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$



#### Definition MM Matrix Multiplication

95

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix with columns  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ . Then the **matrix product** of A with B is the  $m \times p$  matrix where column i is the matrix-vector product  $A\mathbf{B}_i$ . Symbolically,

$$AB = A \left[ \mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[ A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

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#### Theorem EMP Entries of Matrix Products

96

Suppose A is an  $m \times n$  matrix and B =is an  $n \times p$  matrix. Then for  $1 \le i \le m, \ 1 \le j \le p,$  the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

## Theorem MMZM Matrix Multiplication and the Zero Matrix

97

Suppose A is an  $m \times n$  matrix. Then

- 1.  $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$ 2.  $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

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## Theorem MMIM Matrix Multiplication and Identity Matrix

98

Suppose A is an  $m \times n$  matrix. Then

- 1.  $AI_n = A$
- $2. \quad I_m A = A$

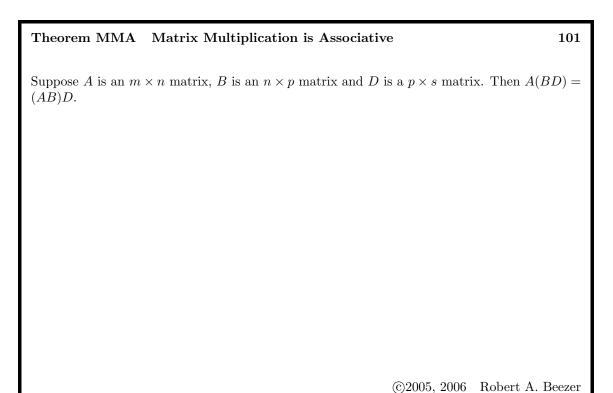
Suppose A is an  $m \times n$  matrix and B and C are  $n \times p$  matrices and D is a  $p \times s$  matrix. Then

- $1. \quad A(B+C) = AB + AC$
- $2. \quad (B+C)D = BD + CD$

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#### Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 100

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Let  $\alpha$  be a scalar. Then  $\alpha(AB) = (\alpha A)B = A(\alpha B)$ .

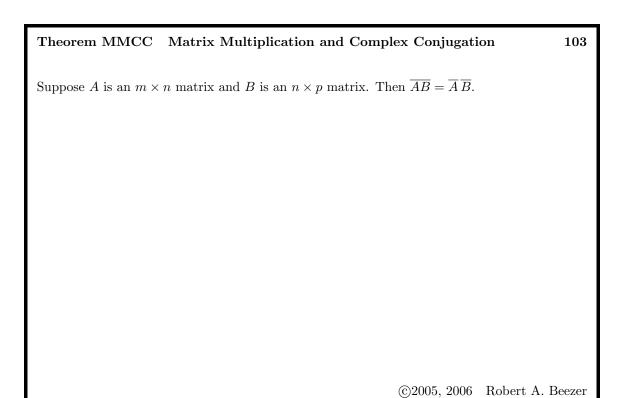


## Theorem MMIP Matrix Multiplication and Inner Products

102

If we consider the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  as  $m \times 1$  matrices then

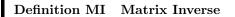
$$\langle \mathbf{u}, \, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$



# Theorem MMT Matrix Multiplication and Transposes

104

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then  $(AB)^t = B^t A^t$ .



Suppose A and B are square matrices of size n such that  $AB = I_n$  and  $BA = I_n$ . Then A is **invertible** and B is the **inverse** of A. In this situation, we write  $B = A^{-1}$ .

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#### Definition SUV Standard Unit Vectors

106

Let  $\mathbf{e}_j \in \mathbb{C}^m$  denote the column vector that is column j of the  $m \times m$  identity matrix  $I_m$ . Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j \mid 1 \le j \le m\}$$

is the set of standard unit vectors in  $\mathbb{C}^m$ .

#### Theorem TTMI Two-by-Two Matrix Inverse

107

108

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

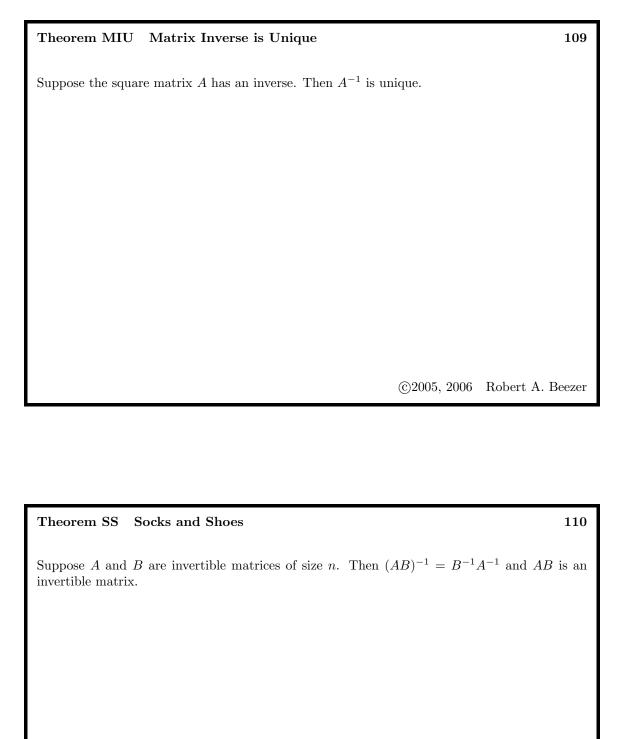
Then A is invertible if and only if  $ad - bc \neq 0$ . When A is invertible, we have

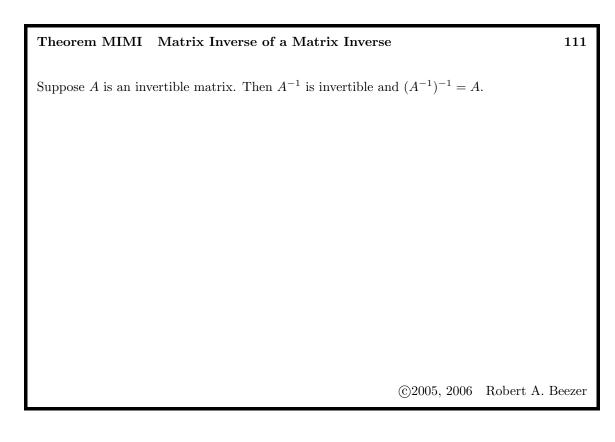
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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#### Theorem CINM Computing the Inverse of a Nonsingular Matrix

Suppose A is a nonsingular square matrix of size n. Create the  $n \times 2n$  matrix M by placing the  $n \times n$  identity matrix  $I_n$  to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then  $AJ = I_n$ .

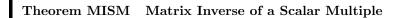




# Theorem MIT Matrix Inverse of a Transpose

112

Suppose A is an invertible matrix. Then  $A^t$  is invertible and  $(A^t)^{-1} = (A^{-1})^t$ .



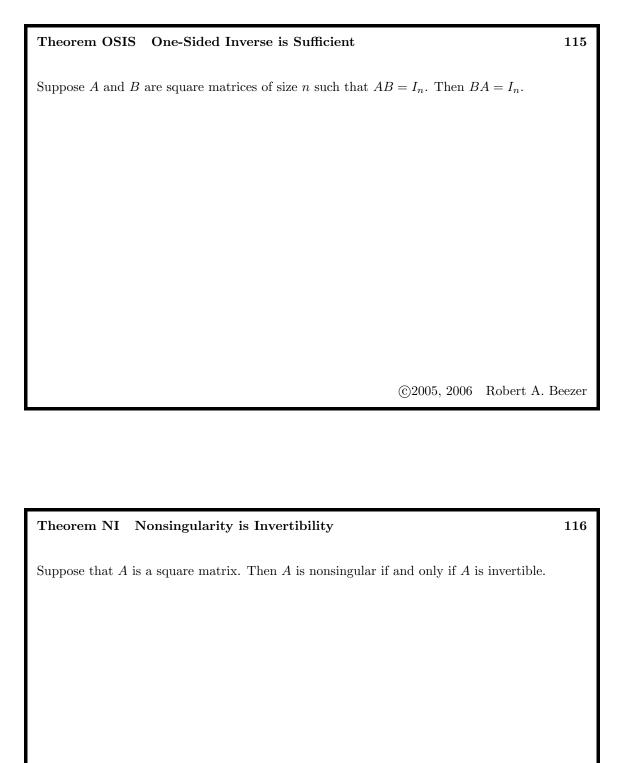
Suppose A is an invertible matrix and  $\alpha$  is a nonzero scalar. Then  $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$  and  $\alpha A$  is invertible.

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## Theorem NPNT Nonsingular Product has Nonsingular Terms

114

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.



#### Theorem NME3 Nonsingular Matrix Equivalences, Round 3

117

Suppose that A is a square matrix of size n. The following are equivalent.

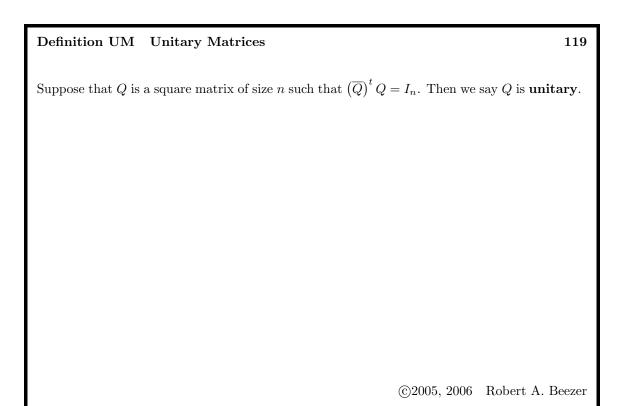
- 1. A is nonsingular.
- $2.\ A$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{0\}$ .
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

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#### Theorem SNCM Solution with Nonsingular Coefficient Matrix

118

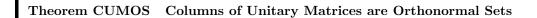
Suppose that A is nonsingular. Then the unique solution to  $\mathcal{LS}(A, \mathbf{b})$  is  $A^{-1}\mathbf{b}$ .



## Theorem UMI Unitary Matrices are Invertible

120

Suppose that Q is a unitary matrix of size n. Then Q is nonsingular, and  $Q^{-1}=(\overline{Q})^t$ .



Suppose that A is a square matrix of size n with columns  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ . Then A is a unitary matrix if and only if S is an orthonormal set.

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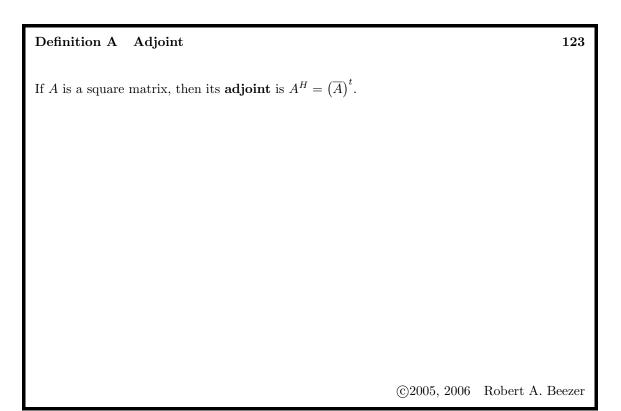
## Theorem UMPIP Unitary Matrices Preserve Inner Products

122

Suppose that Q is a unitary matrix of size n and  $\mathbf{u}$  and  $\mathbf{v}$  are two vectors from  $\mathbb{C}^n$ . Then

$$\langle Q\mathbf{u}, \, Q\mathbf{v} \rangle = \langle \mathbf{u}, \, \mathbf{v} \rangle$$

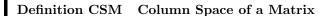
$$\|Q\mathbf{v}\| = \|\mathbf{v}\|$$



## Definition HM Hermitian Matrix

124

The square matrix A is Hermitian (or self-adjoint) if  $A=\left(\overline{A}\right)^t$ 



Suppose that A is an  $m \times n$  matrix with columns  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ . Then the **column space** of A, written  $\mathcal{C}(A)$ , is the subset of  $\mathbb{C}^m$  containing all linear combinations of the columns of A.

