Flash Cards

to accompany

A First Course in Linear Algebra

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Definition SLE System of Linear Equations

1

A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} . ©2005, 2006

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Definition ES Equivalent Systems

 $\mathbf{2}$

Two systems of linear equations are **equivalent** if their solution sets are equal. ©2005, 2006

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Definition EO Equation Operations

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Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

- 1. Swap the locations of two equations in the list.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent. ©2005,

Definition M Matrix

An $m \times n$ matrix is a rectangular layout of numbers from $\mathbb C$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, \ldots) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A. ©2005, 2006 Robert Beezer

Definition AM Augmented Matrix

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Suppose we have a system of m equations in the n variables $x_1, x_2, x_3, \ldots, x_n$ written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

then the augmented matrix of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

Definition RO Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

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Definition REM Row-Equivalent Matrices

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Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations. ©2005, 2006 Robert Beezer

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heorem REMES Row-Equivalent Matrices represent Equivalent Systems 9
appose that A and B are row-equivalent augmented matrices. Then the systems of linear unit unit unit they represent are equivalent systems. ©2005, 2006 Robert Beezer

Definition RREF Reduced Row-Echelon Form

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A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

Definition ZRM Zero Row of a Matrix	11
A row of a matrix where every entry is zero is called a zero row . ©2005, 2006	Robert
Beezer	
Definition LO Leading Ones	12
For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that zero row will be called a leading 1 . ©2005, 2006 Robert	is not a

Definition PC Pivot Columns	13
For a matrix in reduced row-echelon form, a column containing a leading 1 will be called pivot column . ©2005, 2006 Robert Beez	

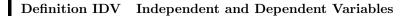
Theorem REMEF Row-Equivalent Matrix in Echelon Form

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Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

Definition RR Row-Reducing	15
To row-reduce the matrix A means to apply row operations to A matrix B in reduced row-echelon form.	row-equivalent Robert Beezer
Definition CS Consistent System	16
Definition CS Consistent System A system of linear equations is consistent if it has at least one s is called inconsistent.	
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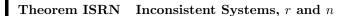


Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**. ©2005, 2006 Robert Beezer

Theorem RCLS Recognizing Consistency of a Linear System

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Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n+1 of B. ©2005, 2006 Robert Beezer

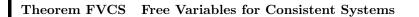


Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent. ©2005, 2006 Robert Beezer

Theorem CSRN Consistent Systems, r and n

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Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions. ©2005, 2006 Robert Beezer



 $\mathbf{21}$

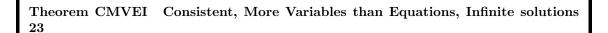
Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.

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Theorem PSSLS Possible Solution Sets for Linear Systems

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A system of linear equations has no solutions, a unique solution or infinitely many solutions.



Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions. ©2005, 2006 Robert Beezer

Definition HS Homogeneous System

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A system of linear equations is homogeneous if each equation has a 0 for its constant term. Such a system then has the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0$$

Theorem HSC Homogeneous Systems are Consistent	25
Suppose that a system of linear equations is homogeneous. Then the system is consistent	
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Definition TSHSE Trivial Solution to Homogeneous Systems of Equations	26
Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = x_2 = 0, \ldots, x_n = 0$ is called the trivial solution . ©2005, 2006 Robert Bed	

tions	 q -access,	27
Suppose that a homogeneous system of linear equations has m $n>m$. Then the system has infinitely many solutions.		variables with Robert Beezer

Homogeneous More Variables than Equations Infinite solu-

Definition CV Column Vector

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde

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underneath the symbol, as in u. To refer to the **entry** or **component** that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$. ©2005, 2006 Robert Beezer

Definition ZV Zero Vector

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The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

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Definition CM Coefficient Matrix

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SV Solution Vector

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

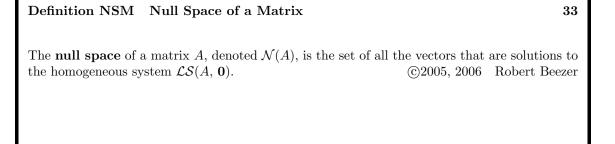
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



Definition SQM Square Matrix

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A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix. ©2005, 2006

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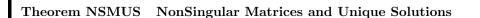
Definition IM Identity Matrix

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The $m \times m$ identity matrix, I_m is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem NSRRI NonSingular matrices Row Reduce to the Identity matrix 37
Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix. ©2005, 2006 Robert
Beezer
BCCZCI
The NETNIC New Committee and the Principle New York 1 New York 20
Theorem NSTNS NonSingular matrices have Trivial Null Spaces 38
Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A , $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{0\}$. ©2005, 2006 Robert Beezer



Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} . ©2005, 2006 Robert Beezer

Theorem NSME1 NonSingular Matrix Equivalences, Round 1

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .



The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} . ©2005, 2006 Robert Beezer

Definition CVE Column Vector Equality

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The vectors \mathbf{u} and \mathbf{v} are equal, written $\mathbf{u} = \mathbf{v}$ provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

Definition	CVA	Column	Vector	Addition

Given the vectors \mathbf{u} and \mathbf{v} the sum of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

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Definition CVSM Column Vector Scalar Multiplication

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Given the vector \mathbf{u} and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of \mathbf{u} by α , $\alpha \mathbf{u}$ is defined by

$$\left[\alpha \mathbf{u}\right]_i = \alpha \left[\mathbf{u}\right]_i$$

$$1 \leq i \leq m$$

Theorem VSPCV Vector Space Properties of Column Vectors

45 Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} +$ $(\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One Column Vectors If $u \in \mathbb{C}^m$ then 1u =

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Definition LCCV Linear Combination of Column Vectors

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Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n$$
.

