Flash Cards

to accompany

A First Course in Linear Algebra

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Definition SLE System of Linear Equations

1

A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .

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Definition ESYS Equivalent Systems

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Two systems of linear equations are $\mathbf{equivalent}$ if their solution sets are equal.

Definition EO Equation Operations

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Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

- 1. Swap the locations of two equations in the list.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M Matrix 5

An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

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Definition CV Column Vector

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A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in \vec{u} . To refer to the **entry** or **component** that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$.

Definition ZCV Zero Column Vector

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The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

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Definition CM Coefficient Matrix

Q

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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Definition VOC Vector of Constants

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SV Solution Vector

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Definition LSMI	R. Matrix	Representation	on of a	Linear S	System

If A is the coefficient matrix of a system of linear equations and \mathbf{b} is the vector of constants, then we will write $\mathcal{L}S(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

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Definition AM Augmented Matrix

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Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants \mathbf{b} . Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (number n+1) is the column vector \mathbf{b} . This matrix will be written as $[A \mid \mathbf{b}]$.

Definition RO Row Operations

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The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

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Definition REM Row-Equivalent Matrices

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Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

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Definition RREF Reduced Row-Echelon Form

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A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot colums will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Theorem REMEF Row-Equivalent Matrix in Echelon Form

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Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

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Definition RR Row-Reducing

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To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

Definition CS Consistent System	19
A system of linear equations is consistent if it has at least one solution. is called inconsistent .	Otherwise, the system

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Definition IDV Independent and Dependent Variables

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Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem	BCLS	Recognizing	Consistency	of a	Linear	System
Theorem	ICLS	rtecognizing	Consistency	or a	Linear	System

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n+1 of B.

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Theorem ISRN Inconsistent Systems, r and n

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Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

Theorem	CSRN	Consistent S	vstems.	r	and	n.
THEOLEIN	CSILI	Consistent S	Aprems.	,	anu	10

Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

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Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.

Theorem PSSLS	Possible Solution Sets for Linear Systems 2	5
A system of linear e	quations has no solutions, a unique solution or infinitely many solutions.	
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Theorem CMVEI 26	Consistent, More Variables than Equations, Infinite solution	ıs
	system of linear equations has m equations in n variables. If $n > m$, the itely many solutions.	n
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Definition HS Homogeneous System	27
	ŀ
A system of linear equations, $\mathcal{L}S(A, \mathbf{b})$ is homogeneous if the vector of constants is the	zero
vector, in other words, $\mathbf{b} = 0$.	ŀ
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Theorem HSC Homogeneous Systems are Consistent	28
Suppose that a system of linear equations is homogeneous. Then the system is consistent.	

Definition TSHSE	Trivial Solution to H	omogeneous Systems	of Equations

Suppose a homogeneous system of linear equations has n variables. The solution $x_1=0$, $x_2=0,\ldots,\,x_n=0$ (i.e. $\mathbf{x}=\mathbf{0}$) is called the **trivial solution**.

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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 30

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

Definition NSM Null Space of a Matrix 3	1
The null space of a matrix A , denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions the homogeneous system $\mathcal{L}S(A, 0)$.	Ю

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Definition SQM Square Matrix

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A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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Definition NM Nonsingular Matrix

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Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{L}S(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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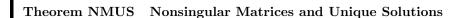
Definition IM Identity Matrix

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The $m \times m$ identity matrix, I_m , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 35
Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.
form. Then A is nonsingular if and only if D is the identity matrix.
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Theorem NMTNS Nonsingular Matrices have Trivial Null Spaces 36
Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A , $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{0\}$.



Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} .

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Theorem NME1 Nonsingular Matrix Equivalences, Round 1

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .

Definition VSCV Vector Space of Column Vectors

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The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} .

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Definition CVE Column Vector Equality

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The vectors \mathbf{u} and \mathbf{v} are equal, written $\mathbf{u} = \mathbf{v}$ provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \le i \le m$$

Definition	CVA	Column	Vector	Addition

Given the vectors \mathbf{u} and \mathbf{v} the sum of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

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Definition CVSM Column Vector Scalar Multiplication

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Given the vector \mathbf{u} and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of \mathbf{u} by α , $\alpha \mathbf{u}$ is defined by

$$[\alpha \mathbf{u}]_i = \alpha [\mathbf{u}]_i$$

$$1 \leq i \leq m$$

Theorem VSPCV Vector Space Properties of Column Vectors

43 Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$. • ACC
- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} +$ $(\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One Column Vectors If $u \in \mathbb{C}^m$ then 1u =

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Definition LCCV Linear Combination of Column Vectors

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Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear **combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n$$
.

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{L}S(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

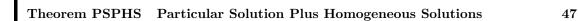
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Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{L}S(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$\begin{split} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} \end{split}$$

Then the set of solutions to the system of equations $\mathcal{L}S(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \right\}$$



Suppose that **w** is one solution to the linear system of equations $\mathcal{L}S(A, b)$. Then **y** is a solution to $\mathcal{L}S(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

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Theorem RREFU Reduced Row-Echelon Form is Unique

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

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Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

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Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle.$$

Definition B	RLDCV	Relation (of Linear	Dependence	for	Column	Vectors
Demindon 1		iteration (oi Lincai	Dependence	101	Column	V CCUOIS

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

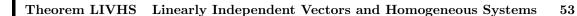
is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.

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Definition LICV Linear Independence of Column Vectors

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The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.



Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{L}S(A, \mathbf{0})$ has a unique solution.

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Theorem LIVRN Linearly Independent Vectors, r and n

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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Theorem MVSLD	\mathbf{More}	Vectors	than	Size	implies	Linear	Dependence

Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that n > m. Then S is a linearly dependent set.

