Flash Cards

to accompany

A First Course in Linear Algebra

by
Robert A. Beezer
Department of Mathematics and Computer Science
University of Puget Sound

Definition SLE   System of Linear Equations 1

A **system of linear equations** is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Definition ESYS   Equivalent Systems 2

Two systems of linear equations are **equivalent** if their solution sets are equal.

Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

1. Swap the locations of two equations in the list.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M   Matrix

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Definition CV   Column Vector

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\underset{\sim}{u}$. To refer to the **entry** or **component** that is number i in the list that is the vector $\mathbf{v}$ we write $[\mathbf{v}]_i$.

Definition ZV   Zero Vector

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \leq i \leq m$.

Definition CM   Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

If A is the coefficient matrix of a system of linear equations and **b** is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants **b**. Then the **augmented matrix** of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (number $n + 1$) is the column vector **b**. This matrix will be written as $[A \,|\, \mathbf{b}]$.

Definition RO   Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.

2. Multiply each entry of a single row by a nonzero quantity.

3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$:   Swap the location of rows i and j.

2. αR_i: Multiply row i by the nonzero scalar α.

3. $\alpha R_i + R_j$:   Multiply row i by the scalar α and add to row j.

©2005, 2006   Robert A. Beezer

Definition REM   Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

©2005, 2006   Robert A. Beezer

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r.
A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1,\ d_2,\ d_3,\ \ldots,\ d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot colums will be denoted as $F = \{f_1,\ f_2,\ f_3,\ \ldots,\ f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Suppose A is a matrix. Then there is a matrix B so that

1. A and B are row-equivalent.

2. B is in reduced row-echelon form.

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS   Possible Solution Sets for Linear Systems   **25**

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI   Consistent, More Variables than Equations, Infinite solutions 26

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE Trivial Solution to Homogeneous Systems of Equations

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem NMRRI   Nonsingular Matrices Row Reduce to the Identity matrix   35

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NMTNS   Nonsingular Matrices have Trivial Null Spaces   36

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

Definition VSCV   Vector Space of Column Vectors

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE   Column Vector Equality

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Definition CVA   Column Vector Addition

Given the vectors $\mathbf{u}$ and $\mathbf{v}$ the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad\qquad 1 \le i \le m$$

Definition CVSM   Column Vector Scalar Multiplication

Given the vector $\mathbf{u}$ and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of $\mathbf{u}$ by α, $\alpha\mathbf{u}$ is defined by

$$[\alpha\mathbf{u}]_i = \alpha\,[\mathbf{u}]_i \qquad\qquad 1 \le i \le m$$

Theorem VSPCV   Vector Space Properties of Column Vectors   **43**

Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC   Additive Closure, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- **SCC   Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.

- **CC   Commutativity, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AAC   Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **ZC   Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.

- **AIC   Additive Inverses, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMAC   Scalar Multiplication Associativity, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVAC   Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSAC   Distributivity across Scalar Addition, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **OC   One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Definition LCCV   Linear Combination of Column Vectors   **44**

Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars α_1, α_2, α_3, $\ldots$, α_n, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$. Then $\mathbf{x}$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n + 1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n + 1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}$, $\mathbf{u}_j$, $1 \leq j \leq n - r$ of size n by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \right\}$$

Theorem PSPHS   Particular Solution Plus Homogeneous Solutions

Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A,\, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Theorem RREFU   Reduced Row-Echelon Form is Unique

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{\, \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_p \mathbf{u}_p \,|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p \}$$
$$= \left\{ \sum_{i=1}^{p} \alpha_i \mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p \right\}$$

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \le j \le n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\, i = f_j \\ 0 & \text{if } i \in F,\, i \neq f_j \\ -[B]_{k, f_j} & \text{if } i \in D,\, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\} \rangle.$$

Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$, a true statement of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.

The set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A form a linearly independent set.

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \le j \le n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \le t \le n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_n$.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \ldots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots \mathbf{v}_{d_r}\}$ is a linearly independent set.

2. $W = \langle T \rangle$.