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

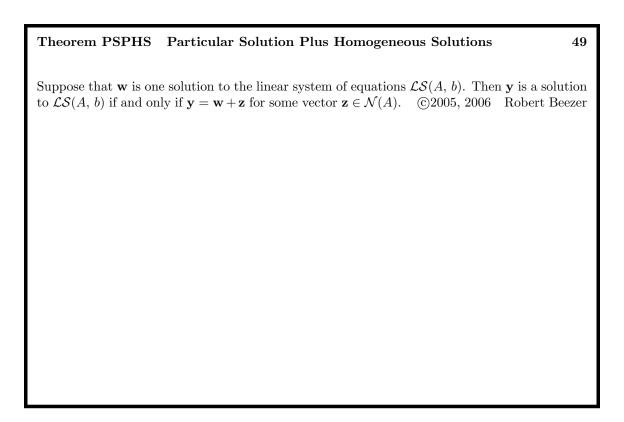
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Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$\begin{split} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} \end{split}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \right\}$$



Theorem RREFU Reduced Row-Echelon Form is Unique

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Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C. ©2005, 2006 Robert Beezer

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_i} & \text{if } i \in D, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle.$$

Definition RLDCV Relation of Linear Dependence for Column Vectors

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.

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Definition LICV Linear Independence of Column Vectors

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The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors. ©2005,

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. ©2005, 2006 Robert Beezer

Theorem LIVRN Linearly Independent Vectors, r and n

56

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r. ©2005, 2006 Robert Beezer

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is the set of vectors in \mathbb{C}^m , and that n > m. Then S is a linearly dependent set. ©2005, 2006 Robert Beezer

Theorem NSLIC NonSingular matrices have Linearly Independent Columns 58

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set. ©2005, 2006 Robert Beezer

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

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Theorem BNS Basis for Null Spaces

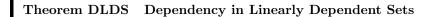
60

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_i , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.



Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$. ©2005, 2006 Robert Beezer

Theorem BS Basis of a Span

62

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

Definition CCCV Complex Conjugate of a Column Vector

63

Suppose that **u** is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

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Theorem CRVA Conjugation Respects Vector Addition

64

Suppose **x** and **y** are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x}+\mathbf{y}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$$

Theorem CRSM Conjugation Respects Vector Scalar Multiplication

65

Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

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Definition IP Inner Product

66

Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[\mathbf{u} \right]_1 \overline{\left[\mathbf{v} \right]_1} + \left[\mathbf{u} \right]_2 \overline{\left[\mathbf{v} \right]_2} + \left[\mathbf{u} \right]_3 \overline{\left[\mathbf{v} \right]_3} + \dots + \left[\mathbf{u} \right]_m \overline{\left[\mathbf{v} \right]_m} = \sum_{i=1}^m \left[\mathbf{u} \right]_i \overline{\left[\mathbf{v} \right]_i}$$

Theorem IPVA Inner Product and Vector Addition

67

Suppose $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1.
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

2.
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

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${\bf Theorem~IPSM~~Inner~Product~and~Scalar~Multiplication}$

68

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

1.
$$\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$$

2.
$$\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$$

Theorem IPAC Inner Product is Anti-Commutative

69

Suppose that \mathbf{u} and \mathbf{v} are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

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Definition NV Norm of a Vector

70

The **norm** of the vector \mathbf{u} is the scalar quantity in \mathbb{C}

$$\|\mathbf{u}\| = \sqrt{\left|\left[\mathbf{u}\right]_{1}\right|^{2} + \left|\left[\mathbf{u}\right]_{2}\right|^{2} + \left|\left[\mathbf{u}\right]_{3}\right|^{2} + \dots + \left|\left[\mathbf{u}\right]_{m}\right|^{2}} = \sqrt{\sum_{i=1}^{m} \left|\left[\mathbf{u}\right]_{i}\right|^{2}}$$



Suppose that **u** is a vector in \mathbb{C}^m . Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$. ©2005, 2006 Robert Beezer

Theorem PIP Positive Inner Products

72

Suppose that \mathbf{u} is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$. ©2005,

Robert Beezer



A pair of vectors, \mathbf{u} and \mathbf{v} , from \mathbb{C}^m are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. ©2005, 2006 Robert Beezer

Definition OSV Orthogonal Set of Vectors

74

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$. ©2005, 2006 Robert Beezer

Theorem OSLI Orthogonal Sets are Linearly Independent

75

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent. ©2005, 2006 Robert Beezer

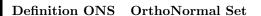
Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

76

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors \mathbf{u}_i , $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$. ©2005, 2006 Robert Beezer



Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an **orthonormal** set of vectors. ©2005, 2006 Robert Beezer

$\textbf{Definition VSM} \quad \textbf{Vector Space of} \ m \times n \ \textbf{Matrices}$

78

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers. ©2005, 2006 Robert Beezer

Definition ME Matrix Equality

The $m \times n$ matrices A and B are **equal**, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$. ©2005, 2006 Robert Beezer

Definition MA Matrix Addition

80

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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Theorem VSPM Vector Space Properties of Matrices

82

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One Matrices If $A \subseteq M$ then 1A = A

Definition	$\mathbf{Z}\mathbf{N}\mathbf{I}$	Zero	Matrix

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$. ©2005, 2006 Robert Beezer

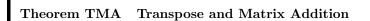
Definition TM Transpose of a Matrix

84

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix		85
The matrix A is symmetric if $A = A^t$.	©2005, 2006	Robert Beezer
Theorem SMS Symmetric Matrices are Square		86
Suppose that A is a symmetric matrix. Then A is square.	©2005, 2006	Robert Beezer



Suppose that A and B are $m \times n$ matrices. Then $(A+B)^t = A^t + B^t$.

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Theorem TMSM Transpose and Matrix Scalar Multiplication

88

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$. ©2005, 2006 Robert

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			_	
Theorem	TT	Transpose	of a	Transpose

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

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Definition CCM Complex Conjugate of a Matrix

90

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written \overline{A} is an $m \times n$ matrix defined by

 $\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$

Theorem CRMA Conjugation Respects Matrix Addition	91
Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$. ©2005, 2006	Robert

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Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication 92

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$. ©2005, 2006 Robert

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Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.