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Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 56

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

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Theorem BNS Basis for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_i , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

Theorem DLDS Dependency in Linearly Dependent Sets

59

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

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Theorem BS Basis of a Span

60

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = \{ \mathbf{v}_{d_1}, \, \mathbf{v}_{d_2}, \, \mathbf{v}_{d_3}, \, \dots \, \mathbf{v}_{d_r} \}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

Definition CCCV Complex Conjugate of a Column Vector

61

Suppose that **u** is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

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Theorem CRVA Conjugation Respects Vector Addition

62

Suppose \mathbf{x} and \mathbf{y} are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Theorem CRSM Conjugation Respects Vector Scalar Multiplication

63

Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

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Definition IP Inner Product

64

Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_1 \overline{[\mathbf{v}]_1} + [\mathbf{u}]_2 \overline{[\mathbf{v}]_2} + [\mathbf{u}]_3 \overline{[\mathbf{v}]_3} + \dots + [\mathbf{u}]_m \overline{[\mathbf{v}]_m} = \sum_{i=1}^m [\mathbf{u}]_i \overline{[\mathbf{v}]_i}$$

Theorem IPVA Inner Product and Vector Addition

65

Suppose $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

- 1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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66

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

- 1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- 2. $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$

Theorem	TPAC	Inner	Product	ic	Anti-	Commutative

Suppose that \mathbf{u} and \mathbf{v} are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

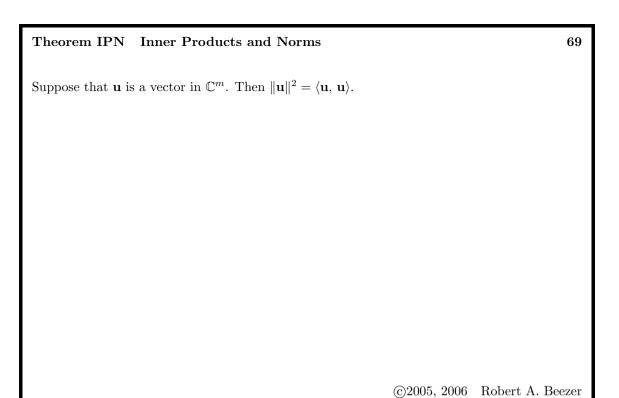
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Definition NV Norm of a Vector

68

The **norm** of the vector ${\bf u}$ is the scalar quantity in ${\mathbb C}$

$$\|\mathbf{u}\| = \sqrt{\left|\left[\mathbf{u}\right]_{1}\right|^{2} + \left|\left[\mathbf{u}\right]_{2}\right|^{2} + \left|\left[\mathbf{u}\right]_{3}\right|^{2} + \dots + \left|\left[\mathbf{u}\right]_{m}\right|^{2}} = \sqrt{\sum_{i=1}^{m} \left|\left[\mathbf{u}\right]_{i}\right|^{2}}$$



Theorem PIP Positive Inner Products

70

Suppose that **u** is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV Orthogonal Vectors

71

A pair of vectors, **u** and **v**, from \mathbb{C}^m are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

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Definition OSV Orthogonal Set of Vectors

72

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Theorem OSLI Orthogonal Sets are Linearly Independent

73

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

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Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

74

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors \mathbf{u}_i , $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Definition ONS OrthoNormal Set

75

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an **orthonormal** set of vectors.

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Definition VSM Vector Space of $m \times n$ Matrices

76

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality

77

The $m \times n$ matrices A and B are **equal**, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

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Definition MA Matrix Addition

78

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \leq i \leq m, \ 1 \leq j \leq n$$

Definition	MSM	Matrix	Scalar	Multiplication	ı

79

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ii} = \alpha [A]_{ii}$$

$$1 \leq i \leq m, \, 1 \leq j \leq n$$

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Theorem VSPM Vector Space Properties of Matrices

80

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One Matrices If $A \in M$ then 1A A

Definition ZM Zero Matrix

81

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

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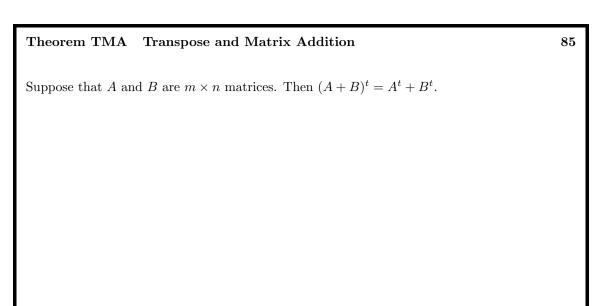
Definition TM Transpose of a Matrix

82

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix		83
The matrix A is symmetric if $A = A^t$.		
-		
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Theorem SMS Symmetric Matrices are Square		84
Theorem SMS Symmetric Matrices are Square		84
Theorem SMS Symmetric Matrices are Square Suppose that A is a symmetric matrix. Then A is square.		84
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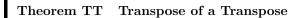


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${\bf Theorem~TMSM~~Transpose~and~Matrix~Scalar~Multiplication}$

86

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.



87

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

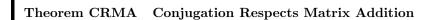
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Definition CCM Complex Conjugate of a Matrix

88

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written \overline{A} is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$



89

Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

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Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

90

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.

Theorem MCT Matrix Conjugation and Transposes

91

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.

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Definition MVP Matrix-Vector Product

92

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

Theorem SLEMM	Systems of Linear Equations as Matrix Multiplication 93
	-
Solutions to the linear	system $\mathcal{L}S(A, \mathbf{b})$ are the solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.
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Theorem EMMVP	Equal Matrices and Matrix-Vector Products 94
1110010111 21.1	Equal matrices and matrix.
Suppose that A and B	B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.
orr.	•

Definition MM Matrix Multiplication

95

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

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Theorem EMP Entries of Matrix Products

96

Suppose A is an $m \times n$ matrix and B =is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix

97

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
- $2. \quad \mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

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Theorem MMIM Matrix Multiplication and Identity Matrix

98

Suppose A is an $m \times n$ matrix. Then

- 1. $AI_n = A$
- $2. \quad I_m A = A$

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

- $1. \quad A(B+C) = AB + AC$
- $2. \quad (B+C)D = BD + CD$

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 100

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA Matrix Multiplication is Associative

101

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then A(BD) = (AB)D.

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Theorem MMIP Matrix Multiplication and Inner Products

102

If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\,\mathbf{v}\rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Theorem MMCC Matrix Multiplication and Complex Conjugation	103
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A} \overline{B}$.	