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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#### Theorem CSCS Column Spaces and Consistent Systems

126

Suppose A is an  $m \times n$  matrix and **b** is a vector of size m. Then  $\mathbf{b} \in \mathcal{C}(A)$  if and only if  $\mathcal{LS}(A, \mathbf{b})$  is consistent.

#### Theorem BCS Basis of the Column Space

127

Suppose that A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the set of column indices where B has leading 1's. Let  $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$ . Then

- 1. T is a linearly independent set.
- 2.  $C(A) = \langle T \rangle$ .

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#### Theorem CSNM Column Space of a Nonsingular Matrix

128

Suppose A is a square matrix of size n. Then A is nonsingular if and only if  $\mathcal{C}(A) = \mathbb{C}^n$ .

#### Theorem NME4 Nonsingular Matrix Equivalences, Round 4

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .

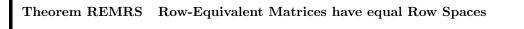
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#### Definition RSM Row Space of a Matrix

130

129

Suppose A is an  $m \times n$  matrix. Then the **row space** of A,  $\mathcal{R}(A)$ , is the column space of  $A^t$ , i.e.  $\mathcal{R}(A) = \mathcal{C}(A^t)$ .



Suppose A and B are row-equivalent matrices. Then  $\mathcal{R}(A) = \mathcal{R}(B)$ .

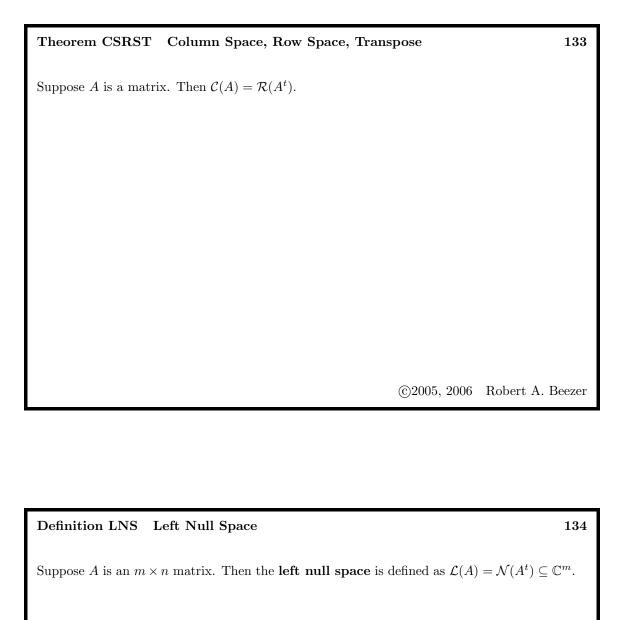
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## Theorem BRS Basis for the Row Space

**132** 

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of  $B^t$ . Then

- 1.  $\mathcal{R}(A) = \langle S \rangle$ .
- $2.\ S$  is a linearly independent set.



Suppose A is an  $m \times n$  matrix. Add m new columns to A that together equal an  $m \times m$  identity matrix to form an  $m \times (n+m)$  matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the  $m \times n$  matrix formed from the first n columns of N and let J denote the  $m \times m$  matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the  $r \times n$  matrix formed from all of the non-zero rows of B. Let K be the  $r \times m$  matrix formed from the first r rows of J, while L will be the  $(m-r) \times m$  matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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#### Theorem PEEF Properties of Extended Echelon Form

136

Suppose that A is an  $m \times n$  matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^m$ , then  $A\mathbf{x} = \mathbf{y}$  if and only if  $B\mathbf{x} = J\mathbf{y}$ .
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

Suppose A is an  $m \times n$  matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C,  $\mathcal{N}(A) = \mathcal{N}(C)$ .
- 2. The row space of A is the row space of C,  $\mathcal{R}(A) = \mathcal{R}(C)$ .
- 3. The column space of A is the null space of L,  $C(A) = \mathcal{N}(L)$ .
- 4. The left null space of A is the row space of L,  $\mathcal{L}(A) = \mathcal{R}(L)$ .

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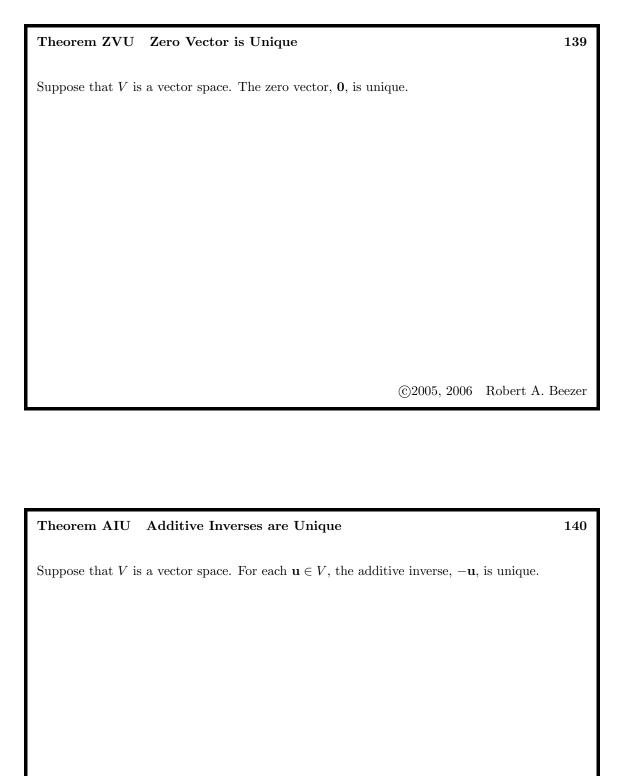
#### Definition VS Vector Space

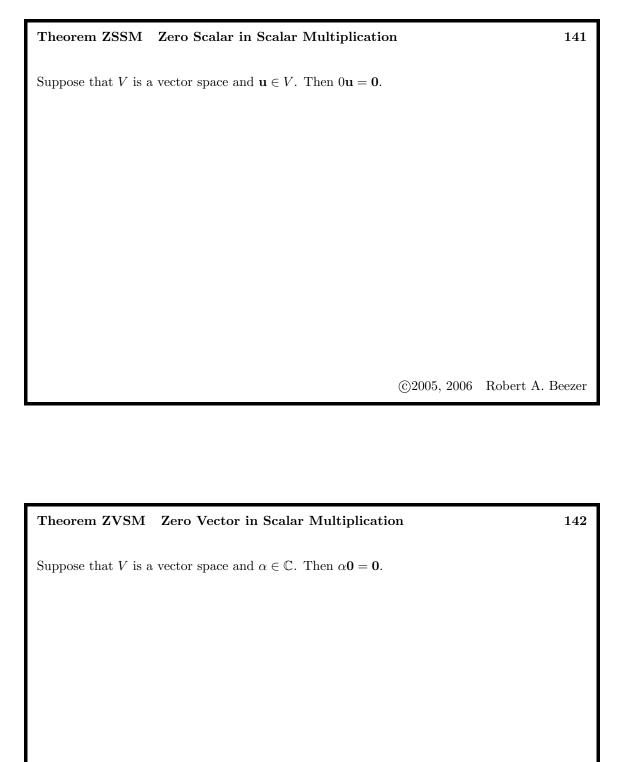
138

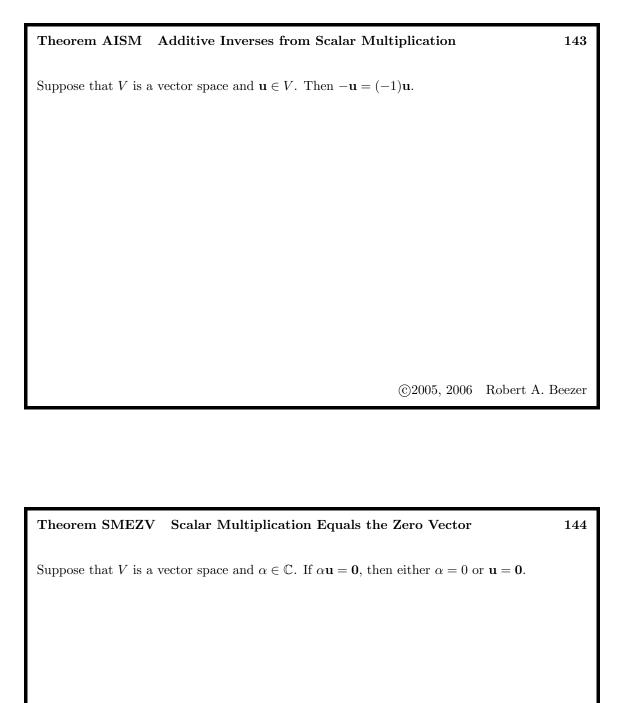
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

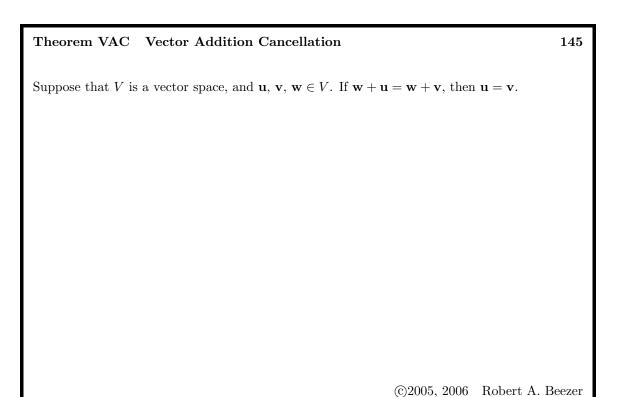
- AC Additive Closure If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} \in V$ .
- SC Scalar Closure If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha \mathbf{u} \in V$ .
- C Commutativity If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AA Additive Associativity If  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w} \in V$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in V$ .
- AI Additive Inverses If  $\mathbf{u} \in V$ , then there exists a vector  $-\mathbf{u} \in V$  so that  $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ .
- SMA Scalar Multiplication Associativity If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVA Distributivity across Vector Addition If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSA Distributivity across Scalar Addition If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .
- One If  $u \in V$  then 1u u

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.





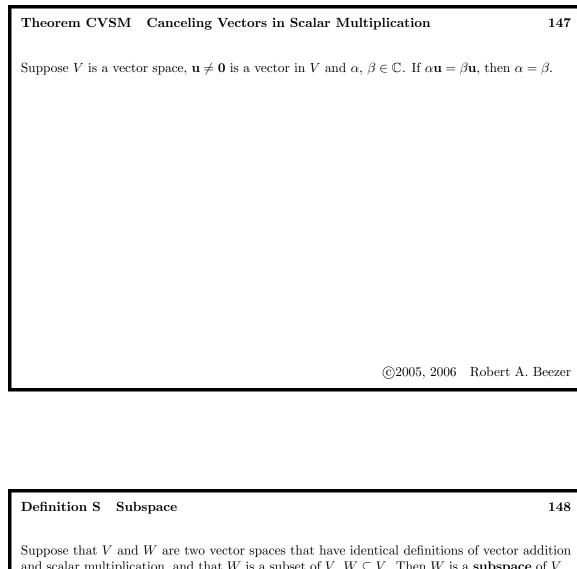




# Theorem CSSM Canceling Scalars in Scalar Multiplication

146

Suppose V is a vector space,  $\mathbf{u}, \mathbf{v} \in V$  and  $\alpha$  is a nonzero scalar from  $\mathbb{C}$ . If  $\alpha \mathbf{u} = \alpha \mathbf{v}$ , then  $\mathbf{u} = \mathbf{v}$ .



and scalar multiplication, and that W is a subset of V,  $W \subseteq V$ . Then W is a subspace of V.