Suppose that $\mathbf{u}$ is a vector from $\mathbb{C}^m$. Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$\left[\overline{\mathbf{u}}\right]_i = \overline{\left[\mathbf{u}\right]_i} \qquad\qquad 1 \le i \le m$$

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha}\, \overline{\mathbf{x}}$$

Given the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\, \mathbf{v} \rangle = [\mathbf{u}]_1 \,\overline{[\mathbf{v}]_1} + [\mathbf{u}]_2 \,\overline{[\mathbf{v}]_2} + [\mathbf{u}]_3 \,\overline{[\mathbf{v}]_3} + \cdots + [\mathbf{u}]_m \,\overline{[\mathbf{v}]_m} = \sum_{i=1}^{m} [\mathbf{u}]_i \,\overline{[\mathbf{v}]_i}$$

Theorem IPVA   Inner Product and Vector Addition

Suppose $\mathbf{u}\,\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

$$1. \quad \langle \mathbf{u} + \mathbf{v},\, \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{w} \rangle + \langle \mathbf{v},\, \mathbf{w} \rangle$$
$$2. \quad \langle \mathbf{u},\, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle + \langle \mathbf{u},\, \mathbf{w} \rangle$$

Theorem IPSM   Inner Product and Scalar Multiplication

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

$$1. \quad \langle \alpha\mathbf{u},\, \mathbf{v} \rangle = \alpha \langle \mathbf{u},\, \mathbf{v} \rangle$$
$$2. \quad \langle \mathbf{u},\, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u},\, \mathbf{v} \rangle$$

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{v} \rangle = \overline{\langle \mathbf{v},\, \mathbf{u} \rangle}$.

The **norm** of the vector $\mathbf{u}$ is the scalar quantity in $\mathbb{C}$

$$\|\mathbf{u}\| = \sqrt{\left|[\mathbf{u}]_1\right|^2 + \left|[\mathbf{u}]_2\right|^2 + \left|[\mathbf{u}]_3\right|^2 + \cdots + \left|[\mathbf{u}]_m\right|^2} = \sqrt{\sum_{i=1}^{m} \left|[\mathbf{u}]_i\right|^2}$$

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3,\, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1},\, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Suppose $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \qquad\qquad 1 \leq i \leq m,\, 1 \leq j \leq n$$

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad\qquad 1 \le i \le m, \, 1 \le j \le n$$

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM   Additive Closure, Matrices** If A, $B \in M_{mn}$, then $A + B \in M_{mn}$.

- **SCM   Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

- **CM   Commutativity, Matrices** If A, $B \in M_{mn}$, then $A + B = B + A$.

- **AAM   Additive Associativity, Matrices** If A, B, $C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

- **ZM   Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.

- **AIM   Additive Inverses, Matrices** If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.

- **SMAM   Scalar Multiplication Associativity, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.

- **DMAM   Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and A, $B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

- **DSAM   Distributivity across Scalar Addition, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

- **OM   One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM   Zero Matrix   **81**

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM   Transpose of a Matrix   **82**

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

Definition SYM   Symmetric Matrix

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS   Symmetric Matrices are Square

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA   Transpose and Matrix Addition   **85**

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Theorem TMSM   Transpose and Matrix Scalar Multiplication   **86**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{\left[A\right]_{ij}}$$

Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Theorem MCT   Matrix Conjugation and Transposes   91

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.

2005, 2006   Robert A. Beezer

Definition MVP   Matrix-Vector Product   92

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \mathbf{A}_n$$

2005, 2006   Robert A. Beezer

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, ..., $\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Suppose A is an $m \times n$ matrix and $B =$ is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of AB are given by

$$
\begin{aligned}
[AB]_{ij} &= [A]_{i1}[B]_{1j} + [A]_{i2}[B]_{2j} + [A]_{i3}[B]_{3j} + \cdots + [A]_{in}[B]_{nj} \\
&= \sum_{k=1}^{n} [A]_{ik}[B]_{kj}
\end{aligned}
$$

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Theorem MMCC   Matrix Multiplication and Complex Conjugation

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT   Matrix Multiplication and Transposes

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m. Then the set
$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \le j \le m\}$$
is the set of **standard unit vectors** in $\mathbb{C}^m$.

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Theorem MIU   Matrix Inverse is Unique   **109**

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS   Socks and Shoes   **110**

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI   Matrix Inverse of a Matrix Inverse   **111**

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Theorem MIT   Matrix Inverse of a Transpose   **112**

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Theorem MISM   Matrix Inverse of a Scalar Multiple   **113**

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem NPNT   Nonsingular Product has Nonsingular Terms   **114**

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^t Q = I_n$. Then we say Q is **orthogonal**.

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Theorem COMOS   Columns of Orthogonal Matrices are Orthonormal Sets   121

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set.

Theorem OMPIP   Orthogonal Matrices Preserve Inner Products   122

Suppose that Q is an orthogonal matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\, Q\mathbf{v} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle \qquad \text{and} \qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|$$

If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$.