Theorem MMT Matrix Multiplication and Transposes

104

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

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Definition MI Matrix Inverse

105

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

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Definition SUV Standard Unit Vectors

106

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m . Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j \, | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

Theorem TTMI Two-by-Two Matrix Inverse

107

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

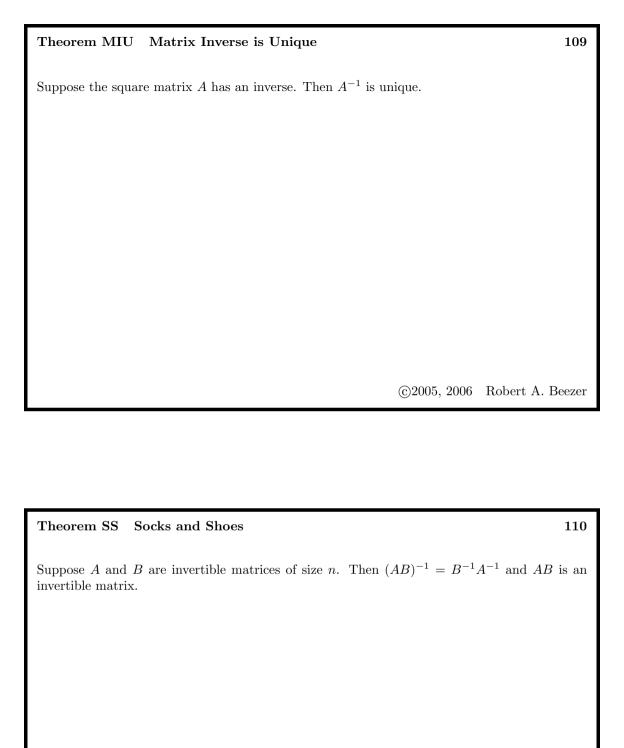
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

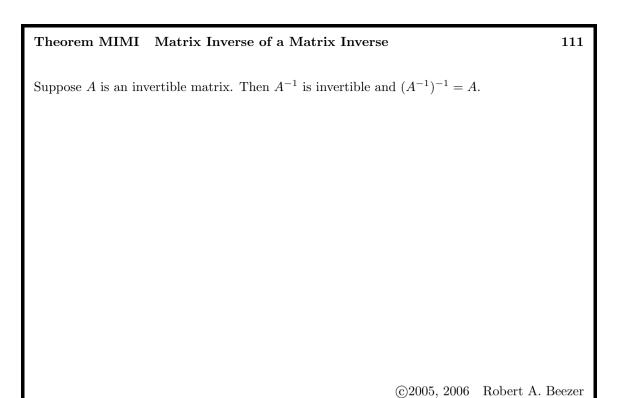
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Theorem CINM Computing the Inverse of a Nonsingular Matrix

108

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.





Theorem MIT Matrix Inverse of a Transpose

112

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Theorem MISM Matrix Inverse of a Scalar Multiple

113

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

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Theorem NPNT Nonsingular Product has Nonsingular Terms

114

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Theorem OSIS One-Sided Inverse is Sufficient	115
Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.	
Suppose II and B are square matrices of size n such that $IIB = I_n$. Then $BII = I_n$.	
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Theorem NI Nonsingularity is Invertibility	116
Theorem NI Nonsingularity is Invertibility Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.	

Theorem NME3 Nonsingular Matrix Equivalences, Round 3

117

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

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Theorem SNCM Solution with Nonsingular Coefficient Matrix

118

Suppose that A is nonsingular. Then the unique solution to $\mathcal{L}S(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.



119

Suppose that Q is a square matrix of size n such that $(\overline{Q})^t Q = I_n$. Then we say Q is **unitary**.

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Theorem UMI Unitary Matrices are Invertible

120

Suppose that Q is a unitary matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Theorem CUMOS	Columns of Unitary Matrices are Orthor	normal Sets

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.

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Theorem UMPIP Unitary Matrices Preserve Inner Products

122

121

Suppose that Q is a unitary matrix of size n and **u** and **v** are two vectors from \mathbb{C}^n . Then

$$\langle Q\mathbf{u}, \, Q\mathbf{v} \rangle = \langle \mathbf{u}, \, \mathbf{v} \rangle$$

$$\|Q\mathbf{v}\| = \|\mathbf{v}\|$$



123

If A is a square matrix, then its **adjoint** is $A^H = (\overline{A})^t$.

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Definition HM Hermitian Matrix

124

The square matrix A is Hermitian (or self-adjoint) if $A=\left(\overline{A}\right)^t$

Definition CSM Column Space of a Matrix

125

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS Column Spaces and Consistent Systems

126

Suppose A is an $m \times n$ matrix and **b** is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{L}S(A, \mathbf{b})$ is consistent.

Theorem BCS Basis of the Column Space

127

Suppose that A is an $m \times n$ matrix with columns \mathbf{A}_1 , \mathbf{A}_2 , \mathbf{A}_3 , ..., \mathbf{A}_n , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $C(A) = \langle T \rangle$.

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Theorem CSNM Column Space of a Nonsingular Matrix

128

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Theorem NME4 Nonsingular Matrix Equivalences, Round 4

129

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

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Definition RSM Row Space of a Matrix

130

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Theorem REMRS	Row-Equivalent Matrices	have equal Row Spaces

131

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

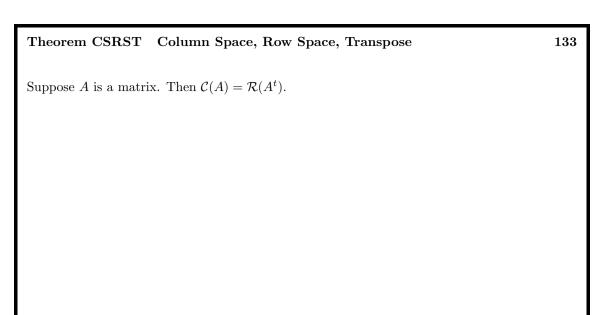
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Theorem BRS Basis for the Row Space

132

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.