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Definition MVP Matrix-Vector Product

94

Suppose A is an $m \times n$ matrix with columns \mathbf{A}_1 , \mathbf{A}_2 , \mathbf{A}_3 , ..., \mathbf{A}_n and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 95
Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.
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Theorem EMMVP Equal Matrices and Matrix-Vector Products 96
Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$. ©2005, 2006 Robert Beezer

Definition MM Matrix Multiplication

97

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

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Theorem EMP Entries of Matrix Products

98

Suppose A is an $m \times n$ matrix and B =is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
- $2. \quad \mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$

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Theorem MMIM Matrix Multiplication and Identity Matrix

100

Suppose A is an $m\times n$ matrix. Then

- 1. $AI_n = A$
- $2. \quad I_m A = A$

Theorem MMDAA	Matrix Multiplication	Distributes Across	Addition

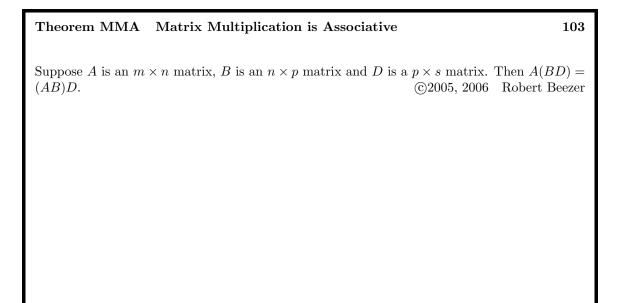
Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

- 1. A(B+C) = AB + AC
- $2. \quad (B+C)D = BD + CD$

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 102

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$. ©2005, 2006 Robert Beezer



Theorem MMIP Matrix Multiplication and Inner Products

104

If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\,\mathbf{v}\rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Matrix Multiplication and Complex Conjugation	105
n matrix and B is an $n\times p$ matrix. Then $\overline{AB}=\overline{A}\overline{B}.$	©2005, 2006
Matrix Multiplication and Transposes	106
n matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.	©2005, 2006
	n matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A} \overline{B}$. Matrix Multiplication and Transposes

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Definition	MI	Matrix	Inverse

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$. ©2005, 2006

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Definition SUV Standard Unit Vectors

108

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m . Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j \, | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

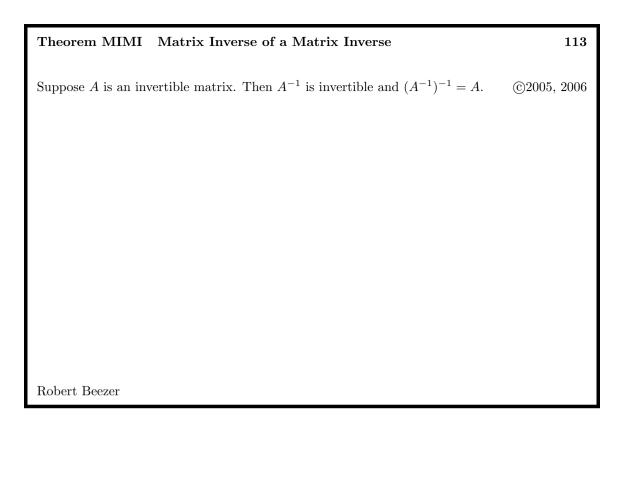
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Theorem CINSM Computing the Inverse of a NonSingular Matrix

110

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Theorem MIU Matrix Inverse is Unique	111
Suppose the square matrix A has an inverse. Then A^{-1} is unique. ©2005, 2006	Robert
Beezer	
Theorem SS Socks and Shoes	112
Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and A	IR is an
Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and A invertible matrix. $(2005, 2006)$ Rober	t Beezer

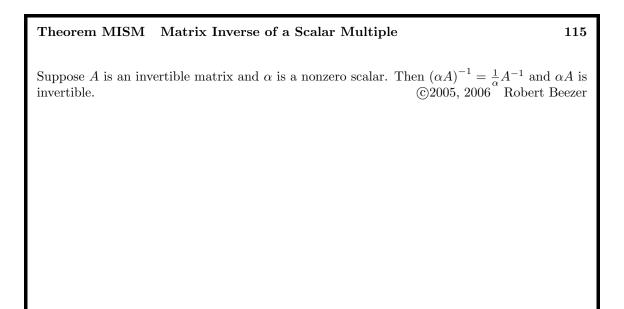


Theorem MIT Matrix Inverse of a Transpose

114

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. ©2005, 2006

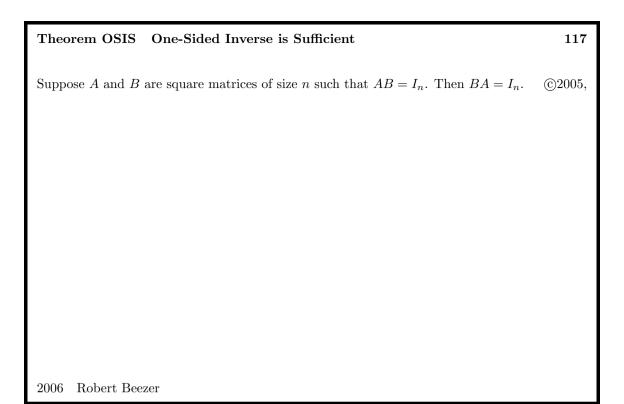
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Theorem NPNT Nonsingular Product has Nonsingular Terms

116

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular. ©2005, 2006 Robert Beezer



Theorem NSI NonSingularity is Invertibility

118

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NSME3 NonSingular Matrix Equivalences, Round 3

119

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

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Theorem SNSCM Solution with NonSingular Coefficient Matrix

120

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$. ©2005, 2006

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Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^tQ=I_n$. Then we say Q is **orthogonal**. ©2005, 2006 Robert

Beezer

Theorem OMI Orthogonal Matrices are Invertible

122

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1}=(\overline{Q})^t$.