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = \left(\overline{A}\right)^t$

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}\rangle$$

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.

2. $\mathcal{C}(A) = \langle T \rangle$.

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Definition EEF   Extended Echelon Form   135

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom $m - r$ rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array} \right]$$

Theorem PEEF   Properties of Extended Echelon Form   136

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. J is nonsingular.

2. $B = JA$.

3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.

4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.

5. L is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m - r$ rows. Then

1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.

2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.

3. The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.

4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS   Vector Space   138
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "$+$", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- **AC   Additive Closure** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

- **SC   Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

- **C   Commutativity** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AA   Additive Associativity** If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Z   Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

- **AI   Additive Inverses** If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMA   Scalar Multiplication Associativity** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVA   Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSA   Distributivity across Scalar Addition** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **O   One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue

Theorem ZVU   Zero Vector is Unique

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU   Additive Inverses are Unique

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CVSM   Canceling Vectors in Scalar Multiplication   **147**

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and $\alpha,\ \beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Definition S   Subspace   **148**

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS   Testing Subsets for Subspaces

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.

2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.

3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Definition TS   Trivial Subspaces

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Suppose that V is a vector space.   Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

Definition SS   Span of a Set

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^{t} \alpha_i \mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}$$

Theorem SSS   Span of a Set is a Subspace

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$.

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Suppose that A is a square matrix of size m. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Theorem NME5   Nonsingular Matrix Equivalences, Round 5   161

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

Theorem COB   Coordinates and Orthonormal Bases   162

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.

Theorem BIS   Bases have Identical Sizes

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM   Dimension of $\mathbb{C}^m$

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP   Dimension of P_n   **167**

The dimension of P_n (Example VSP) is $n + 1$.

Theorem DM   Dimension of M_{mn}   **168**

The dimension of M_{mn} (Example VSM) is mn.

Definition NOM   Nullity Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM   Rank Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Theorem RNNM   Rank and Nullity of a Nonsingular Matrix   **173**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. The rank of A is n, $r(A) = n$.

3. The nullity of A is zero, $n(A) = 0$.

Theorem NME6   Nonsingular Matrix Equivalences, Round 6   **174**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.

2. If $m < t$, then S does not span V.

3. If $m = t$ and S is linearly independent, then S spans V.

4. If $m = t$ and S spans V, then S is linearly independent.

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem DFS   Dimensions of Four Subspaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim\left(\mathcal{N}(A)\right) = n - r$

2. $\dim\left(\mathcal{C}(A)\right) = r$

3. $\dim\left(\mathcal{R}(A)\right) = r$

4. $\dim\left(\mathcal{L}(A)\right) = m - r$

Definition ELEM   Elementary Matrices

1. $E_{i,j}$ is the square matrix of size n with

$$
[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}
$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$
[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}
$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$
[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}
$$

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.

2. If the row operation multiplies row i by α, then $B = E_i\left(\alpha\right)A$.

3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}\left(\alpha\right)A$.

If E is an elementary matrix, then E is nonsingular.

Theorem NMPEM   Nonsingular Matrices are Products of Elementary Matrices 183

Suppose that A is a nonsingular matrix.   Then there exists elementary matrices E_1, E_2, E_3, ..., E_t so that $A = E_1 E_2 E_3 \ldots E_t$.

Definition SM   SubMatrix 184

Suppose that A is an $m \times n$ matrix. Then the **submatrix** $A\,(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM   Determinant of a Matrix   **185**

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - \cdots + (-1)^{n+1} [A]_{1n} \det(A(1|n))$$

Theorem DMST   Determinant of Matrices of Size Two   **186**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Suppose that A is a square matrix of size n. Then

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2))$$
$$+ (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \qquad 1 \leq i \leq n$$

which is known as **expansion** about row i.

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Theorem DEC   Determinant Expansion about Columns Suppose that A is a square matrix of size n. Then

$$
\det(A) = (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j))
$$
$$
+ (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j)) \qquad 1 \le j \le n
$$

which is known as **expansion** about column j.

©2005, 2006   Robert A. Beezer

Theorem DZRC   Determinant with Zero Row or Column Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

©2005, 2006   Robert A. Beezer

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $\det(B) = \alpha \det(A)$.

Theorem DERC   Determinant with Equal Rows or Columns   193

Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Theorem DRCMA   Determinant for Row or Column Multiples and Addition 194

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

Theorem DIM   Determinant of the Identity Matrix   **195**

For every $n \geq 1$, $\det(I_n) = 1$.