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Definition LNS Left Null Space

134

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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Theorem PEEF Properties of Extended Echelon Form

136

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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Definition VS Vector Space

138

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One If $u \in V$ then 1u u

Theorem ZVU Zero vector is Unique	139
Suppose that V is a vector space. The zero vector, 0 , is unique.	
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Theorem AIII Additive Inverses are Unique	140
Theorem AIU Additive Inverses are Unique	140
Theorem AIU Additive Inverses are Unique $ \text{Suppose that } V \text{ is a vector space. For each } \mathbf{u} \in V \text{, the additive inverse, } -\mathbf{u} $	

Theorem ZSSM Zero Scalar	in Scalar Multiplication	141
Suppose that V is a vector space a	and $\mathbf{u} \in V$. Then $0\mathbf{u} = 0$.	
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Theorem ZVSM Zero Vecto		Robert A. Beezer
Theorem ZVSM Zero Vecto		
Theorem ZVSM Zero Vector Suppose that V is a vector space a	r in Scalar Multiplication	
	r in Scalar Multiplication	

Theorem AISM Additive Inverses from Scalar Multiplication	143
Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.	
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Theorem SMEZV Scalar Multiplication Equals the Zero Vector	144

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha \mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Theorem	$V\Delta C$	Vector	Addition	Cancellation

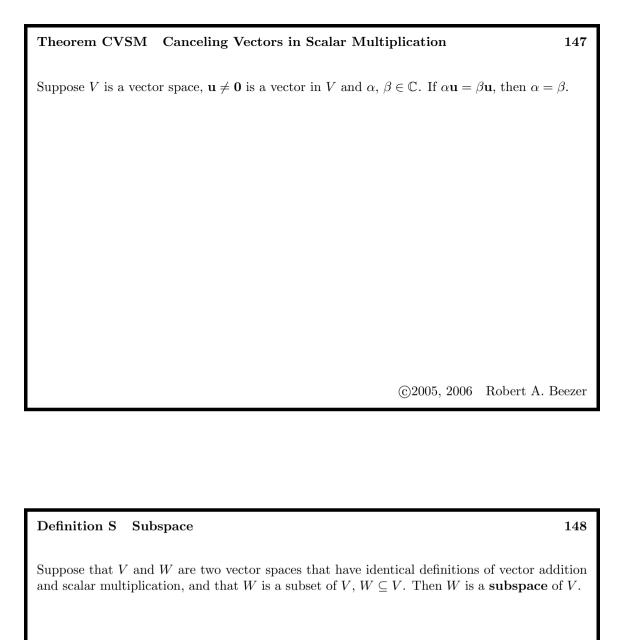
Suppose that V is a vector space, and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

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Theorem CSSM Canceling Scalars in Scalar Multiplication

146

Suppose V is a vector space, $\mathbf{u}, \mathbf{v} \in V$ and α is a nonzero scalar from \mathbb{C} . If $\alpha \mathbf{u} = \alpha \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.



Theorem TSS Testing Subsets for Subspaces

149

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

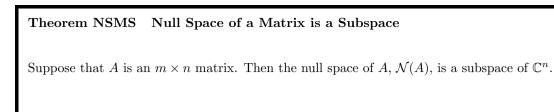
- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

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Definition TS Trivial Subspaces

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Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.



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Definition LC Linear Combination

152

151

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

Definition SS Span of a Set

153

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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Theorem SSS Span of a Set is a Subspace

154

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Theorem CSMS Column Space of a Matrix is a Subspace	155
Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .	
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Theorem RSMS Row Space of a Matrix is a Subspace	156
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .	

Theorem LNSMS Left I	Null Space of a	Matrix is a Subspace

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .

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Definition RLD Relation of Linear Dependence

158

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LI Linear Independence

159

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Definition TSVS To Span a Vector Space

160

Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V.

Theorem VR	RB Vector	Representation	Relative to	a Basis
THEOLEM ATO	LCD A GC LOI	1 tepresemanon	iterative to	a Dasis

162

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let \mathbf{w} be any vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \dots, a_m$ such that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$

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Definition B Basis

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem	CIIIID	Standard	T T : 4	T/ootoma		Dagia
THEORETT	31 V D	Standard		VECTORS	are a	Dasis

The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .

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Theorem CNMB Columns of Nonsingular Matrix are a Basis

164

Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

Theorem NME5 Nonsingular Matrix Equivalences, Round 5

165

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

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Theorem COB Coordinates and Orthonormal Bases

166

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$

Definition D	Dimension	167

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

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Theorem SSLD Spanning Sets and Linear Dependence

168

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of t+1 or more vectors from V is linearly dependent.

Theorem BIS Bases have Identical Sizes	169
Suppose that V is a vector space with a finite basis B and a second basis C	. Then B and C
have the same size.	
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Theorem DCM Dimension of \mathbb{C}^m	170
The dimension of \mathbb{C}^m (Example VSCV) is m .	

Theorem DP Dimension of P_n		171
The dimension of P_n (Example VSP) is $n+1$.		
		1
		I
		1
		I
		1
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The same DM Dimension of M		179
Theorem DM Dimension of M_{mn}		172
The dimension of M_{mn} (Example VSM) is mn .		



Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

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Definition ROM Rank Of a Matrix

174

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Theorem CRN Computing Rank and Nullity

175

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

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Theorem RPNC Rank Plus Nullity is Columns

176

Suppose that A is an $m \times n$ matrix. Then r(A) + n(A) = n.

Theorem RNNM Rank and Nullity of a Nonsingular Matrix

177

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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Theorem NME6 Nonsingular Matrix Equivalences, Round 6

178

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.

Theorem ELIS Extending Linearly Independent Sets

179

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

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Theorem G Goldilocks

180

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

Theorem PSSD	Proper Subspaces have Smaller Dimension	181
	V are subspaces of the vector space $W,$ such that $U \subsetneq V.$ Then dim (U) <
$\dim(V)$.		
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Theorem EDYES	Equal Dimensions Yields Equal Subspaces	182
Suppose that U and $\dim(V)$. Then $U = V$	V are subspaces of the vector space W , such that $U \subseteq V$ and dim (V).	U) =

Theorem RMRT	Rank of a Matrix i	is the Rank of the	Transpose

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

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Theorem DFS Dimensions of Four Subspaces

184

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. $\dim (\mathcal{R}(A)) = r$
- 4. $\dim (\mathcal{L}(A)) = m r$