Theorem COMOS Columns of Outhers and Matrices are Outhers and Sets 12			
Theorem COMOS Columns of Orthogonal Matrices are Orthonormal Sets 12	Theorem COMOS	olumns of Orthogonal Matrices are Orthonormal Sets 12	13

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set. ©2005, 2006 Robert Beezer

Theorem OMPIP Orthogonal Matrices Preserve Inner Products

124

Suppose that Q is an orthogonal matrix of size n and \mathbf{u} and \mathbf{v} are two vectors from \mathbb{C}^n . Then

$$\langle Q\mathbf{u},\,Q\mathbf{v}\rangle = \langle \mathbf{u},\,\mathbf{v}\rangle$$

$$\|Q\mathbf{v}\| = \|\mathbf{v}\|$$



If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$. ©2005, 2006 Robert Beezer

Definition HM **Hermitian Matrix**

126

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = (\overline{A})^t$ ©2005, 2006 Robert

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Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

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Theorem CSCS Column Spaces and Consistent Systems

128

Suppose A is an $m \times n$ matrix and **b** is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent. ©2005, 2006 Robert Beezer

Theorem BCS Basis of the Column Space

129

Suppose that A is an $m \times n$ matrix with columns \mathbf{A}_1 , \mathbf{A}_2 , \mathbf{A}_3 , ..., \mathbf{A}_n , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $C(A) = \langle T \rangle$.

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Theorem CSNSM Column Space of a NonSingular Matrix

130

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

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Definition RSM Row Space of a Matrix

132

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$. ©2005, 2006 Robert Beezer

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$. ©2005, 2006 Robert

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Theorem BRS Basis for the Row Space

134

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- $2.\ S$ is a linearly independent set.

Theorem CSRST Column Space, Row Space, Transpo	se	135
Suppose A is a matrix. Then $C(A) = \mathcal{R}(A^t)$.	©2005, 2006	Robert Beezer

Definition LNS Left Null Space

136

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$. ©2005, 2006 Robert Beezer

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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Theorem PEEF Properties of Extended Echelon Form

138

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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Definition VS Vector Space

140

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

Theorem ZVU Zero Vector is Unique	141
Suppose that V is a vector space. The zero vector, 0 , is unique. ©2005, 2006	6 Robert
Suppose vilate V is a vector apartite and a vector and a vector apartite and a vector and a vector apartite an	
Beezer	
Theorem AIU Additive Inverses are Unique	142
Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.	©2005,

Theorem ZSSM	Zero Scalar in Scalar Multiplication		143
Suppose that V is a	vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = 0$.	©2005, 2006	Robert Beezer
Theorem ZVSM	Zero Vector in Scalar Multiplication		144
	Zero Vector in Scalar Multiplication vector space and $\alpha \in \mathbb{C}$. Then $\alpha 0 = 0$.	©2005, 2006	144 Robert Beezer
		©2005, 2006	
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		©2005, 2006	

Theorem AISM	Additive	Inverses	from	Scalar	Multiplication

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$. ©2005, 2006

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Scalar Multiplication Equals the Zero Vector Theorem SMEZV

146

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha \mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$. ©2005,

Robert Beezer 2006

Theorem VAC Vector Addition Cancellation	147
Suppose that V is a vector space, and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.	©2005,
2006 Robert Beezer	
Theorem CSSM Canceling Scalars in Scalar Multiplication	148
	.,
Suppose V is a vector space, $\mathbf{u}, \mathbf{v} \in V$ and α is a nonzero scalar from \mathbb{C} . If $\alpha \mathbf{u} = \mathbf{u} = \mathbf{v}$. ©2005, 2006 Robe	$\alpha \mathbf{v}$, then ert Beezer

Theorem CVSM	Canceling Vectors in Scalar Multiplication	

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and $\alpha, \beta \in \mathbb{C}$. If $\alpha \mathbf{u} = \beta \mathbf{u}$, then $\alpha = \beta$.

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Definition S Subspace

150

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V. ©2005, 2006 Robert Beezer

Theorem TSS Testing Subsets for Subspaces

151

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

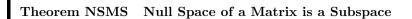
- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

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Definition TS Trivial Subspaces

152

Given the vector space V, the subspaces V and $\{0\}$ are each called a **trivial subspace**.



Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of \mathbb{C}^n .

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Definition LC Linear Combination

154

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

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Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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${\bf Theorem~SSS~~Span~of~a~Set~is~a~Subspace}$

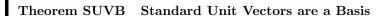
156

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace. ©2005, 2006 Robert Beezer

		1
Theorem CSMS Column Space of a Matrix is a Subspace	;e	157
Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb C$	\mathbb{C}^m . ©2005, 2006	Robert
Suppose that A is an $m \wedge n$ matrix. Then $C(n)$ is a subspace of	, . <u>©</u> 2000, 2000	TODGL
5		
Beezer		
Theorem RSMS Row Space of a Matrix is a Subspace		158
	2 - 2 - 2 - 2 - 2 - 2 - 2 - 2 - 2 - 2 -	
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \emptyset	\mathbb{C}^n . ©2005, 2006	Robert

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Theorem LNSMS Left Null Space of a Matrix is a Subspace	159
Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m . ©2005, 2006	Robert
Beezer	
Definition B Basis	160
Suppose V is a vector space. Then a subset $S \subseteq V$ is a basis of V if it is linearly independent and spans V . ©2005, 2006 Robert	pendent Beezer



The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m . ©2005, 2006 Robert Beezer

${\bf Theorem~CNSMB~~Columns~of~NonSingular~Matrix~are~a~Basis}$

162

Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular. ©2005, 2006 Robert Beezer

Theorem NSME5 NonSingular Matrix Equivalences, Round 5

163

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

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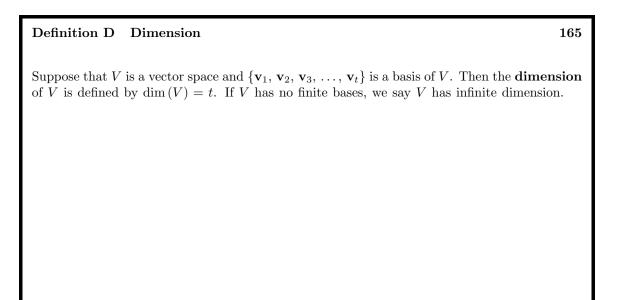
Theorem COB Coordinates and Orthonormal Bases