# Theorem TSS Testing Subsets for Subspaces

149

Suppose that V is a vector space and W is a subset of V,  $W \subseteq V$ . Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

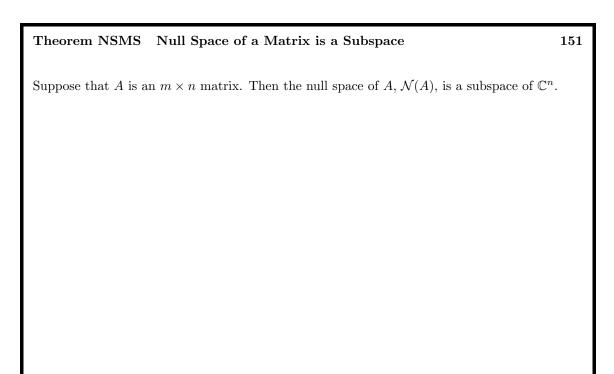
- 1. W is non-empty,  $W \neq \emptyset$ .
- 2. If  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ , then  $\mathbf{x} + \mathbf{y} \in W$ .
- 3. If  $\alpha \in \mathbb{C}$  and  $\mathbf{x} \in W$ , then  $\alpha \mathbf{x} \in W$ .

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#### Definition TS Trivial Subspaces

150

Given the vector space V, the subspaces V and  $\{0\}$  are each called a **trivial subspace**.



#### Definition LC Linear Combination

152

Suppose that V is a vector space. Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

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#### Definition SS Span of a Set

153

Suppose that V is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ . Symbolically,

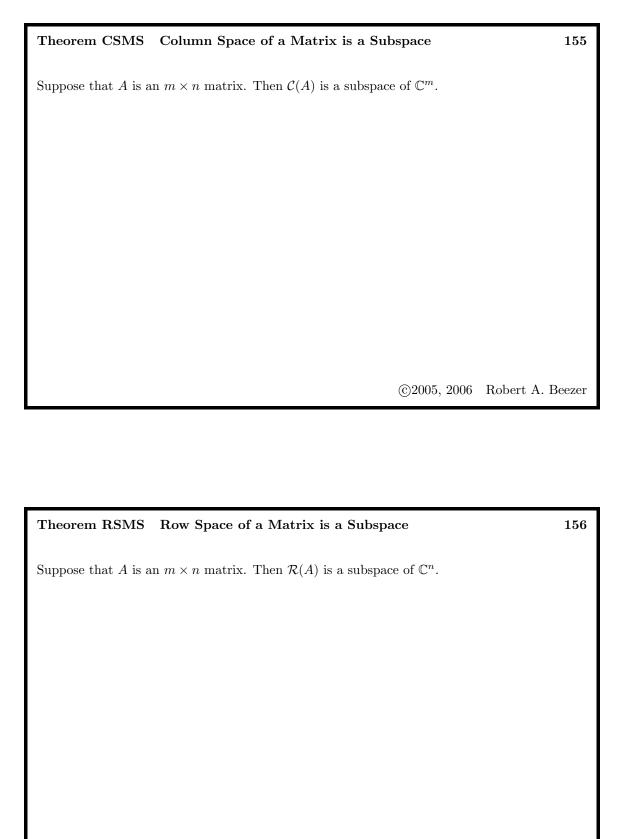
$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

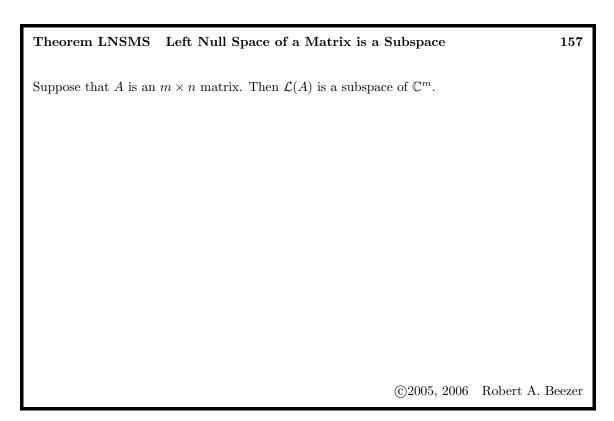
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#### Theorem SSS Span of a Set is a Subspace

154

Suppose V is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$ , their span,  $\langle S \rangle$ , is a subspace.





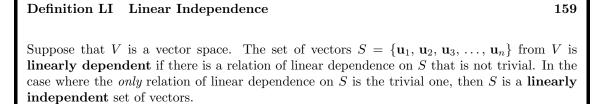
#### Definition RLD Relation of Linear Dependence

**158** 

Suppose that V is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_n\}$ , an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is a **trivial relation of linear dependence** on S.



#### Definition TSVS To Span a Vector Space

160

Suppose V is a vector space. A subset S of V is a **spanning set** for V if  $\langle S \rangle = V$ . In this case, we also say S **spans** V.

Theorem VRRB Vector Representation Relative to a Ba	heorem VKKI	VKKB Vecto	r Representation	Relative	to a	Basis
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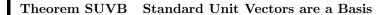
Suppose that V is a vector space and  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  is a linearly independent set that spans V. Let  $\mathbf{w}$  be any vector in V. Then there exist *unique* scalars  $a_1, a_2, a_3, \dots, a_m$  such that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$

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#### Definition B Basis 162

Suppose V is a vector space. Then a subset  $S \subseteq V$  is a **basis** of V if it is linearly independent and spans V.



The set of standard unit vectors for  $\mathbb{C}^m$  (Definition SUV),  $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$  is a basis for the vector space  $\mathbb{C}^m$ .

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#### Theorem CNMB Columns of Nonsingular Matrix are a Basis

164

Suppose that A is a square matrix of size m. Then the columns of A are a basis of  $\mathbb{C}^m$  if and only if A is nonsingular.

#### Theorem NME5 Nonsingular Matrix Equivalences, Round 5

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{0\}$ .
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .

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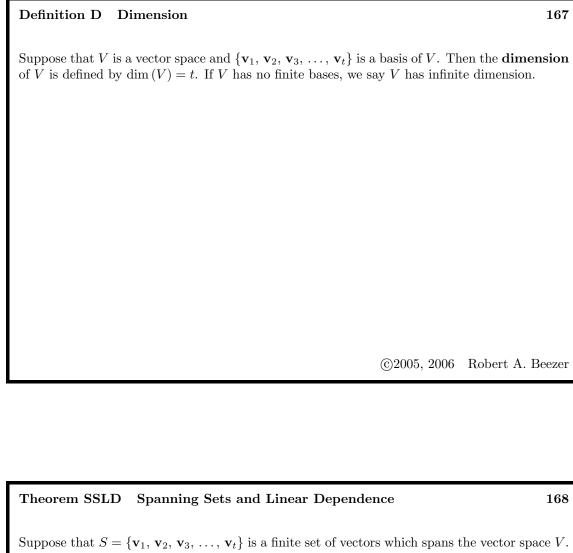
#### Theorem COB Coordinates and Orthonormal Bases

166

165

Suppose that  $B = \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_p\}$  is an orthonormal basis of the subspace W of  $\mathbb{C}^m$ . For any  $\mathbf{w} \in W$ ,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$



Then any set of t+1 or more vectors from V is linearly dependent.

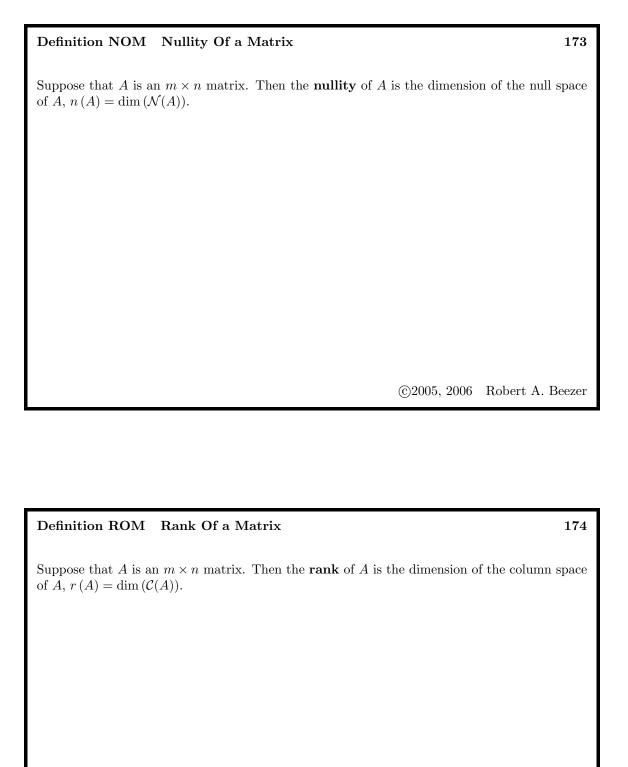
Theorem BIS Bases have Identical Sizes	169
Suppose that $V$ is a vector space with a finite basis $B$ and a second basis $C$ . The have the same size.	Then $B$ and $C$
have the same size.	
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Theorem DCM Dimension of $\mathbb{C}^m$	170
Theorem DCM Dimension of $\mathbb{C}^m$ The dimension of $\mathbb{C}^m$ (Example VSCV) is $m$ .	170
	170
	170
	170

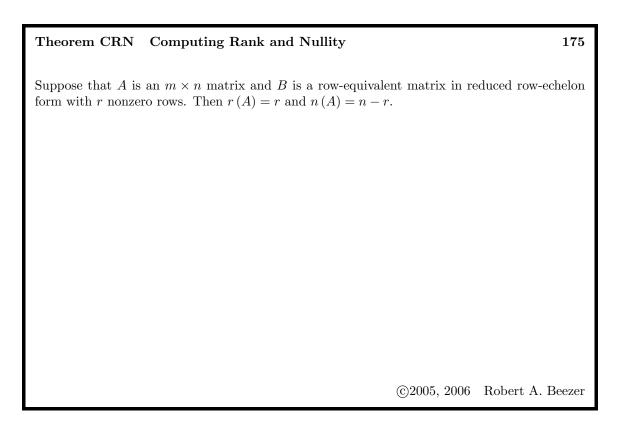
Theorem DP Dimension of $P_n$		171
The dimension of $P_n$ (Example VSP) is $n+1$ .		
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# Theorem DM Dimension of $M_{mn}$

172

The dimension of  $M_{mn}$  (Example VSM) is mn.





# Theorem RPNC Rank Plus Nullity is Columns

176

Suppose that A is an  $m \times n$  matrix. Then r(A) + n(A) = n.

#### Theorem RNNM Rank and Nullity of a Nonsingular Matrix

177

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

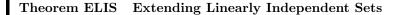
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#### Theorem NME6 Nonsingular Matrix Equivalences, Round 6

178

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.



Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose  $\mathbf{w}$  is a vector such that  $\mathbf{w} \notin \langle S \rangle$ . Then the set  $S' = S \cup \{\mathbf{w}\}$  is linearly independent.

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#### Theorem G Goldilocks

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Suppose that V is a vector space of dimension t. Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

# Theorem EB Extending a Basis

181

Suppose that W is a subspace of the vector space V. Then there exists a subspace X of V such that

- 1. For every  $\mathbf{v} \in V$  there exists vectors  $\mathbf{w} \in W$ ,  $\mathbf{x} \in X$  such that  $\mathbf{v} = \mathbf{w} + \mathbf{x}$ .
- 2.  $W \cap X = \{0\}.$

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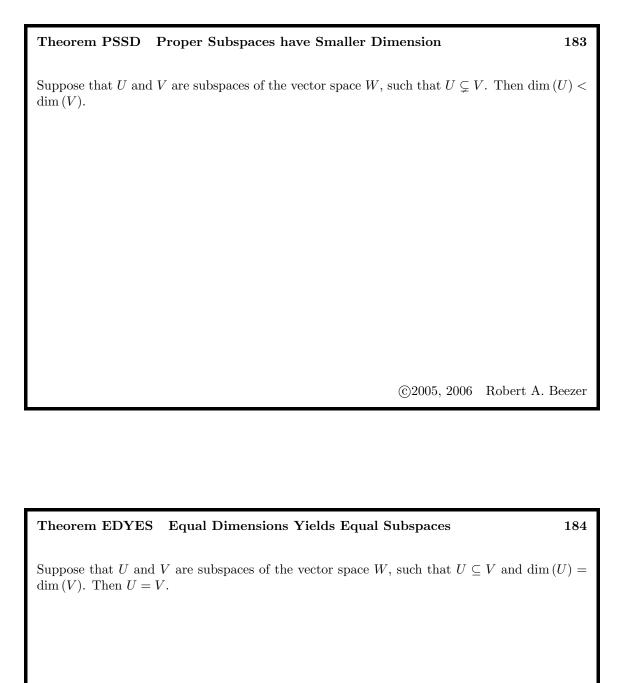
#### Definition DS Direct Sum

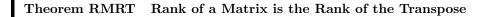
182

Suppose that V is a vector space with two subspaces U and W such that

- 1. For every  $\mathbf{v} \in V$  there exists vectors  $\mathbf{u} \in U$ ,  $\mathbf{w} \in W$  such that  $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2.  $U \cap W = \{0\}$

Then V is the **direct sum** of U and W and we write  $V = U \oplus W$ .