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

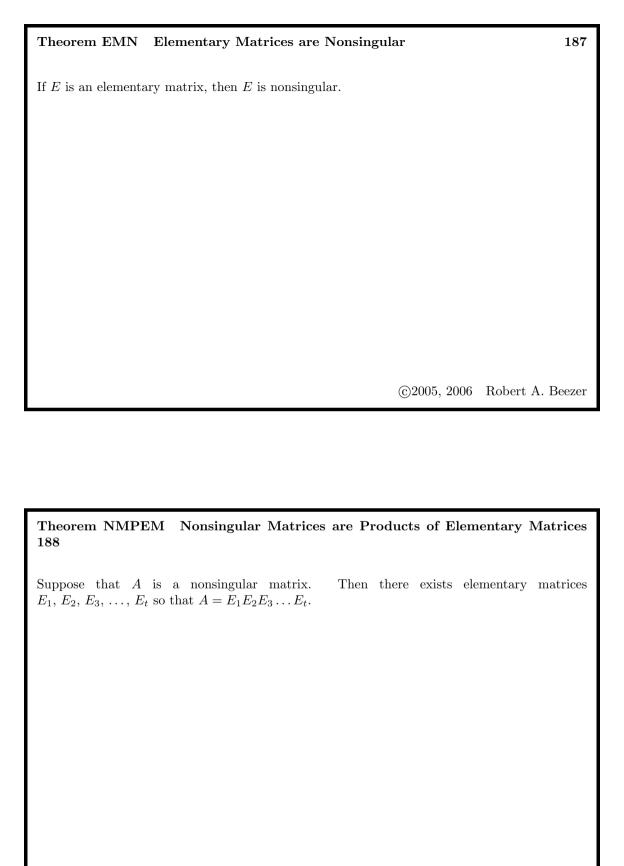
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Theorem EMDRO Elementary Matrices Do Row Operations

186

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B=E_{i,j}\left(\alpha\right)A$.



Definition SM SubMatrix

189

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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Definition DM Determinant of a Matrix

190

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

If A is a 1×1 matrix, then $det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - [A]_{14} \det(A(1|4)) + \dots + (-1)^{n+1} [A]_{1n} \det(A(1|n))$$

Theorem DMST Determinant of Matrices of Size Two

191

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

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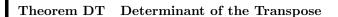
Theorem DER Determinant Expansion about Rows

192

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A\left(i|1\right))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A\left(i|2\right))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A\left(i|3\right))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A\left(i|n\right))} \qquad 1 \leq i \leq n \end{split}$$

which is known as **expansion** about row i.



Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

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Theorem DEC Determinant Expansion about Columns

194

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

which is known as **expansion** about column j.

Theorem DZRC Determinant with Zero Row or Column

195

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

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Theorem DRCS Determinant for Row or Column Swap

196

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM	Determinant for Row or Column Multiples	197
	square matrix. Let B be the square matrix obtained from calar α , or by multiplying a single column by the scalar	
$\alpha \det(A)$.	calai α , or by muniplying a single column by the scalar	α . Then $\det(D) =$
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Theorem DERC	Determinant with Equal Rows or Columns	198
	Determinant with Equal Rows or Columns quare matrix with two equal rows, or two equal columns.	

Theorem DRCMA	Determinant for Row or Column	n Multiples and Addition 199
a row by the scalar α	uare matrix. Let B be the square matrix and then adding it to another row, ong it to another column. Then $\det(B)$	r by multiplying a column by the
		©2005, 2006 Robert A. Beezer
Theorem DIM De	terminant of the Identity Matrix	200
For every $n \ge 1$, det (I	(n) = 1.	

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$

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Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication 202

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det\left(EA\right)=\det\left(E\right)\det\left(A\right)$$

Theorem SMZD Singular Matrices have Zero Determinants

203

Let A be a square matrix. Then A is singular if and only if det(A) = 0.

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Theorem NME7 Nonsingular Matrix Equivalences, Round 7

204

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.

Theorem DRMM	Determinant Respects Matrix N	Iultiplication	205
Suppose that A and B	B are square matrices of the same size.	Then $\det(AB) = \det(A) \det(A)$	(B).

Definition EEM Eigenvalues and Eigenvectors of a Matrix

206

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda \mathbf{x}$$

Theorem EMHE Every Matrix Has an Eigenvalue	207
Suppose A is a square matrix. Then A has at least one eigenvalue.	
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Definition CP Characteristic Polynomial	208
Suppose that A is a square matrix of size n. Then the characteristic polynomial of A polynomial $n_{+}(x)$ defined by	1 is the

polynomial $p_A(x)$ defined by

$$p_A(x) = \det\left(A - xI_n\right)$$

Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials 209
Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_{A}(\lambda) = 0$.
○2007 2006 Palant A. Paran
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Definition EM Eigenspace of a Matrix 210
Suppose that A is a square matrix and λ is an eigenvalue of A . Then the eigenspace of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.

Theorem EMS Eigenspace for a Matrix is a Subspace

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $\mathcal{E}_{A}(\lambda)$ is a subspace of the vector space \mathbb{C}^{n} .

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Theorem EMNS Eigenspace of a Matrix is a Null Space

212

211

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Definition AME	Algebraic Multiplicity of an Eigenvalue 213
	square matrix and λ is an eigenvalue of A . Then the algebraic multiplicity nighest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.
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Definition GME	Geometric Multiplicity of an Eigenvalue 214
Suppose that A is a	Geometric Multiplicity of an Eigenvalue 214 square matrix and λ is an eigenvalue of A . Then the geometric multi is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.
Suppose that A is a	square matrix and λ is an eigenvalue of A . Then the geometric multi -
Suppose that A is a	square matrix and λ is an eigenvalue of A . Then the geometric multi -
Suppose that A is a	square matrix and λ is an eigenvalue of A . Then the geometric multi -

Theorem EDELI	Eigenvectors with Distinct Eigenvalues are Linearly Independent
215	

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

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Theorem SMZE Singular Matrices have Zero Eigenvalues

216

Suppose A is a square matrix. Then A is singular if and only if $\lambda=0$ is an eigenvalue of A.

Theorem NME8 Nonsingular Matrix Equivalences, Round 8

217

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

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Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix

218

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA .

Theorem EOMP	Eigenvalues Of Matrix Powers	219
Suppose A is a squareigenvalue of A^s .	re matrix, λ is an eigenvalue of A , and $s \geq 0$ is an integer.	Then λ^s is an

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Theorem EPM Eigenvalues of the Polynomial of a Matrix

220

Suppose A is a square matrix and λ is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix q(A).