Theorem DEM   Determinants of Elementary Matrices   **196**

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1. $\det(E_{i,j}) = -1$

2. $\det(E_i(\alpha)) = \alpha$

3. $\det(E_{i,j}(\alpha)) = 1$

Theorem DEMMM   Determinants, Elementary Matrices, Matrix Multiplication 197

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det(EA) = \det(E)\det(A)$$

Theorem SMZD   Singular Matrices have Zero Determinants   198

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NME7   Nonsingular Matrix Equivalences, Round 7   **199**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

Theorem DRMM   Determinant Respects Matrix Multiplication   **200**

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in $\mathbb{C}^n$, and λ is a scalar in $\mathbb{C}$. Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

Suppose A is a square matrix. Then A has at least one eigenvalue.

Definition CP   Characteristic Polynomial

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

Theorem EMRCP   Eigenvalues of a Matrix are Roots of Characteristic Polynomials

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM   Eigenspace of a Matrix

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $E_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Theorem EMS   Eigenspace for a Matrix is a Subspace

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multi-plicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$.

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM   Eigenvalues of a Scalar Multiple of a Matrix

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP   Eigenvalues Of Matrix Powers

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Suppose that A is a square matrix of size n with distinct eigenvalues λ_1, λ_2, λ_3, ..., λ_k. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Theorem ME   Multiplicities of an Eigenvalue   **221**

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Theorem MNEM   Maximum Number of Eigenvalues of a Matrix   **222**

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)

2. If A is similar to B, then B is similar to A. (Symmetric)

3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.

Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$.

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMLE   Diagonalizable Matrices have Large Eigenspaces   231

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED   Distinct Eigenvalues implies Diagonalizable   232

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Definition LT   Linear Transformation   **234**

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T\left(\mathbf{u}_1 + \mathbf{u}_2\right) = T\left(\mathbf{u}_1\right) + T\left(\mathbf{u}_2\right)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$

2. $T\left(\alpha\mathbf{u}\right) = \alpha T\left(\mathbf{u}\right)$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV   Matrix of a Linear Transformation, Column Vectors

Suppose that $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC   Linear Transformations and Linear Combinations

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, $\ldots$, a_t are scalars from $\mathbb{C}$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t)$$

Suppose that $T \colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ be the scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T\left(\mathbf{w}\right) = a_1 T\left(\mathbf{u}_1\right) + a_2 T\left(\mathbf{u}_2\right) + a_3 T\left(\mathbf{u}_3\right) + \cdots + a_n T\left(\mathbf{u}_n\right)$$

Suppose that $T \colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{\, \mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v}\,\right\}$$

Definition LTA   Linear Transformation Addition**241**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT   Sum of Linear Transformations is a Linear Transformation 242

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Suppose that $T : U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T : U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Suppose that $T : U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T) : U \mapsto V$ is a linear transformation.

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\text{LT}(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T): U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT   Composition of Linear Transformations is a Linear Transformation   **247**
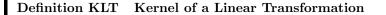
Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a linear transformation.

Definition ILT   Injective Linear Transformation   **248**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{\, \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \,\}$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then
$$T^{-1}(\mathbf{v}) = \{\,\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\,\} = \mathbf{u} + \mathcal{K}(T)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Theorem ILTLI   Injective Linear Transformations and Linear Independence   253

Suppose that $T\colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB   Injective Linear Transformations and Bases   254

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is an injective linear transformation.

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) \,|\, \mathbf{u} \in U \}$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT   Spanning Set for Range of a Linear Transformation   261

Suppose that $T\colon U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.

Theorem RPI   Range and Pre-Image   262

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a spanning set for V.

Suppose that $T \colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

**Theorem CSLTS   Composition of Surjective Linear Transformations is Surjective
265**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT   Identity Linear Transformation   266

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

Suppose that $T: U \mapsto V$ is a linear transformation. If there is a function $S: V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}: V \mapsto U$ is a linear transformation.

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT   Composition of Invertible Linear Transformations   **271**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T)\colon U \mapsto W$ is an invertible linear transformation.