Suppose A is an  $m \times n$  matrix. Then  $r(A) = r(A^t)$ .

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#### Theorem DFS Dimensions of Four Subspaces

186

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim  $(\mathcal{N}(A)) = n r$
- 2. dim  $(\mathcal{C}(A)) = r$
- 3. dim  $(\mathcal{R}(A)) = r$
- 4. dim  $(\mathcal{L}(A)) = m r$

1.  $E_{i,j}$  is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2.  $E_i(\alpha)$ , for  $\alpha \neq 0$ , is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3.  $E_{i,j}(\alpha)$  is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \end{cases}$$

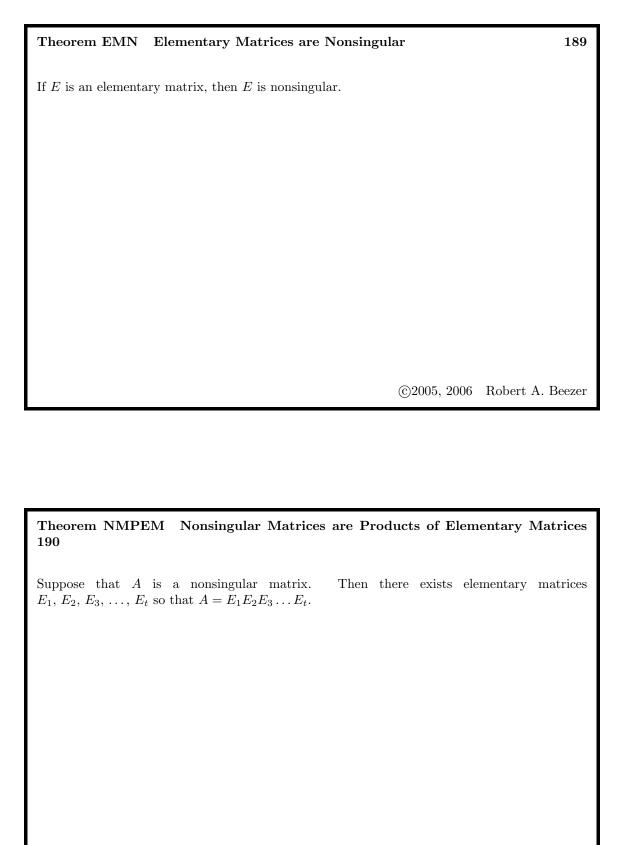
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#### Theorem EMDRO Elementary Matrices Do Row Operations

188

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

- 1. If the row operation swaps rows i and j, then  $B = E_{i,j}A$ .
- 2. If the row operation multiplies row i by  $\alpha$ , then  $B = E_i(\alpha) A$ .
- 3. If the row operation multiplies row i by  $\alpha$  and adds the result to row j, then  $B = E_{i,j}(\alpha) A$ .





Suppose that A is an  $m \times n$  matrix. Then the **submatrix** A(i|j) is the  $(m-1) \times (n-1)$  matrix obtained from A by removing row i and column j.

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#### Definition DM Determinant of a Matrix

192

Suppose A is a square matrix. Then its **determinant**,  $\det(A) = |A|$ , is an element of  $\mathbb{C}$  defined recursively by:

If A is a  $1 \times 1$  matrix, then  $\det(A) = [A]_{11}$ .

If A is a matrix of size n with  $n \geq 2$ , then

$$\begin{split} \det{(A)} &= [A]_{11} \det{(A\,(1|1))} - [A]_{12} \det{(A\,(1|2))} + [A]_{13} \det{(A\,(1|3))} - \\ & [A]_{14} \det{(A\,(1|4))} + \dots + (-1)^{n+1} \, [A]_{1n} \det{(A\,(1|n))} \end{split}$$

Theorem DMST	Determinant	of Matricos	of Sizo	Two
Theorem Divisi	Determinant	or matrices	or Size	$\perp$ wo

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $\det{(A)} = ad - bc$ 

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# Theorem DER Determinant Expansion about Rows

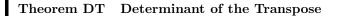
**194** 

193

Suppose that A is a square matrix of size n. Then

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2))$$
$$+ (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \dots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \qquad 1 \le i \le n$$

which is known as **expansion** about row i.



Suppose that A is a square matrix. Then  $\det(A^t) = \det(A)$ .

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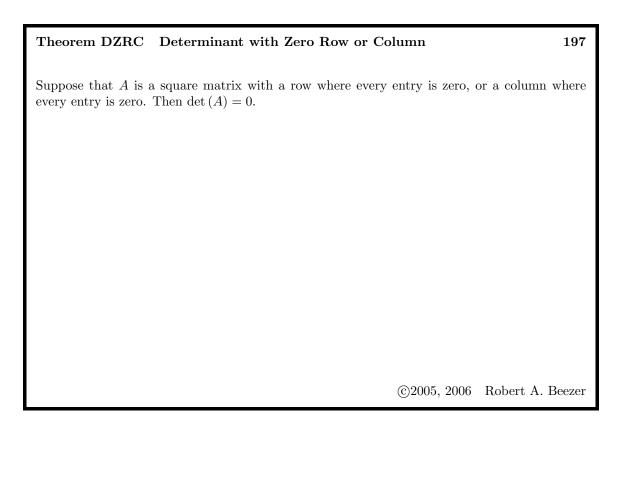
#### Theorem DEC Determinant Expansion about Columns

196

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[ A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[ A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[ A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[ A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

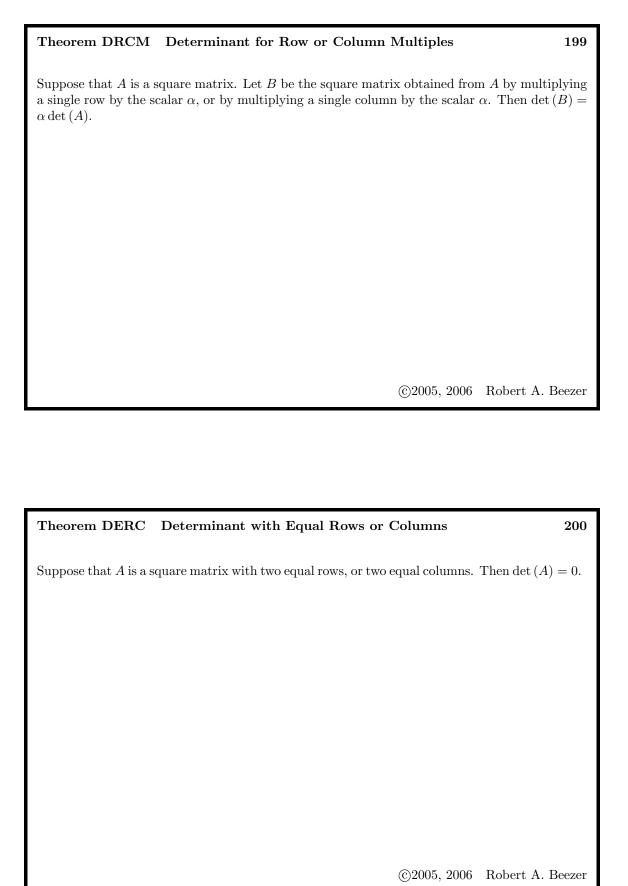
which is known as **expansion** about column j.

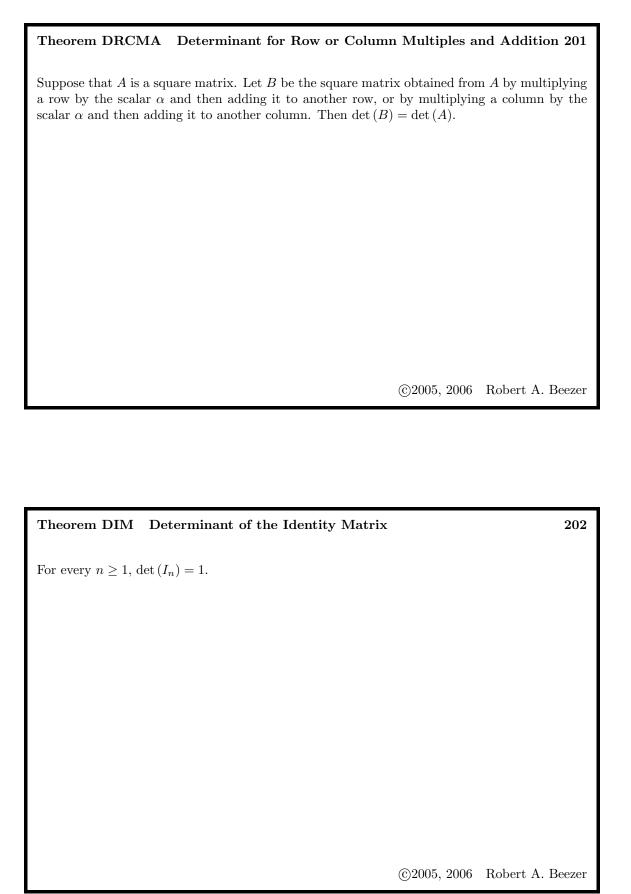


#### Theorem DRCS Determinant for Row or Column Swap

198

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then  $\det(B) = -\det(A)$ .





For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

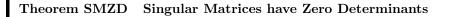
- 1.  $\det(E_{i,j}) = -1$
- 2.  $\det (E_i(\alpha)) = \alpha$
- 3.  $\det (E_{i,j}(\alpha)) = 1$

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# Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 204

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det\left(EA\right) = \det\left(E\right)\det\left(A\right)$$



Let A be a square matrix. Then A is singular if and only if det(A) = 0.

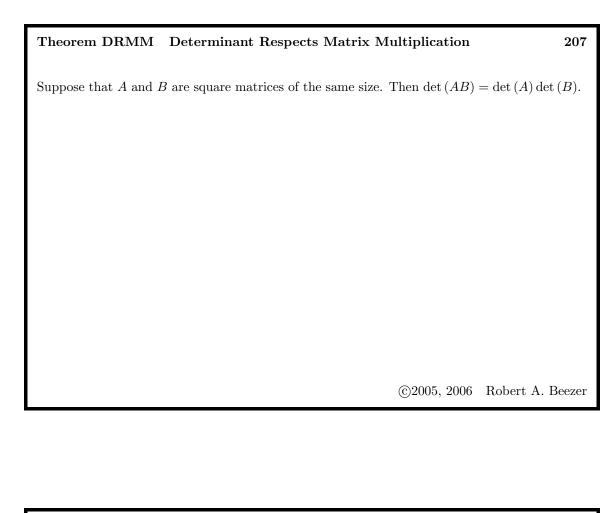
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#### Theorem NME7 Nonsingular Matrix Equivalences, Round 7

206

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $det(A) \neq 0$ .