Theorem EIM Eigenvalues of the Inverse of a Matrix	221
Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then	$\frac{1}{\lambda}$ is an eigenvalue
of the matrix A^{-1} .	
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Theorem ETM Eigenvalues of the Transpose of a Matrix	222
Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenv	
Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenv	
Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenv	
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Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenv	

Theorem ERMCP I	Eigenvalues of Real Matrices con	me in Conjuga	te Pairs 223
	matrix with real entries and \mathbf{x} is an ector of A for the eigenvalue $\overline{\lambda}$.	igenvector of A	for the eigenvalue
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Theorem DCP Degree of the Characteristic Polynomial

224

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_{A}\left(x\right)$, has degree n.

Theorem NEM Number of Eigenvalues of a Matrix

225

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

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Theorem ME Multiplicities of an Eigenvalue

226

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matri	x 227
Suppose that A is a square matrix of size n . Then A cannot have more t	han n distinct eigen-
values.	
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Theorem HMRE Hermitian Matrices have Real Eigenvalues	228
Theorem HMRE Hermitian Matrices have Real Eigenvalues Suppose that A is a Hermitian matrix and λ is an eigenvalue of A . Then	

Theorem HMOE Hermitian Matrices	have Orthogonal Eigenvectors 229
	\mathbf{x} and \mathbf{y} are two eigenvectors of A for different
eigenvalues. Then ${\bf x}$ and ${\bf y}$ are orthogonal vec	tors.
	©2005, 2006 Robert A. Beezer
Definition SIM Similar Matrices	230
Suppose A and B are two square matrices of a nonsingular matrix of size n , S , such that A	size n . Then A and B are similar if there exists $A = S^{-1}BS$.

Theorem SER Similarity is an Equivalence Relation

231

Suppose A, B and C are square matrices of size n. Then

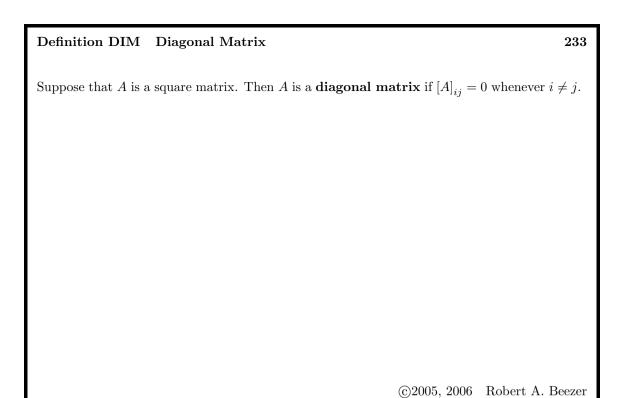
- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE Similar Matrices have Equal Eigenvalues

232

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.



Definition DZM Diagonalizable Matrix

234

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC	Diagonalization Characterization

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

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Theorem DMFE Diagonalizable Matrices have Full Eigenspaces

236

235

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED Distinct Eigenvalues implies Diagonalizable	237
Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonal	lizable.
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r	

Definition LT Linear Transformation

239

A linear transformation, $T : U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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Theorem LTTZZ Linear Transformations Take Zero to Zero

240

Suppose $T \colon U \mapsto V$ is a linear transformation. Then $T\left(\mathbf{0}\right) = \mathbf{0}$.

Theorem MBLT Matrices Build Linear Transformations

241

Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors

242

Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations

243

Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$

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Theorem LTDB Linear Transformation Defined on a Basis

244

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U and \mathbf{w} is a vector from U. Let $a_1, a_2, a_3, \dots, a_n$ be the scalars from $\mathbb C$ such that

$$\mathbf{w} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_n \mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$$

Definition PI Pre-Image

245

Suppose that $T: U \mapsto V$ is a linear transformation. For each \mathbf{v} , define the **pre-image** of \mathbf{v} to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v} \}$$

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Definition LTA Linear Transformation Addition

246

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T+S\colon U\mapsto V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT Si	um of Lincon Tro	neformatione is a	Lincor Tron	cformation	247

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are two linear transformations with the same domain and codomain. Then $T+S\colon U\mapsto V$ is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication

248

Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T: U \mapsto V$ whose outputs are defined by

$$\left(\alpha T\right)\left(\mathbf{u}\right)=\alpha T\left(\mathbf{u}\right)$$

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 249
Suppose that $T \colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T) \colon U \mapsto V$ is a linear transformation.

Theorem VSLT Vector Space of Linear Transformations

250

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, LT (U, V) is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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Definition LTC	Linear	Transformation	Composition
Demindon Li C	Linear	Transior manon	Composition

251

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T) \colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

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Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 252

Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are linear transformations. Then $(S\circ T)\colon U\mapsto W$ is a linear transformation.

Definition ILT Injective Linear Transformation

253

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

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Definition KLT Kernel of a Linear Transformation

254

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the \mathbf{kernel} of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$$

Theorem KLTS Kernel of a Linear Transformation is a Subspace

255

Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

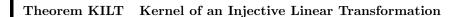
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Theorem KPI Kernel and Pre-Image

256

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}\left(\mathbf{v}\right) = \left\{ \left. \mathbf{u} + \mathbf{z} \, \right| \, \mathbf{z} \in \mathcal{K}(T) \right\} = \mathbf{u} + \mathcal{K}(T)$$



257

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}.$

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Theorem ILTLI Injective Linear Transformations and Linear Independence 258

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases

259

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

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Theorem ILTD Injective Linear Transformations and Dimension

260

Suppose that $T \colon U \mapsto V$ is an injective linear transformation. Then $\dim (U) \leq \dim (V)$.

Theorem CILTI Composition of Injective Linear T	ransformation	s is Injective 261
-		-
Commence that To II and I call to IV and in institution linear te		T $= (C - T) \cdot II \cdot$
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are injective linear t W is an injective linear transformation.	ransformations.	Then $(S \circ I): U \mapsto$
w is an injective linear transformation.		
	O 200 F 200 A	D 1 (A D
	©2005, 2006	Robert A. Beezer
Definition SLT Surjective Linear Transformation		262
•		
	•	- T7 /1

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Definition RLT Range of a Linear Transformation

263

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

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Theorem RLTS Range of a Linear Transformation is a Subspace

264

Suppose that $T: U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT Range of a Surjective Linear Transformation

265

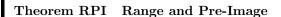
Suppose that $T \colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

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Theorem SSRLT Spanning Set for Range of a Linear Transformation

266

Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.