164

Suppose that $B = \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$

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Theorem SSLD Spanning Sets and Linear Dependence

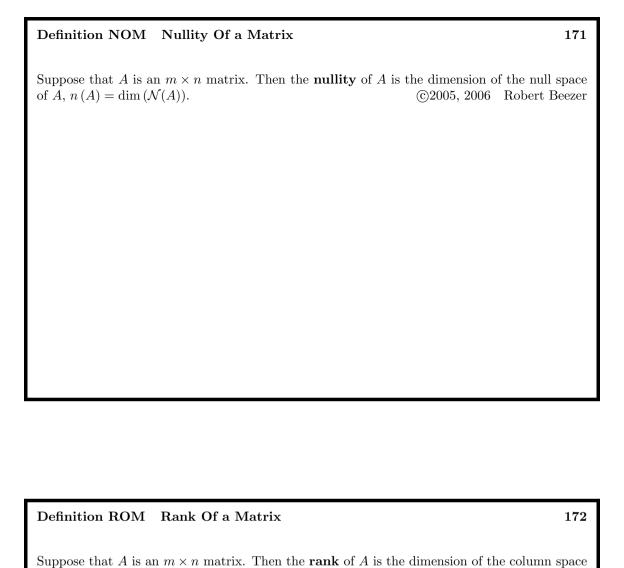
166

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of t+1 or more vectors from V is linearly dependent. ©2005, 2006 Robert

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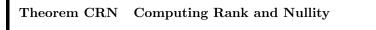
Theorem BIS Bases have Identical Sizes	167
Suppose that V is a vector space with a finite basis B and a second basis C . have the same size. ©2005, 2006	Then B and C Robert Beezer
Theorem DCM Dimension of \mathbb{C}^m	168
The dimension of \mathbb{C}^m (Example VSCV) is m . ©2005, 2006	Robert Beezer

Theorem DP Dimension of P_n		169
The dimension of P_n (Example VSP) is $n+1$.	©2005, 2006	Robert Beezer
Theorem DM Dimension of M_{mn}		170
The dimension of M_{mn} (Example VSM) is mn .	@2005_2006	Robert Beezer
The difficultion of M_{mn} (Example VSM) is not.	©2009, 2000	ROBERT BECZET



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of A, $r(A) = \dim(\mathcal{C}(A))$.



Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r. ©2005, 2006 Robert Beezer

Theorem RPNC Rank Plus Nullity is Columns

174

Suppose that A is an $m \times n$ matrix. Then r(A) + n(A) = n. ©2005, 2006 Robert Beezer

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

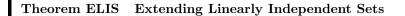
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Theorem NSME6 NonSingular Matrix Equivalences, Round 6

176

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.



Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent. ©2005, 2006

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Theorem G Goldilocks

178

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

Theorem EDYES Equal Dimensions Yields Equal Sub	spaces	179
Suppose that U and V are subspaces of the vector space W , sudim (V) . Then $U=V$.		and $\dim(U) =$ Robert Beezer

Theorem RMRT Rank of a Matrix is the Rank of the Transpose

180

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$. ©2005, 2006 Robert Beezer

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

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Definition ELEM Elementary Matrices

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Theorem EMDRO Elementary Matrices Do Row Operations

183

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.

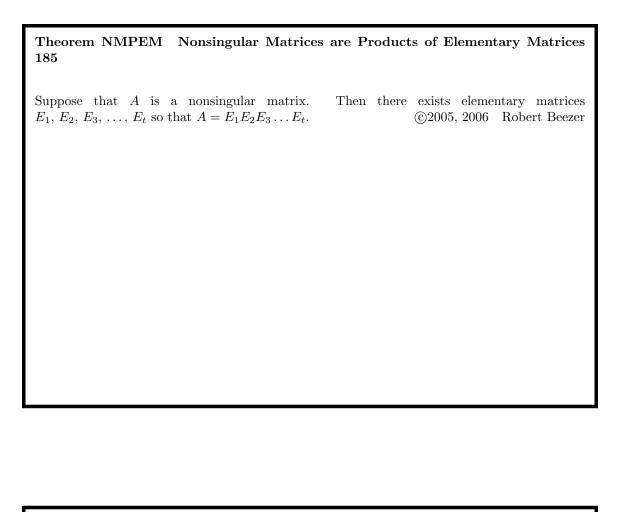
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Theorem EMN Elementary Matrices are Nonsingular

184

If E is an elementary matrix, then E is nonsingular.

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Definition SM SubMatrix

186

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j. ©2005, 2006 Robert Beezer

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det\left(A\right) = [A]_{11} \det\left(A\left(1|1\right)\right) - [A]_{12} \det\left(A\left(1|2\right)\right) + [A]_{13} \det\left(A\left(1|3\right)\right) - \dots + (-1)^{n+1} [A]_{1n} \det\left(A\left(1|n\right)\right)$$

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Theorem DMST Determinant of Matrices of Size Two

188

Suppose that
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Then $\det(A) = ad - bc$

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Theorem DER Determinant Expansion about Rows

189

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A \, (i|1))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A \, (i|2))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A \, (i|3))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A \, (i|n))} \qquad 1 \leq i \leq n \end{split}$$

which is known as **expansion** about row i.