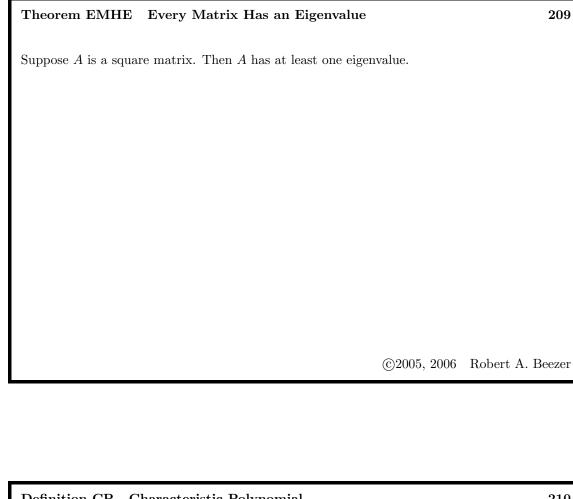


## Definition EEM Eigenvalues and Eigenvectors of a Matrix

**208** 

Suppose that A is a square matrix of size n,  $\mathbf{x} \neq \mathbf{0}$  is a vector in  $\mathbb{C}^n$ , and  $\lambda$  is a scalar in  $\mathbb{C}$ . Then we say  $\mathbf{x}$  is an **eigenvector** of A with **eigenvalue**  $\lambda$  if

$$A\mathbf{x} = \lambda \mathbf{x}$$

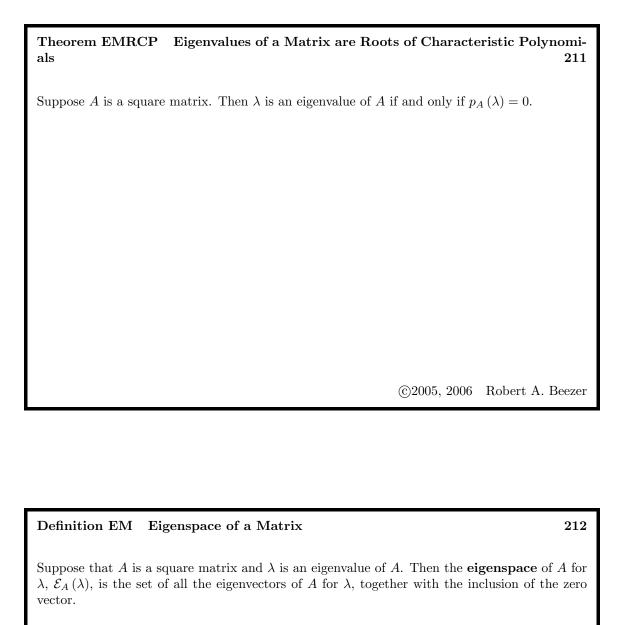


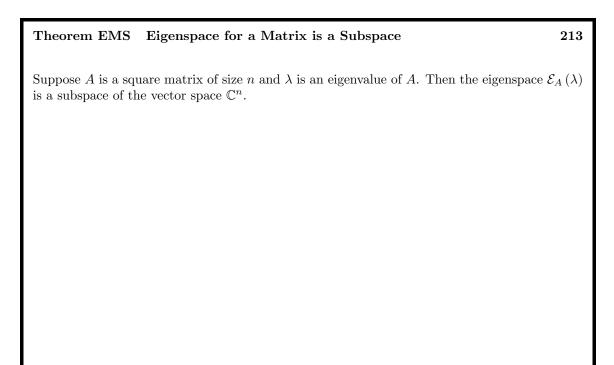
## Definition CP Characteristic Polynomial

**210** 

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial  $p_A(x)$  defined by

$$p_A(x) = \det(A - xI_n)$$





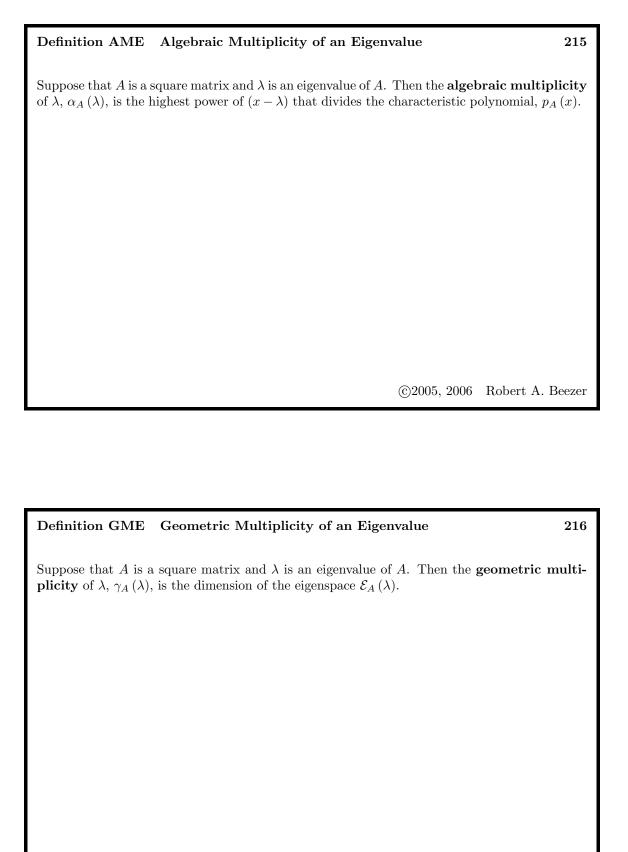
# Theorem EMNS Eigenspace of a Matrix is a Null Space

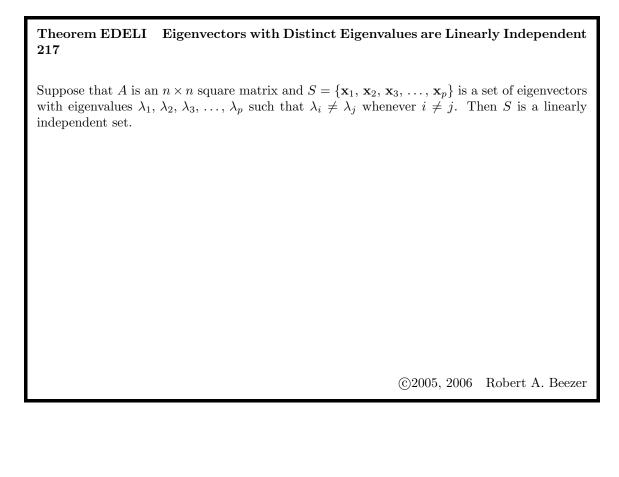
214

Suppose A is a square matrix of size n and  $\lambda$  is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

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# Theorem SMZE Singular Matrices have Zero Eigenvalues

218

Suppose A is a square matrix. Then A is singular if and only if  $\lambda = 0$  is an eigenvalue of A.

### Theorem NME8 Nonsingular Matrix Equivalences, Round 8

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .
- 12.  $\lambda = 0$  is not an eigenvalue of A.

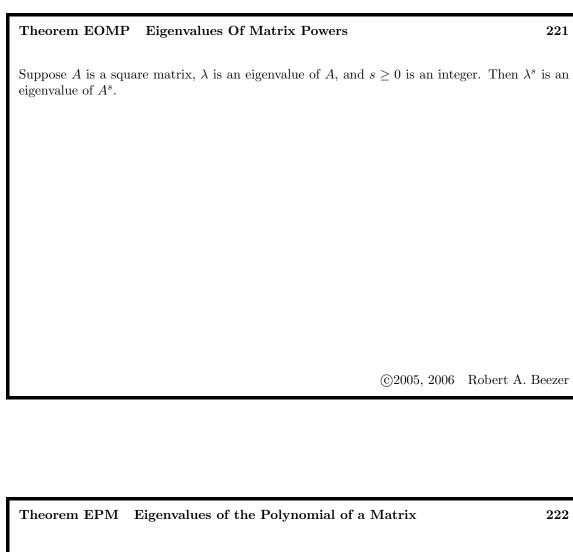
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## Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix

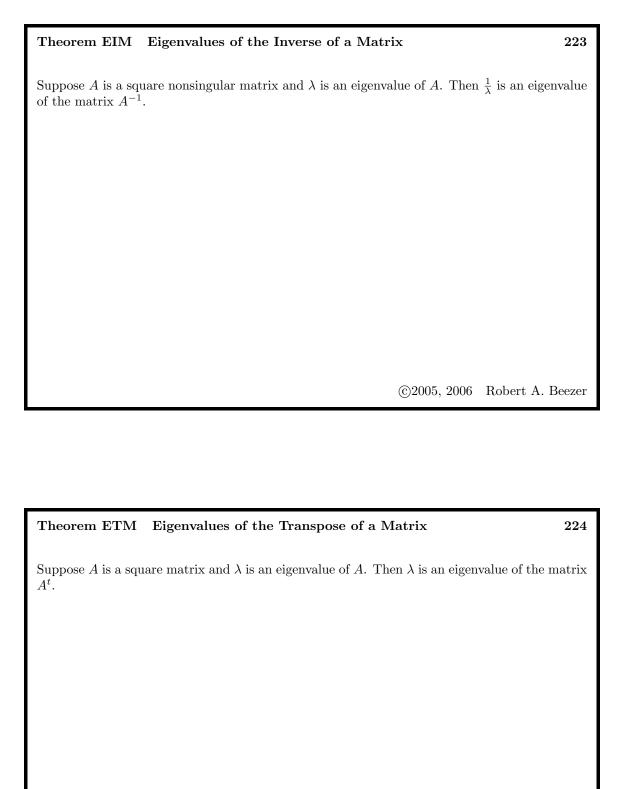
220

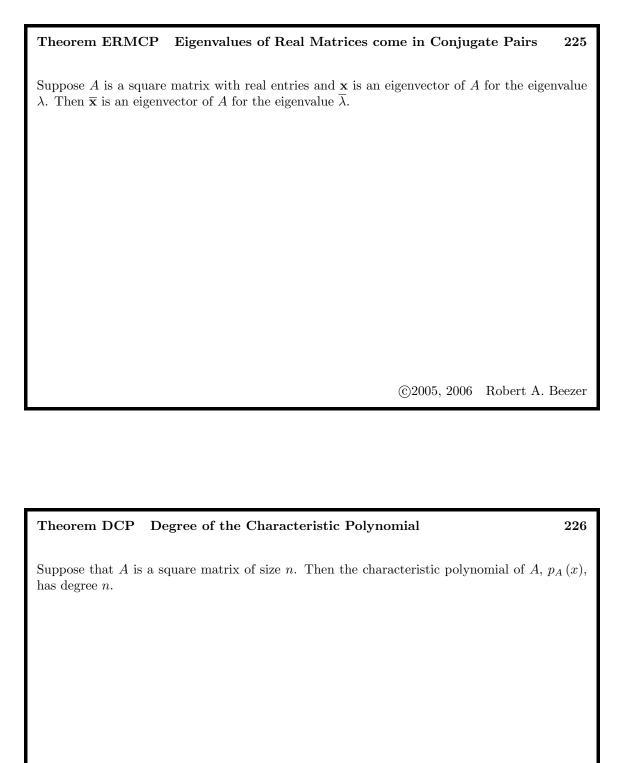
219

Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Then  $\alpha\lambda$  is an eigenvalue of  $\alpha A$ .



Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then  $q(\lambda)$  is an eigenvalue of the matrix q(A).