267

Suppose that $T: U \mapsto V$ is a linear transformation. Then

 $\mathbf{v} \in \mathcal{R}(T)$ if and only if $T^{-1}(\mathbf{v}) \neq \emptyset$

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Theorem SLTB Surjective Linear Transformations and Bases

268

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.

Theorem SLTD	Surjective Linear Transformations and Dimension 20	269
Suppose that $T: U$	$U\mapsto V$ is a surjective linear transformation. Then $\dim\left(U\right)\geq\dim\left(V\right)$.	•
	©2005, 2006 Robert A. Beez	Tor
	©2005, 2006 Robert A. Beez	zei
Theorem CSLTS 270	S Composition of Surjective Linear Transformations is Surjective	ve
	$U\mapsto V$ and $S\colon V\mapsto W$ are surjective linear transformations. Then (Surjective linear transformation.	<i>;</i> 0
1		

Definition IDLT Identity Linear Transformation

271

The identity linear transformation on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W (\mathbf{w}) = \mathbf{w}$$

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Definition IVLT Invertible Linear Transformations

272

Suppose that $T\colon U\mapsto V$ is a linear transformation. If there is a function $S\colon V\mapsto U$ such that

$$S \circ T = I_U$$

$$T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT	Inverse of a Linear	Transformation i	s a Linear	Transformation
273				

Suppose that $T\colon U\mapsto V$ is an invertible linear transformation. Then the function $T^{-1}\colon V\mapsto U$ is a linear transformation.

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Theorem IILT Inverse of an Invertible Linear Transformation

274

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$.

Theorem ICLT	Inverse of a	Composition	of Linear	Transformations

277

Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are invertible linear transformations. Then $S\circ T$ is invertible and $(S\circ T)^{-1}=T^{-1}\circ S^{-1}$.

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Definition IVS Isomorphic Vector Spaces

278

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED	Isomorphic Vector Spaces have Eq	ual Dimensio	on 279
Suppose U and V are	e isomorphic vector spaces. Then $\dim (U$	$(V) = \dim(V).$	
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Definition ROLT	Rank Of a Linear Transformation		280

Suppose that $T:U\mapsto V$ is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

$$r\left(T\right)=\dim\left(\mathcal{R}(T)\right)$$

Definition NOLT Nullity Of a Linear Transformation

281

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

$$n(T) = \dim (\mathcal{K}(T))$$

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Theorem ROSLT Rank Of a Surjective Linear Transformation

282

Suppose that $T:U\mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r\left(T\right)=\dim\left(V\right)$, if and only if T is surjective.

Theorem NOILT Nullity Of an Injective Linear Transformation

283

Suppose that $T:U\mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n\left(T\right)=0$, if and only if T is injective.

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Theorem RPNDD Rank Plus Nullity is Domain Dimension

284

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

$$r\left(T\right)+n\left(T\right)=\dim\left(U\right)$$

Definition VR Vector Representation

285

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \dots, a_n$ so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i$$

$$1 \le i \le n$$

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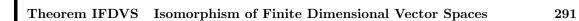
Theorem VRLT Vector Representation is a Linear Transformation

286

The function ρ_B (Definition VR) is a linear transformation.

The function ρ_B (Definition VR) is an injective linear transformation.	
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Theorem VRS Vector Representation is Surjective	288
The function ρ_B (Definition VR) is a surjective linear transformation.	
The function ρ_B (Definition VR) is a surjective linear transformation.	
The function ρ_B (Definition VR) is a surjective linear transformation.	
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The function ρ_B (Definition VR) is a surjective linear transformation.	
The function ρ_B (Definition VR) is a surjective linear transformation.	

Theorem VRILT Vector Representation is an Invertible Linear Transformation 289
The function of (Definition VD) is an inventible linear transformation
The function ρ_B (Definition VR) is an invertible linear transformation.
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Theorem CFDVS Characterization of Finite Dimensional Vector Spaces 290
Theorem CFDVS Characterization of Finite Dimensional Vector Spaces 290 Suppose that V is a vector space with dimension n . Then V is isomorphic to \mathbb{C}^n .



Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

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Theorem CLI Coordinatization and Linear Independence

292

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n .

Theorem CSS Coordinatization and Spanning Sets

293

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\} \rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\} \rangle$.

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Definition MR Matrix Representation

294

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\left.\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right)\right|\left.\rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right)\right|\left.\rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right)\right|...\left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$$

Suppose that $T\colon U\mapsto V$ is a linear transformation, B is a basis for $U,\,C$ is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u}\in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations296

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

Theorem MRMLT	Matrix Representation of a Multiple of a Linear	Transforma-
${f tion}$		297

Suppose that $T\colon U\mapsto V$ is a linear transformation, $\alpha\in\mathbb{C},\,B$ is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 298

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S\circ T}=M_{C,D}^SM_{B,C}^T$$

Theorem KNSI Kernel and Null Space Isomorphism

299

Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

300

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

Theorem IMR Invertible Matrix Representations

301

Suppose that $T: U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation

302

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{L}S(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation

304

Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

Definition CBM Change-of-Basis Matrix

305

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$

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Theorem CB Change-of-Basis

306

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_{C}\left(\mathbf{v}\right)=C_{B,C}\rho_{B}\left(\mathbf{v}\right)$$

Theorem ICBM Inverse of Change-of-Basis Matrix

307

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

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Theorem MRCB Matrix Representation and Change of Basis

308

Suppose that $T\colon U\mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

Theorem SCB Similarity and Change of Basis

309

Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

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Theorem EER Eigenvalues, Eigenvectors, Representations

310

Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

Definition NLT Nilpotent Linear Transformation

311

Suppose that $T: V \mapsto V$ is a linear transformation such that there is an integer p > 0 such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T.