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Theorem DT Determinant of the Transpose

190

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$. ©2005, 2006 Robert Beezer

Theorem	DEC	Determinant	Expansion	about	Columns
THEOLEIN		Determinant	Lapansion	about	Columb

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det\left(A\right) &= (-1)^{1+j} \left[A\right]_{1j} \det\left(A\left(1|j\right)\right) + (-1)^{2+j} \left[A\right]_{2j} \det\left(A\left(2|j\right)\right) \\ &+ (-1)^{3+j} \left[A\right]_{3j} \det\left(A\left(3|j\right)\right) + \dots + (-1)^{n+j} \left[A\right]_{nj} \det\left(A\left(n|j\right)\right) \qquad 1 \leq j \leq n \end{split}$$

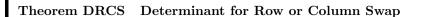
which is known as **expansion** about column j.

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Theorem DZRC Determinant with Zero Row or Column

192

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$. ©2005, 2006 Robert Beezer



Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$. ©2005, 2006 Robert Beezer

Theorem DRCM Determinant for Row or Column Multiples

194

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α , or by multiplying a single column by the scalar α . Then det $(B) = \alpha \det(A)$. ©2005, 2006 Robert Beezer

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Theorem DRCMA Determinant for Row or Column Multiples and Addition 196
Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying
a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$. ©2005, 2006

Theorem DERC Determinant with Equal Rows or Columns

195

Theorem DIM	Determinant of	the Identity Matrix

For every $n \ge 1$, $\det(I_n) = 1$.

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Theorem DEM Determinants of Elementary Matrices

198

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$

Suppose that A is a square matrix of size n and E is any elementary matrix of size n . Then			
$\det\left(EA\right) = \det\left(E\right)\det\left(A\right)$			
	©2005, 2006	Robert Beezer	
Theorem SMZD Singular Matrices have Zero Determine	inants	200	
Theorem SMZD Singular Matrices have Zero Determined Let A be a square matrix. Then A is singular if and only if det		200 ©2005, 2006	

Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication

199

Theorem NSME7 NonSingular Matrix Equivalences, Round 7

201

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.

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Theorem DRMM Determinant Respects Matrix Multiplication

202

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A) \det(B)$. © 2005, 2006 Robert Beezer

Definition EEM	Eigenvalues	and Eigenvectors	ofa	Matrix

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda \mathbf{x}$$

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Theorem EMHE Every Matrix Has an Eigenvalue

204

Suppose A is a square matrix. Then A has at least one eigenvalue. ©2005, 2006 Robert

Beezer

Definition CP	Characteristic	Polynomial

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_{A}(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

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Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials 206

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$. ©2005,

Definition EM Eigenspace of a Matrix	207
Suppose that A is a square matrix and λ is an eigenvalue of A . λ , $E_A(\lambda)$, is the set of all the eigenvectors of A for λ , together vector.	
vector.	©2000, 2000 1000010 Ecc.
Theorem EMS Eigenspace for a Matrix is a Subspace	208
Suppose A is a square matrix of size n and λ is an eigenvalue of is a subspace of the vector space \mathbb{C}^n .	A. Then the eigenspace $E_A(\lambda)$ ©2005, 2006 Robert Beezer
•	

Theorem EMNS	Eigenspace of	of a	Matrix	is a	a Null	Space

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

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Definition AME Algebraic Multiplicity of an Eigenvalue

210

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$. ©2005, 2006 Robert Beezer

Definition GME Geometric Multiplicity of an Eigenvalue	211
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the geometric in	nulti-
plicity of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$. ©2005, 2006 F	Robert

Beezer

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 212

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \, \mathbf{x}_2, \, \mathbf{x}_3, \, \dots, \, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \, \lambda_2, \, \lambda_3, \, \dots, \, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set. ©2005, 2006 Robert

Beezer

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

©2005, 2006 Robert Beezer

Theorem NSME8 NonSingular Matrix Equivalences, Round 8

214

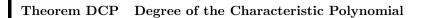
Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix	215
Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of	αA .
©2005, 2006 Robert Beezer	
Theorem EOMP Eigenvalues Of Matrix Powers	216
Suppose A is a square matrix, λ is an eigenvalue of A , and $s \geq 0$ is an integer. Then λ eigenvalue of A^s . $\textcircled{c}2005, 2006$ Robert	

Theorem EPM Eigenvalues of the Polynomial of a Matrix	217
Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in variable of Theory $q(\lambda)$ is an eigenvalue of the matrix $q(\lambda)$.	
variable x . Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$. ©2005, 2006 Robert B	eezer
Theorem EIM Eigenvalues of the Inverse of a Matrix	218
Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigen	value
of the matrix A^{-1} . ©2005, 2006 Robert B	Seezer

Theorem ETM Eigenvalues of the Transpose of a Matrix	219
Suppose A is a square matrix and λ is an eigenvalue of A . Then λ is an eigenvalue of the result. A^t .	
Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs	220
Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue $\overline{\lambda}$. Color, 2005, 2006 Robert 1	



Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_{A}\left(x\right)$, has degree n. ©2005, 2006 Robert Beezer

Theorem NEM Number of Eigenvalues of a Matrix

222

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

Theorem ME Multiplicities of an Eigenvalue

223

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

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Theorem MNEM Maximum Number of Eigenvalues of a Matrix

 $\mathbf{224}$

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues. ©2005, 2006 Robert

Beezer

Theorem HMRE I	Hermitian Matrices have Real Eigenvalues	225
Suppose that A is a He	ermitian matrix and λ is an eigenvalue of A . Then $\lambda \in \mathbb{R}$.	2005,
2006 Robert Beezer		
Theorem HMOE I	Hermitian Matrices have Orthogonal Eigenvectors	226
	Hermitian matrix and \mathbf{x} and \mathbf{y} are two eigenvectors of A for different \mathbf{y} are orthogonal vectors. ©2005, 2006 Robert Be	