# Theorem NEM Number of Eigenvalues of a Matrix

227

Suppose that A is a square matrix of size n with distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ . Then

$$\sum_{i=1}^{k} \alpha_A \left( \lambda_i \right) = n$$

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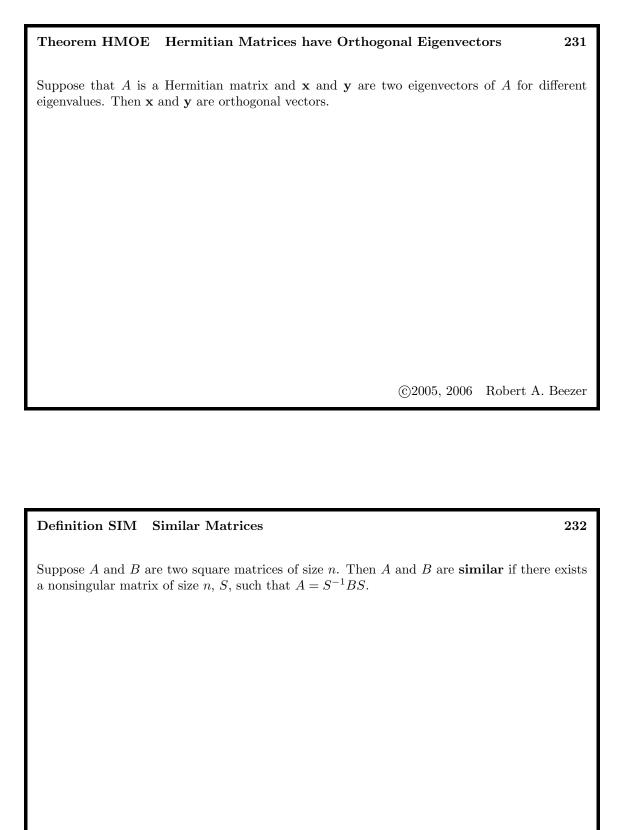
## Theorem ME Multiplicities of an Eigenvalue

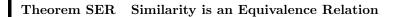
228

Suppose that A is a square matrix of size n and  $\lambda$  is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix	229
Suppose that $A$ is a square matrix of size $n$ . Then $A$ cannot have more than	n distinct eigen-
values.	
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Theorem HMRE Hermitian Matrices have Real Eigenvalues	230
Theorem HMRE Hermitian Matrices have Real Eigenvalues	
Theorem HMRE Hermitian Matrices have Real Eigenvalues Suppose that $A$ is a Hermitian matrix and $\lambda$ is an eigenvalue of $A$ . Then $\lambda \in$	





Suppose A, B and C are square matrices of size n. Then

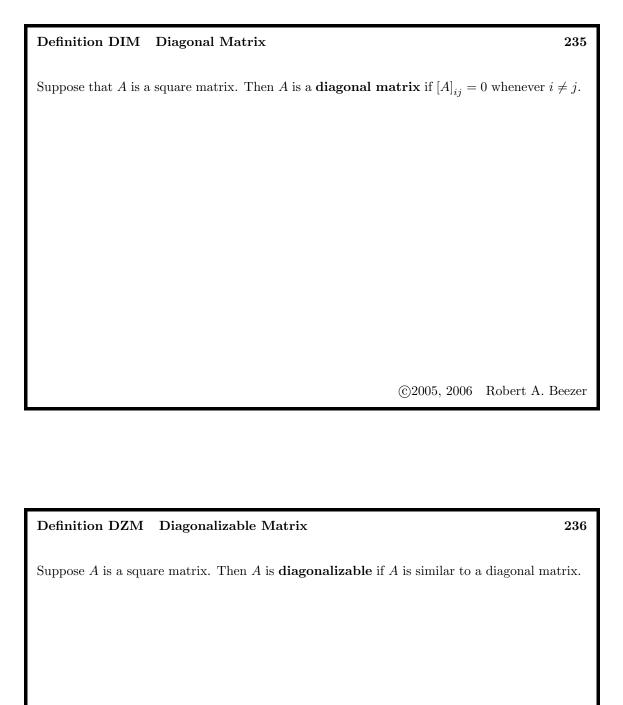
- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

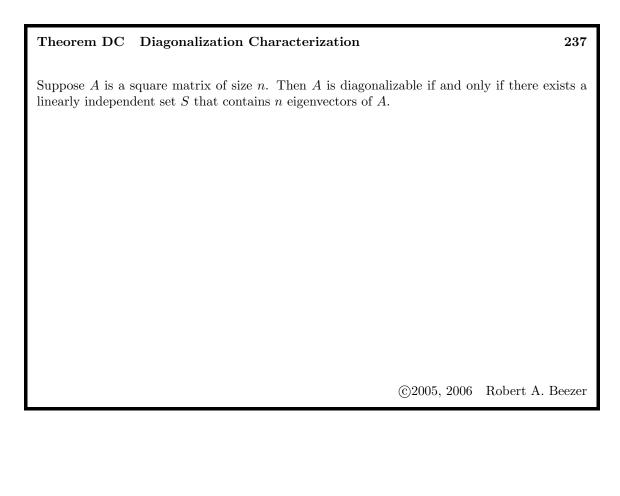
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### Theorem SMEE Similar Matrices have Equal Eigenvalues

234

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is  $p_A(x) = p_B(x)$ .





# Theorem DMFE Diagonalizable Matrices have Full Eigenspaces

**238** 

Suppose A is a square matrix. Then A is diagonalizable if and only if  $\gamma_A(\lambda) = \alpha_A(\lambda)$  for every eigenvalue  $\lambda$  of A.

Theorem DED	Distinct Eigenvalues implies Diagonalizable	239
Suppose $A$ is a sq	quare matrix of size $n$ with $n$ distinct eigenvalues. Then $A$ is diagonalizab	ole.
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		<u>,                                    </u>
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# Definition LT Linear Transformation

241

A linear transformation,  $T \colon U \mapsto V$ , is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1.  $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$  for all  $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2.  $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$  for all  $\mathbf{u} \in U$  and all  $\alpha \in \mathbb{C}$

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### Theorem LTTZZ Linear Transformations Take Zero to Zero

 $\mathbf{242}$ 

Suppose  $T: U \mapsto V$  is a linear transformation. Then  $T(\mathbf{0}) = \mathbf{0}$ .



244

Suppose that A is an  $m \times n$  matrix. Define a function  $T: \mathbb{C}^n \mapsto \mathbb{C}^m$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Then T is a linear transformation.

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### Theorem MLTCV Matrix of a Linear Transformation, Column Vectors

Suppose that  $T: \mathbb{C}^n \mapsto \mathbb{C}^m$  is a linear transformation. Then there is an  $m \times n$  matrix A such that  $T(\mathbf{x}) = A\mathbf{x}$ .

### Theorem LTLC Linear Transformations and Linear Combinations

245

Suppose that  $T: U \mapsto V$  is a linear transformation,  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$  are vectors from U and  $a_1, a_2, a_3, \ldots, a_t$  are scalars from  $\mathbb{C}$ . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$

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### Theorem LTDB Linear Transformation Defined on a Basis

246

Suppose that  $T: U \mapsto V$  is a linear transformation,  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a basis for U and  $\mathbf{w}$  is a vector from U. Let  $a_1, a_2, a_3, \dots, a_n$  be the scalars from  $\mathbb{C}$  such that

$$\mathbf{w} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_n \mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$$

# Definition PI Pre-Image

247

Suppose that  $T: U \mapsto V$  is a linear transformation. For each  $\mathbf{v}$ , define the **pre-image** of  $\mathbf{v}$  to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v} \}$$

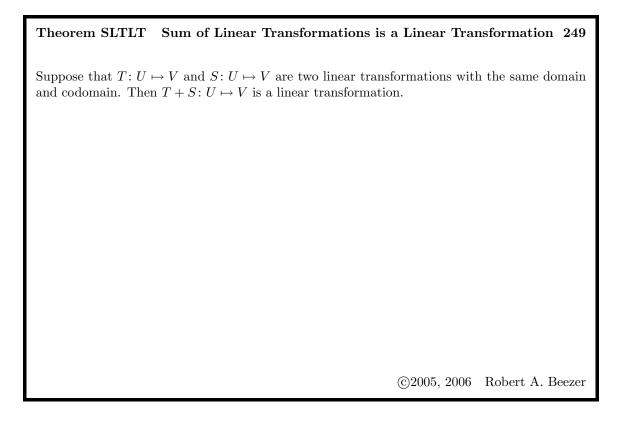
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### Definition LTA Linear Transformation Addition

248

Suppose that  $T\colon U\mapsto V$  and  $S\colon U\mapsto V$  are two linear transformations with the same domain and codomain. Then their **sum** is the function  $T+S\colon U\mapsto V$  whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

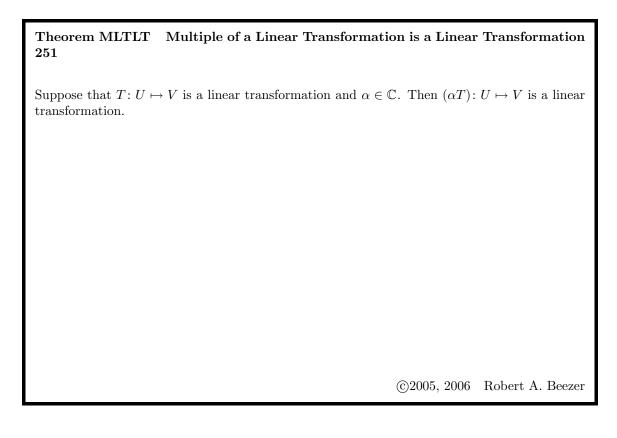


## Definition LTSM Linear Transformation Scalar Multiplication

**250** 

Suppose that  $T: U \mapsto V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then the **scalar multiple** is the function  $\alpha T: U \mapsto V$  whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$



## Theorem VSLT Vector Space of Linear Transformations

252

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V,  $\mathrm{LT}\,(U,V)$  is a vector space when the operations are those given in Definition LTA and Definition LTSM.



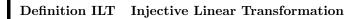
Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations. Then the **composition** of S and T is the function  $(S \circ T): U \mapsto W$  whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

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# Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 254

Suppose that  $T\colon U\mapsto V$  and  $S\colon V\mapsto W$  are linear transformations. Then  $(S\circ T)\colon U\mapsto W$  is a linear transformation.



Suppose  $T: U \mapsto V$  is a linear transformation. Then T is **injective** if whenever  $T(\mathbf{x}) = T(\mathbf{y})$ , then  $\mathbf{x} = \mathbf{y}$ .

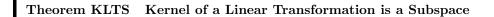
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# Definition KLT Kernel of a Linear Transformation

256

Suppose  $T \colon U \mapsto V$  is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$$



Suppose that  $T: U \mapsto V$  is a linear transformation. Then the kernel of T,  $\mathcal{K}(T)$ , is a subspace of U.

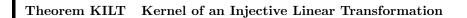
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## Theorem KPI Kernel and Pre-Image

258

Suppose  $T: U \mapsto V$  is a linear transformation and  $\mathbf{v} \in V$ . If the preimage  $T^{-1}(\mathbf{v})$  is non-empty, and  $\mathbf{u} \in T^{-1}(\mathbf{v})$  then

$$T^{-1}(\mathbf{v}) = {\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)} = \mathbf{u} + \mathcal{K}(T)$$

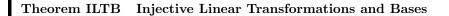


Suppose that  $T: U \mapsto V$  is a linear transformation. Then T is injective if and only if the kernel of T is trivial,  $\mathcal{K}(T) = \{\mathbf{0}\}.$ 

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### Theorem ILTLI Injective Linear Transformations and Linear Independence 260

Suppose that  $T: U \mapsto V$  is an injective linear transformation and  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$  is a linearly independent subset of U. Then  $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$  is a linearly independent subset of V.



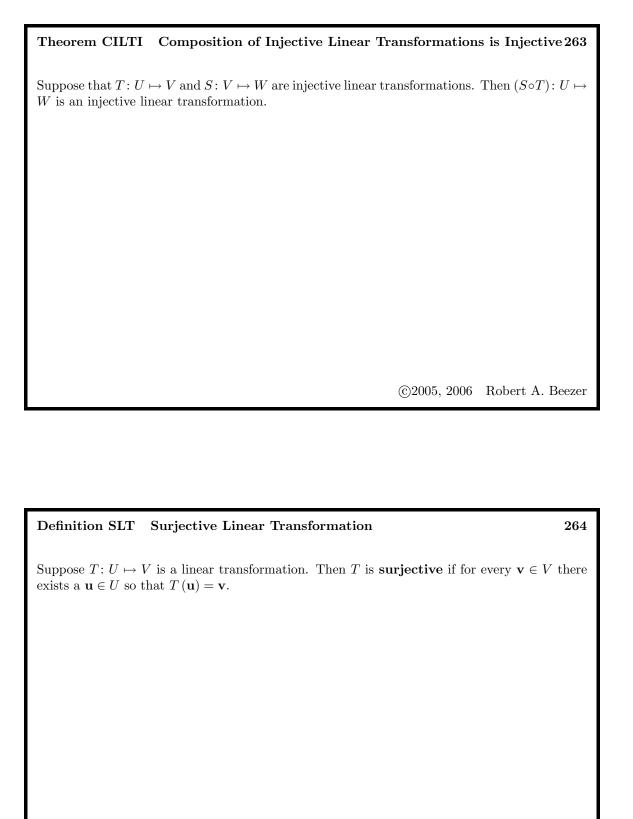
Suppose that  $T: U \mapsto V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is injective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a linearly independent subset of V.