Theorem ICLT   Inverse of a Composition of Linear Transformations   **272**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS   Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T \colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED   Isomorphic Vector Spaces have Equal Dimension

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,
$$r(T) = \dim\left(\mathcal{R}(T)\right)$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n(T)$, is the dimension of the kernel of T,
$$n(T) = \dim\left(\mathcal{K}(T)\right)$$

Theorem ROSLT   Rank Of a Surjective Linear Transformation

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT   Nullity Of an Injective Linear Transformation

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Suppose that $T : U \mapsto V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define a function $\rho_B : V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \ldots, a_n$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$[\rho_B(\mathbf{w})]_i = a_i \qquad\qquad 1 \leq i \leq n$$

Theorem VRLT   Vector Representation is a Linear Transformation

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI   Vector Representation is Injective

The function ρ_B (Definition VR) is an injective linear transformation.

The function ρ_B (Definition VR) is a surjective linear transformation.

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS   Characterization of Finite Dimensional Vector Spaces   285

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS   Isomorphism of Finite Dimensional Vector Spaces   286

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI   Coordinatization and Linear Independence   287

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B\left(\mathbf{u}_1\right),\, \rho_B\left(\mathbf{u}_2\right),\, \rho_B\left(\mathbf{u}_3\right),\, \ldots,\, \rho_B\left(\mathbf{u}_k\right)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS   Coordinatization and Spanning Sets   288

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}\rangle$ if and only if $\rho_B\left(\mathbf{u}\right) \in \langle\{\rho_B\left(\mathbf{u}_1\right),\, \rho_B\left(\mathbf{u}_2\right),\, \rho_B\left(\mathbf{u}_3\right),\, \ldots,\, \rho_B\left(\mathbf{u}_k\right)\}\rangle$.

Definition MR   Matrix Representation Suppose that $T \colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = [\,\rho_C\left(T\left(\mathbf{u}_1\right)\right)|\ \rho_C\left(T\left(\mathbf{u}_2\right)\right)|\ \rho_C\left(T\left(\mathbf{u}_3\right)\right)|\ldots|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\,]$$

© 2005, 2006   Robert A. Beezer

Theorem FTMR   Fundamental Theorem of Matrix Representation Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

© 2005, 2006   Robert A. Beezer

Theorem MRSLT   Matrix Representation of a Sum of Linear Transformations291

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

Theorem MRMLT   Matrix Representation of a Multiple of a Linear Transformation   292

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^{S} M_{B,C}^{T}$$

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^{T}$,

$$\mathcal{K}(T) \cong \mathcal{N}\left(M_{B,C}^{T}\right)$$

Theorem RCSI **Range and Column Space Isomorphism**

Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^{T}$,

$$\mathcal{R}(T) \cong \mathcal{C}\left(M_{B,C}^{T}\right)$$

Theorem IMR **Invertible Matrix Representations**

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^{T}$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^{T}\right)^{-1}$$

Suppose that A is a square matrix of size n and $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

13. The linear transformation $T\colon \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Definition EELT   Eigenvalue and Eigenvector of a Linear Transformation   299

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM   Change-of-Basis Matrix   300

Suppose that V is a vector space, and $I_V\colon V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M_{B,C}^{I_V} \\
&= \left[\, \rho_C\left(I_V\left(\mathbf{v}_1\right)\right)|\ \rho_C\left(I_V\left(\mathbf{v}_2\right)\right)|\ \rho_C\left(I_V\left(\mathbf{v}_3\right)\right)|\ldots|\rho_C\left(I_V\left(\mathbf{v}_n\right)\right)\,\right] \\
&= \left[\, \rho_C\left(\mathbf{v}_1\right)|\ \rho_C\left(\mathbf{v}_2\right)|\ \rho_C\left(\mathbf{v}_3\right)|\ldots|\rho_C\left(\mathbf{v}_n\right)\,\right]
\end{aligned}
$$

Suppose that $\mathbf{v}$ is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^{T} = C_{E,D} M_{C,E}^{T} C_{B,C}$$

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^{T} = C_{B,C}^{-1} M_{C,C}^{T} C_{B,C}$$

Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ.

The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

Suppose that c and d are complex numbers. Then $\overline{c+d} = \overline{c} + \overline{d}$.

Suppose that c and d are complex numbers. Then $\overline{cd} = \overline{c}\,\overline{d}$.

Theorem CCT   Complex Conjugation Twice   **309**

Suppose that c is a complex number. Then $\overline{\overline{c}} = c$.

Definition MCN   Modulus of a Complex Number   **310**

The **modulus** of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|c| = \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}.$$

Definition SET   Set   **311**

A **set** is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. we refer to the objects in a set as its **elements**.

Definition SSET   Subset   **312**

If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Definition ES   Empty Set   313

The empty set is the set with no elements. Its is denoted by $\emptyset$.

Definition SE   Set Equality   314

Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. in this case, we write $S = T$.