Definition SIM Similar Matrices

227

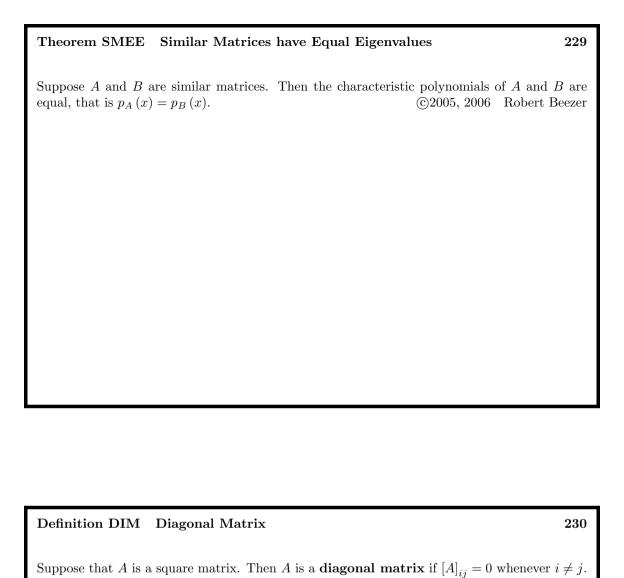
Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$. ©2005, 2006 Robert Beezer

Theorem SER Similarity is an Equivalence Relation

228

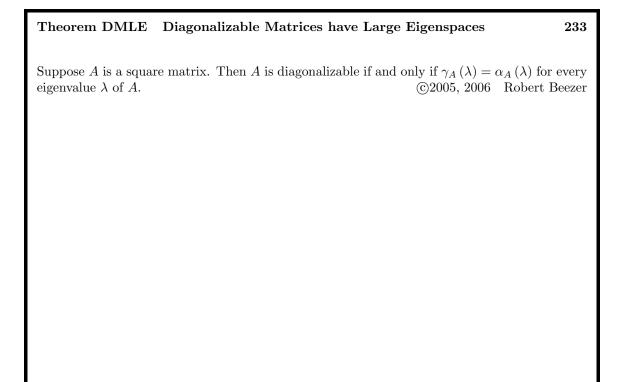
Suppose A, B and C are square matrices of size n. Then

- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)



Definition DZM Diagonalizable Matrix	231
Suppose A is a square matrix. Then A is diagonalizable if A is similar to a diagonalizable $\textcircled{0}2005, 2006$ Rob	nal matrix. bert Beezer
Theorem DC Diagonalization Characterization	232
Theorem DC Diagonalization Characterization Suppose A is a square matrix of size n . Then A is diagonalizable if and only if the	
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Suppose A is a square matrix of size n . Then A is diagonalizable if and only if the	ere exists a
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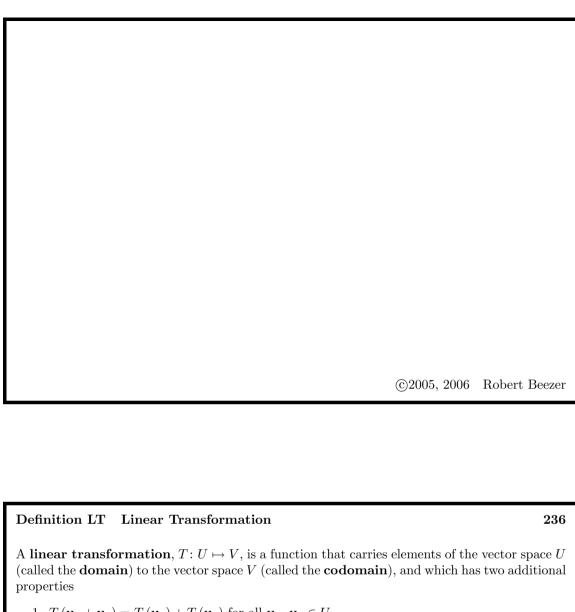
г



Theorem DED Distinct Eigenvalues implies Diagonalizable

234

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.



1.
$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$$
 for all $\mathbf{u}_1, \mathbf{u}_2 \in U$

2.
$$T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$$
 for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ	Linear Transformations Take Zero to Zero

Suppose $T\colon U\mapsto V$ is a linear transformation. Then $T\left(\mathbf{0}\right)=\mathbf{0}.$ ©2005, 2006 Robert

Beezer

Theorem MBLT Matrices Build Linear Transformations

238

237

Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation. ©2005, 2006 Robert Beezer

Theorem MLTCV	Matrix of a Linear	Transformation.	Column Vectors

Suppose that $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. ©2005, 2006 Robert Beezer

Theorem LTLC Linear Transformations and Linear Combinations

240

Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

$$T\left(a_{1}\mathbf{u}_{1}+a_{2}\mathbf{u}_{2}+a_{3}\mathbf{u}_{3}+\cdots+a_{t}\mathbf{u}_{t}\right)=a_{1}T\left(\mathbf{u}_{1}\right)+a_{2}T\left(\mathbf{u}_{2}\right)+a_{3}T\left(\mathbf{u}_{3}\right)+\cdots+a_{t}T\left(\mathbf{u}_{t}\right)$$

Theorem LTDB Linear Transformation Defined on a Basis

241

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U and \mathbf{w} is a vector from U. Let $a_1, a_2, a_3, \dots, a_n$ be the scalars from $\mathbb C$ such that

$$\mathbf{w} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_n \mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$$

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Definition PI Pre-Image

242

Suppose that $T: U \mapsto V$ is a linear transformation. For each \mathbf{v} , define the **pre-image** of \mathbf{v} to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{ \mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v} \right\}$$

Definition LTA	Linear Transformatio	n Addition

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T+S: U \mapsto V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 244

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T+S: U \mapsto V$ is a linear transformation. ©2005, 2006 Robert Beezer

Definition	LTSM	Linear	Transformation	Scalar	Multiplication
Demmon	TIDIVI	Linear	11 ansion mation	Scarai	Munipheanon

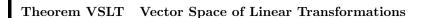
Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T: U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

©2005, 2006 Robert Beezer

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 246

Suppose that $T\colon U\mapsto V$ is a linear transformation and $\alpha\in\mathbb{C}$. Then $(\alpha T)\colon U\mapsto V$ is a linear transformation. ©2005, 2006 Robert Beezer



Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, LT (U, V) is a vector space when the operations are those given in Definition LTA and Definition LTSM. ©2005, 2006 Robert Beezer