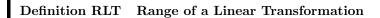
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### 

**262** 

Suppose that  $T: U \mapsto V$  is an injective linear transformation. Then  $\dim(U) \leq \dim(V)$ .





Suppose  $T \colon U \mapsto V$  is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

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### Theorem RLTS Range of a Linear Transformation is a Subspace

**266** 

Suppose that  $T \colon U \mapsto V$  is a linear transformation. Then the range of T,  $\mathcal{R}(T)$ , is a subspace of V.



**268** 

Suppose that  $T \colon U \mapsto V$  is a linear transformation. Then T is surjective if and only if the range of T equals the codomain,  $\mathcal{R}(T) = V$ .

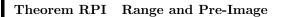
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### Theorem SSRLT Spanning Set for Range of a Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation and  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$  spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

spans  $\mathcal{R}(T)$ .



Suppose that  $T \colon U \mapsto V$  is a linear transformation. Then

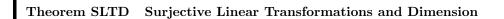
 $\mathbf{v} \in \mathcal{R}(T)$  if and only if  $T^{-1}(\mathbf{v}) \neq \emptyset$ 

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### Theorem SLTB Surjective Linear Transformations and Bases

270

Suppose that  $T: U \mapsto V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is surjective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a spanning set for V.

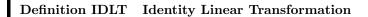


Suppose that  $T: U \mapsto V$  is a surjective linear transformation. Then  $\dim(U) \geq \dim(V)$ .

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# Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 272

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are surjective linear transformations. Then  $(S \circ T): U \mapsto W$  is a surjective linear transformation.



The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W \left( \mathbf{w} \right) = \mathbf{w}$$

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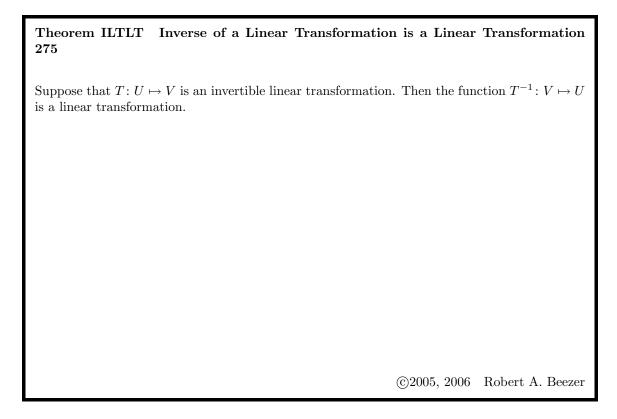
### Definition IVLT Invertible Linear Transformations

274

Suppose that  $T\colon U\mapsto V$  is a linear transformation. If there is a function  $S\colon V\mapsto U$  such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write  $S = T^{-1}$ .

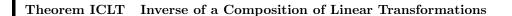


### Theorem IILT Inverse of an Invertible Linear Transformation

276

Suppose that  $T: U \mapsto V$  is an invertible linear transformation. Then  $T^{-1}$  is an invertible linear transformation and  $\left(T^{-1}\right)^{-1} = T$ .

Theorem 12115 In	vertible Linear Trans	formations are Injective an	d Surjective277
	linear transformation.	Then $T$ is invertible if and on	ly if $T$ is injective
and surjective.			
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The energy CIVIT C	lama ariti an af Inven	ible Linger Transformation	279
Theorem CIVLT C	Composition of Invert	tible Linear Transformation	ns 278
	$V$ and $S \colon V \mapsto W$ a	re invertible linear transforms	
Suppose that $T: U \mapsto$	$V$ and $S \colon V \mapsto W$ a	re invertible linear transforms	
Suppose that $T: U \mapsto$	$V$ and $S \colon V \mapsto W$ a	re invertible linear transforms	
Suppose that $T: U \mapsto$	$V$ and $S \colon V \mapsto W$ a	re invertible linear transforms	
Suppose that $T: U \mapsto$	$V$ and $S \colon V \mapsto W$ a	re invertible linear transforms	



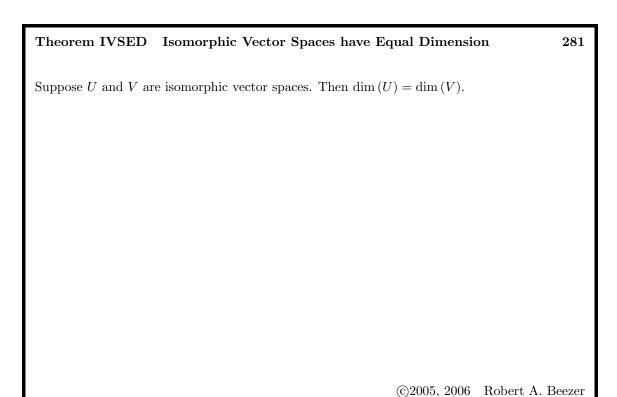
Suppose that  $T\colon U\mapsto V$  and  $S\colon V\mapsto W$  are invertible linear transformations. Then  $S\circ T$  is invertible and  $(S\circ T)^{-1}=T^{-1}\circ S^{-1}$ .

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### Definition IVS Isomorphic Vector Spaces

280

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V,  $T: U \mapsto V$ . In this case, we write  $U \cong V$ , and the linear transformation T is known as an **isomorphism** between U and V.

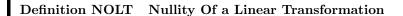


# Definition ROLT Rank Of a Linear Transformation

282

Suppose that  $T:U\mapsto V$  is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

$$r(T) = \dim (\mathcal{R}(T))$$



Suppose that  $T:U\mapsto V$  is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

$$n(T) = \dim (\mathcal{K}(T))$$

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### Theorem ROSLT Rank Of a Surjective Linear Transformation

 $\mathbf{284}$ 

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the rank of T is the dimension of V,  $r(T) = \dim(V)$ , if and only if T is surjective.



Suppose that  $T: U \mapsto V$  is an injective linear transformation. Then the nullity of T is zero, n(T) = 0, if and only if T is injective.

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## Theorem RPNDD Rank Plus Nullity is Domain Dimension

286

Suppose that  $T\colon U\mapsto V$  is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

# Definition VR Vector Representation

287

Suppose that V is a vector space with a basis  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ . Define a function  $\rho_B \colon V \mapsto \mathbb{C}^n$  as follows. For  $\mathbf{w} \in V$ , find scalars  $a_1, a_2, a_3, \dots, a_n$  so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

$$[\rho_B(\mathbf{w})]_i = a_i \qquad 1 \le i \le n$$

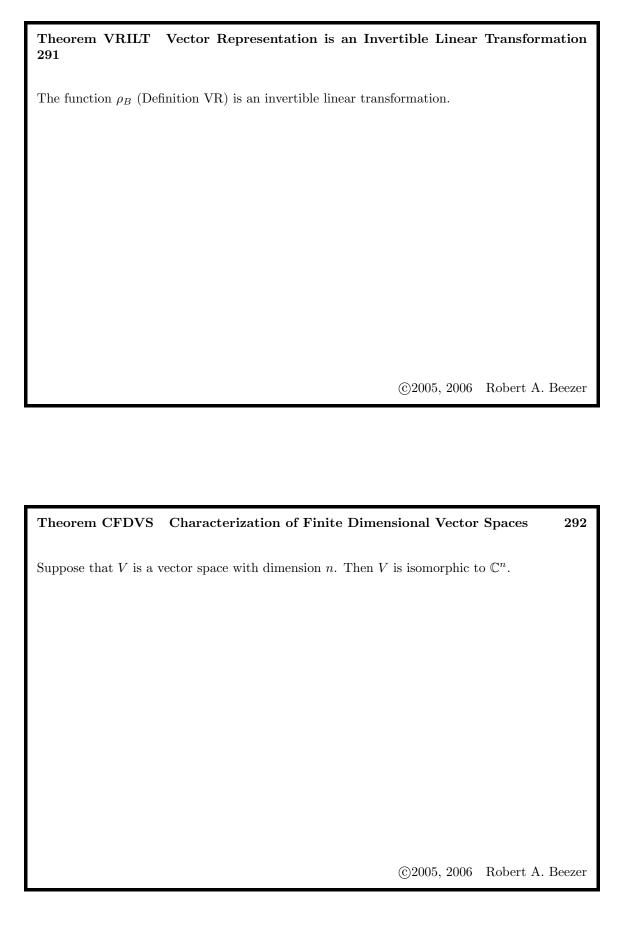
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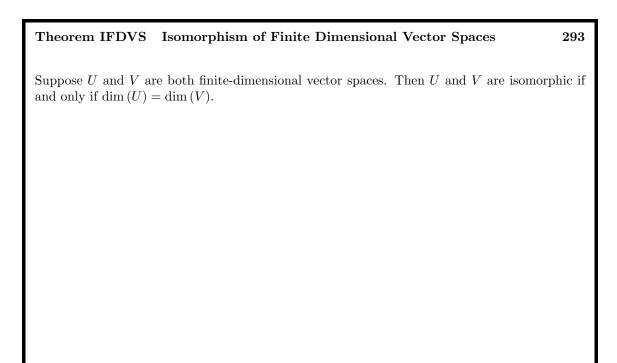
## Theorem VRLT Vector Representation is a Linear Transformation

288

The function  $\rho_B$  (Definition VR) is a linear transformation.

Theorem VRI	Vector Representation is Injective		289		
The function $\rho_B$ (Definition VR) is an injective linear transformation.					
		_			
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Theorem VDC	Vector Depresentation is Service time		900		
Theorem VRS	Vector Representation is Surjective		290		
The function $\rho_B$ (Definition VR) is a surjective linear transformation.					



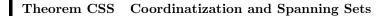


## Theorem CLI Coordinatization and Linear Independence

294

Suppose that U is a vector space with a basis B of size n. Then  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$  is a linearly independent subset of U if and only if  $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$  is a linearly independent subset of  $\mathbb{C}^n$ .

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Suppose that U is a vector space with a basis B of size n. Then  $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$  if and only if  $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$ .

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## Definition MR Matrix Representation

296

Suppose that  $T: U \mapsto V$  is a linear transformation,  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the  $m \times n$  matrix,

$$M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right)\right|\rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right)\right|\rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right)\right|\ldots\left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$$

#### Theorem FTMR Fundamental Theorem of Matrix Representation

297

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U, C is a basis for V and  $M_{B,C}^T$  is the matrix representation of T relative to B and C. Then, for any  $\mathbf{u} \in U$ ,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

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#### Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 298

Suppose that  $T\colon U\mapsto V$  and  $S\colon U\mapsto V$  are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

# Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 299

Suppose that  $T\colon U\mapsto V$  is a linear transformation,  $\alpha\in\mathbb{C},\,B$  is a basis of U and C is a basis of V. Then

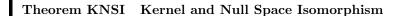
$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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# Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 300

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$



Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of  $M_{B,C}^T$ ,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

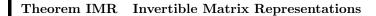
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## Theorem RCSI Range and Column Space Isomorphism

**302** 

Suppose that  $T \colon U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of  $M_{B,C}^T$ ,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$



Suppose that  $T: U \mapsto V$  is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C,  $M_{B,C}^T$  is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

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## Theorem IMILT Invertible Matrices, Invertible Linear Transformation

304

Suppose that A is a square matrix of size n and  $T: \mathbb{C}^n \to \mathbb{C}^n$  is the linear transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ . Then A is invertible matrix if and only if T is an invertible linear transformation.

#### Theorem NME9 Nonsingular Matrix Equivalences, Round 9

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .
- 12.  $\lambda = 0$  is not an eigenvalue of A.
- 13. The linear transformation  $T: \mathbb{C}^n \mapsto \mathbb{C}^n$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is invertible.