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Definition JB Jordan Block

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Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$\left[J_{n}\left(\lambda\right)\right]_{ij} = \begin{cases} \lambda & i = j\\ 1 & j = i+1\\ 0 & \text{otherwise} \end{cases}$$

Theorem ENLT Eigenvalu	es of Nilpotent Linear Transformations	313
Suppose that $T: V \mapsto V$ is a lin	ear transformation and λ is an eigenvalue of T . Then	$\lambda = 0.$
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Theorem DNLT Diagonali	zable Nilpotent Linear Transformations	314
Suppose the linear transformation T is the zero linear transformation.	on $T \colon V \mapsto V$ is nilpotent. Then T is diagonalizable if ion.	and only

Theorem KPLT Kernels of Powers of Linear Transformations

Suppose $T: V \mapsto V$ is a linear transformation, where dim (V) = n. Then there is an integer m, $0 \le m \le n$, such that

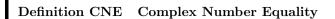
$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

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Theorem KPNLT Kernels of Powers of Nilpotent Linear Transformations 316

Suppose $T\colon V\mapsto V$ is a nilpotent linear transformation with index p and $\dim\left(V\right)=n.$ Then $0\leq p\leq n$ and

$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$



The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are **equal**, denoted $\alpha = \beta$, if a = c and b = d.

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Definition CNA Complex Number Addition

318

The **sum** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is (a + c) + (b + d)i.

Definition	CNM	Complex	Number	Multiplication
Denningn	CITIVI	Complex	14 milliner	Multiplication

The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is (ac - bd) + (ad + bc)i.

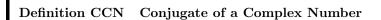
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Theorem PCNA Properties of Complex Number Arithmetic

320

The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Commutativity, Complex Numbers For any α , $\beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- MCCN Multiplicative Commutativity, Complex Numbers For any $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- **ZCN Zero, Complex Numbers** There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\frac{1}{\alpha}\alpha = 1$.



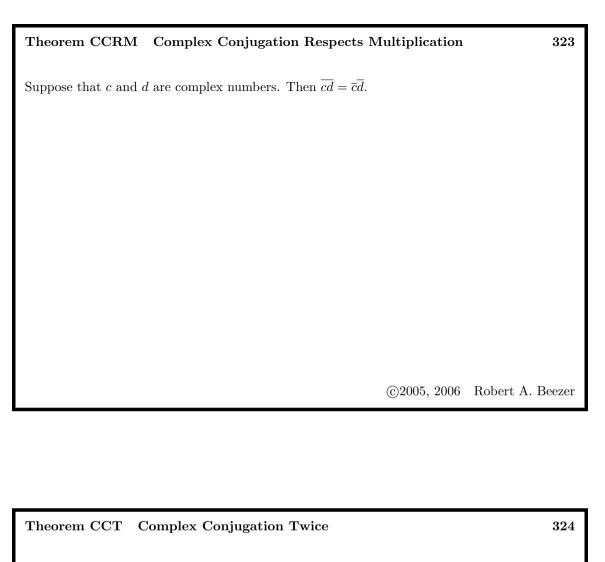
The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

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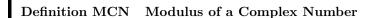
Theorem CCRA Complex Conjugation Respects Addition

322

Suppose that c and d are complex numbers. Then $\overline{c+d}=\overline{c}+\overline{d}.$



Suppose that c is a complex number. Then $\overline{\overline{c}}=c.$



The **modulus** of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

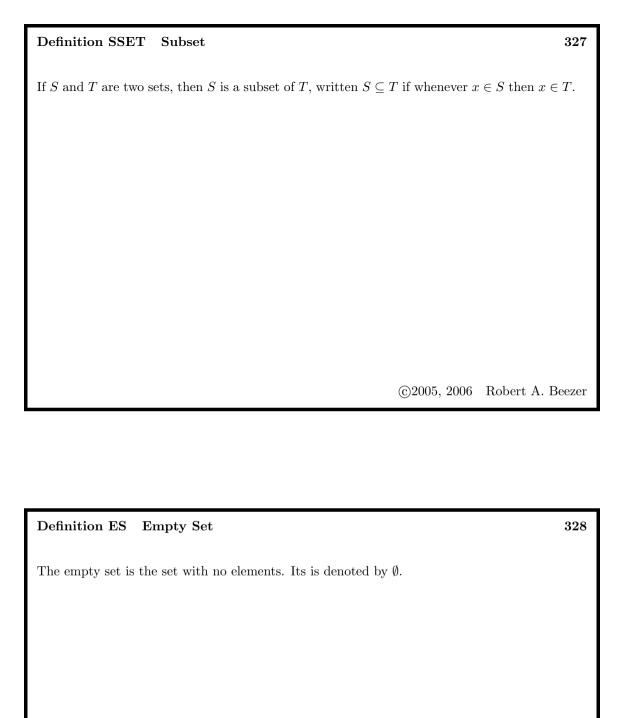
$$|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}.$$

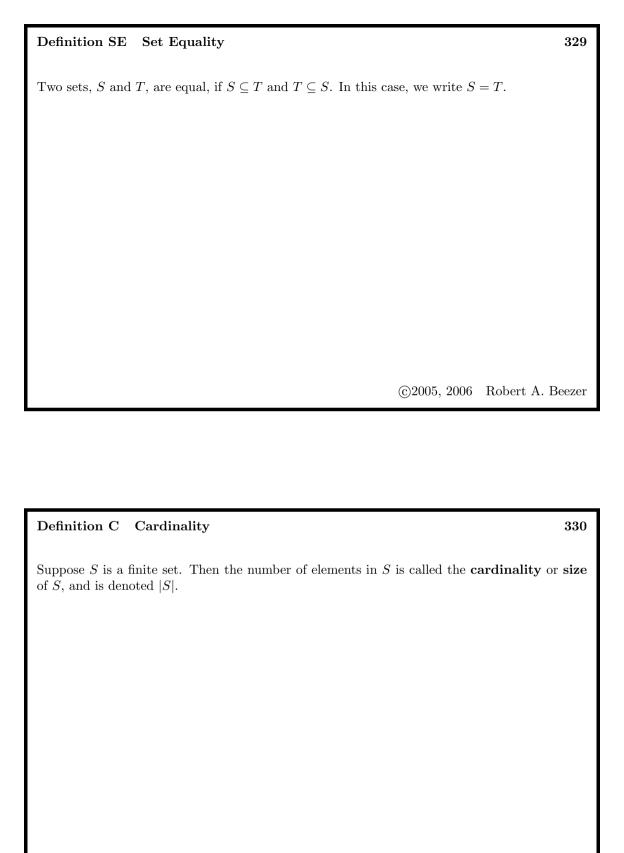
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Definition SET Set

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A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.





Definition	SII	Sat	Linion

Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$

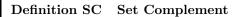
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Definition SI Set Intersection

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Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$ if and only if $x \in S$ and $x \in T$



Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$