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## Definition EELT Eigenvalue and Eigenvector of a Linear Transformation

Suppose that  $T: V \mapsto V$  is a linear transformation. Then a nonzero vector  $\mathbf{v} \in V$  is an **eigenvector** of T for the **eigenvalue**  $\lambda$  if  $T(\mathbf{v}) = \lambda \mathbf{v}$ .

#### Definition CBM Change-of-Basis Matrix

307

Suppose that V is a vector space, and  $I_V: V \mapsto V$  is the identity linear transformation on V. Let  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of  $I_V$  relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[ \rho_C \left( I_V \left( \mathbf{v}_1 \right) \right) \middle| \rho_C \left( I_V \left( \mathbf{v}_2 \right) \right) \middle| \rho_C \left( I_V \left( \mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left( I_V \left( \mathbf{v}_n \right) \right) \right]$$

$$= \left[ \rho_C \left( \mathbf{v}_1 \right) \middle| \rho_C \left( \mathbf{v}_2 \right) \middle| \rho_C \left( \mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left( \mathbf{v}_n \right) \right]$$

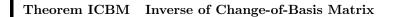
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#### Theorem CB Change-of-Basis

308

Suppose that  $\mathbf{v}$  is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C(\mathbf{v}) = C_{B,C}\rho_B(\mathbf{v})$$



Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix  $C_{B,C}$  is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

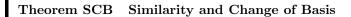
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## Theorem MRCB Matrix Representation and Change of Basis

310

Suppose that  $T\colon U\mapsto V$  is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$



Suppose that  $T: V \mapsto V$  is a linear transformation and B and C are bases of V. Then

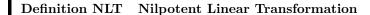
$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

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## Theorem EER Eigenvalues, Eigenvectors, Representations

312

Suppose that  $T: V \mapsto V$  is a linear transformation and B is a basis of V. Then  $\mathbf{v} \in V$  is an eigenvector of T for the eigenvalue  $\lambda$  if and only if  $\rho_B(\mathbf{v})$  is an eigenvector of  $M_{B,B}^T$  for the eigenvalue  $\lambda$ .



Suppose that  $T: V \mapsto V$  is a linear transformation such that there is an integer p > 0 such that  $T^p(\mathbf{v}) = \mathbf{0}$  for every  $\mathbf{v} \in V$ . The smallest p for which this condition is met is called the **index** of T.

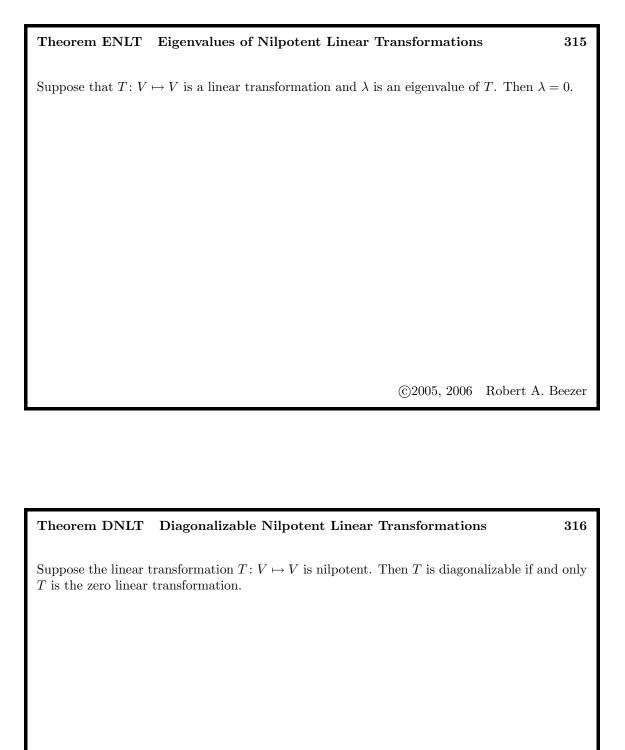
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#### Definition JB Jordan Block

**314** 

Given the scalar  $\lambda \in \mathbb{C}$ , the Jordan block  $J_n(\lambda)$  is the  $n \times n$  matrix defined by

$$\left[J_{n}\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$



#### Theorem KPLT Kernels of Powers of Linear Transformations

317

Suppose  $T: V \mapsto V$  is a linear transformation, where dim (V) = n. Then there is an integer m,  $0 \le m \le n$ , such that

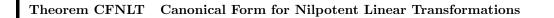
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

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#### Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 318

Suppose  $T: V \mapsto V$  is a nilpotent linear transformation with index p and dim (V) = n. Then  $0 \le p \le n$  and

$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$



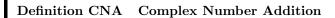
Suppose that  $T\colon V\mapsto V$  is a nilpotent linear transformation of index d. Then there is a basis for V so that the matrix representation,  $M_{B,B}^T$ , is block diagonal with each block being a Jordan block,  $J_n(0)$ . The size of the largest block is the index d, and the total number of blocks is the nullity of T, n(T).

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## Definition CNE Complex Number Equality

**320** 

The complex numbers  $\alpha = a + bi$  and  $\beta = c + di$  are **equal**, denoted  $\alpha = \beta$ , if a = c and b = d.



The **sum** of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha + \beta$ , is (a + c) + (b + d)i.

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## Definition CNM Complex Number Multiplication

322

The **product** of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha\beta$ , is (ac - bd) + (ad + bc)i.

#### Theorem PCNA Properties of Complex Number Arithmetic

323

The operations of addition and multiplication of complex numbers have the following properties.

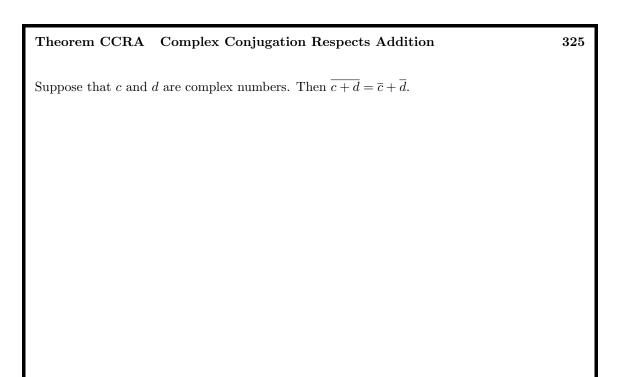
- ACCN Additive Commutativity, Complex Numbers For any  $\alpha$ ,  $\beta \in \mathbb{C}$ ,  $\alpha + \beta = \beta + \alpha$ .
- MCCN Multiplicative Commutativity, Complex Numbers For any  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha\beta = \beta\alpha$ .
- AACN Additive Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .
- MACN Multiplicative Associativity, Complex Numbers For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ .
- DCN Distributivity, Complex Numbers For any  $\alpha$ ,  $\beta$ ,  $\gamma \in \mathbb{C}$ ,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ .
- **ZCN Zero, Complex Numbers** There is a complex number 0 = 0 + 0i so that for any  $\alpha \in \mathbb{C}$ ,  $0 + \alpha = \alpha$ .
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any  $\alpha \in \mathbb{C}$ ,  $1\alpha = \alpha$ .
- AICN Additive Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$  there exists  $-\alpha \in \mathbb{C}$  so that  $\alpha + (-\alpha) = 0$ .
- MICN Multiplicative Inverse, Complex Numbers For every  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$  there exists  $\frac{1}{\alpha} \in \mathbb{C}$  so that  $\frac{1}{\alpha}\alpha = 1$ .

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#### Definition CCN Conjugate of a Complex Number

324

The **conjugate** of the complex number  $c = a + bi \in \mathbb{C}$  is the complex number  $\overline{c} = a - bi$ .

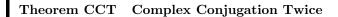


## Theorem CCRM Complex Conjugation Respects Multiplication

326

Suppose that c and d are complex numbers. Then  $\overline{cd} = \overline{c}\overline{d}$ .

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Suppose that c is a complex number. Then  $\overline{\overline{c}} = c$ .

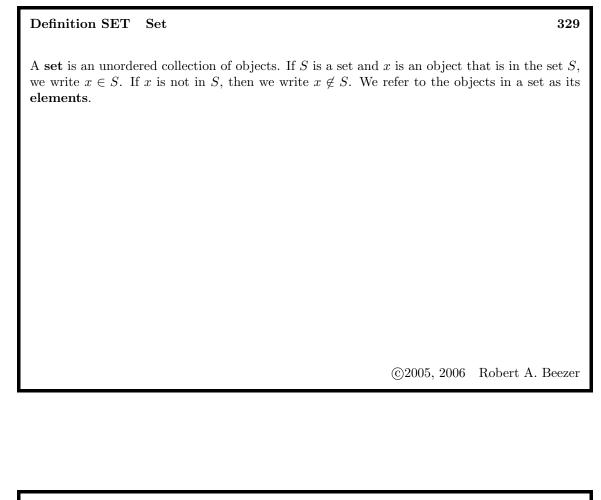
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## Definition MCN Modulus of a Complex Number

328

The **modulus** of the complex number  $c = a + bi \in \mathbb{C}$ , is the nonnegative real number

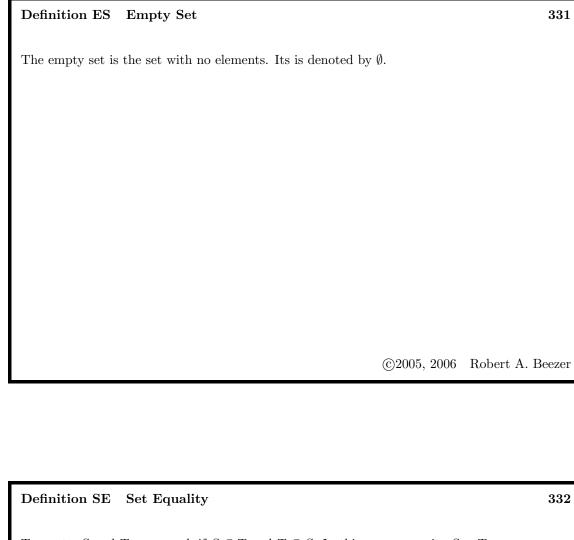
$$|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}.$$



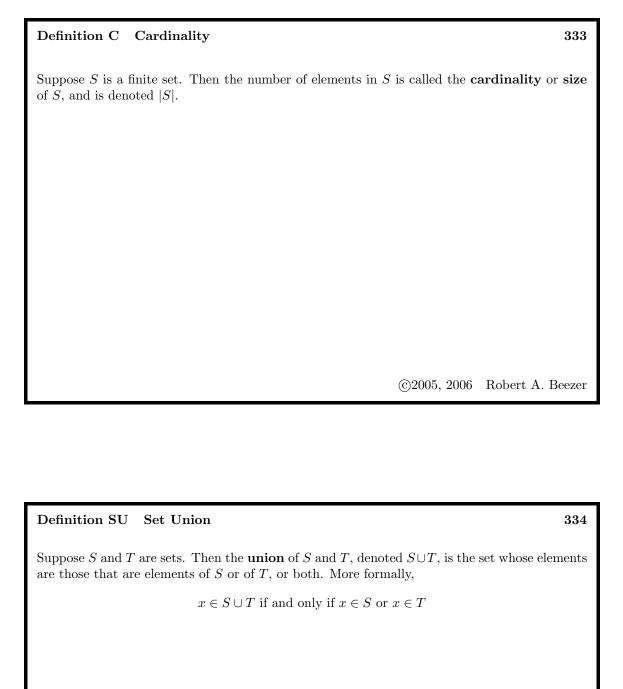
## Definition SSET Subset

330

If S and T are two sets, then S is a subset of T, written  $S \subseteq T$  if whenever  $x \in S$  then  $x \in T$ .



Two sets, S and T, are equal, if  $S \subseteq T$  and  $T \subseteq S$ . In this case, we write S = T.





Suppose S and T are sets. Then the **intersection** of S and T, denoted  $S \cap T$ , is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$  if and only if  $x \in S$  and  $x \in T$ 

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#### Definition SC Set Complement

336

Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted  $\overline{S}$ , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$  if and only if  $x \in U$  and  $x \notin S$