Definition LTC Linear Transformation Composition

248

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T) \colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 249
Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are linear transformations. Then $(S\circ T)\colon U\mapsto W$ is a linear transformation. ©2005, 2006 Robert Beezer
Definition ILT Injective Linear Transformation 250
Suppose $T: U \mapsto V$ is a linear transformation. Then T is injective if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$. ©2005, 2006 Robert Beezer

Definition KLT	Kernel of a Linear	Transformation

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$$

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Theorem KLTS Kernel of a Linear Transformation is a Subspace

252

Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U. ©2005, 2006 Robert Beezer

Theorem	\mathbf{KPI}	Kernel	and	Pre-	Image

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$$

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Theorem KILT Kernel of an Injective Linear Transformation

254

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$. ©2005, 2006 Robert Beezer

Theorem ILTLI	Injective Linea	r Transformations and	Linear In	dependence	255
THOUSE THE LEFT	III CCUIVC LINCA	i ii ansioi maddons and	Linca in	acpenachee	400

Suppose that $T \colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V. ©2005, 2006 Robert Beezer

256

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V. ©2005, 2006 Robert Beezer

Theorem ILTD Injective Linear Transformations and Dimension 257
Suppose that $T: U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.
Suppose that $T: C \mapsto V$ is an injective linear transformation. Then $\dim(C) \leq \dim(V)$.
©2005, 2006 Robert Beezer
Theorem CILTI Composition of Injective Linear Transformations is Injective 258
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are injective linear transformations. Then $(S \circ T): U \mapsto W$
W is an injective linear transformation. ©2005, 2006 Robert Beezer

Definition	SLT	Surjective	Linear	Transf	formation

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$. ©2005, 2006 Robert Beezer

Definition RLT Range of a Linear Transformation

260

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the ${\bf range}$ of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

Theorem RLTS Range of a Linear Transformation is a Subspace	261
Suppose that $T: U \mapsto V$ is a linear transformation. Then the range of T , $\mathcal{R}(T)$, is a sub of V . ©2005, 2006 Robert B	
Theorem RSLT Range of a Surjective Linear Transformation	262
Suppose that $T: U \mapsto V$ is a linear transformation. Then T is surjective if and only if the of T equals the codomain, $\mathcal{R}(T) = V$. ©2005, 2006 Robert B	

Theorem SSRLT	Spanning Set for	Range of a Linear	Transformation

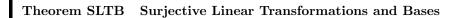
Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$. ©2005, 2006 Robert Beezer

Theorem RPI Range and Pre-Image

264

Suppose that $T\colon U\mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T)$$
 if and only if $T^{-1}(\mathbf{v}) \neq \emptyset$



Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.

Theorem SLTD Surjective Linear Transformations and Dimension

266

Suppose that $T: U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS Composition of Surjective Linear Trans 267	sformations is Surjective
Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are surjective linear t $T)\colon U\mapsto W$ is a surjective linear transformation.	transformations. Then $(S \circ 2005, 2006]$ Robert Beezer

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Identity Linear Transformation

 $I_W \colon W \mapsto W, \qquad I_W (\mathbf{w}) = \mathbf{w}$

The identity linear transformation on the vector space W is defined as

Definition IDLT

Suppose that $T\colon U\mapsto V$ is a linear transformation. If there is a function $S\colon V\mapsto U$ such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$. ©2005,

2006 Robert Beezer

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 270

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}: V \mapsto U$ is a linear transformation. ©2005, 2006 Robert Beezer

Theorem IILT	Inverse of an Invertible Linear Transformation

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$. ©2005, 2006 Robert Beezer

Theorem ILTIS Invertible Linear Transformations are Injective and Surjective 272

Suppose $T: U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective. ©2005, 2006 Robert Beezer

Theorem CIVLT	Composition	of Invertible Li	near Transformations

Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are invertible linear transformations. Then the composition, $(S\circ T)\colon U\mapsto W$ is an invertible linear transformation. ©2005, 2006 Robert

Beezer

Theorem ICLT Inverse of a Composition of Linear Transformations

274

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. ©2005, 2006 Robert Beezer

Definition	IVS	Isomorphic	Vector	Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V. ©2005, 2006 Robert

Beezer

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension 276

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$. ©2005, 2006

Robert Beezer

Definition	ROLT	Rank Of a	Linear	Transformation

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

$$r(T) = \dim \left(\mathcal{R}(T) \right)$$

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278

Suppose that $T:U\mapsto V$ is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

$$n(T) = \dim (\mathcal{K}(T))$$

Theorem ROSLT	Rank Of a Surjective Linear Transformation	

Suppose that $T\colon U\mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r\left(T\right)=\dim\left(V\right)$, if and only if T is surjective.

${\bf Theorem~NOILT~~Nullity~Of~an~Injective~Linear~Transformation}$

280

279

Suppose that $T\colon U\mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n\left(T\right)=0$, if and only if T is injective.

Theorem RPNDD Rank Plus Nullity is Domain Dimension

281

Suppose that $T: U \mapsto V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

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Definition VR Vector Representation

282

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \dots, a_n$ so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i$$

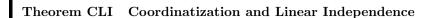
$$1 \le i \le n$$

Theorem VRLT Vector Representation is a Linear Transform	nation	283
The function ρ_B (Definition VR) is a linear transformation. ©2008	5, 2006 Rober	t Beezer
Theorem VRI Vector Representation is Injective		284
The function ρ_B (Definition VR) is an injective linear transformation.	©2005, 2006	Robert

 Beezer

Theorem VRS Vector Representation is Surjective	285
The function ρ_B (Definition VR) is a surjective linear transformation. ©2005, 2006	Robert
Beezer	
Theorem VRILT Vector Representation is an Invertible Linear Transform	nation
286	
The function ρ_B (Definition VR) is an invertible linear transformation. ©200	5, 2006

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces	287
Suppose that V is a vector space with dimension n . Then V is isomorphic to \mathbb{C}^n .	©2005,
2006 Robert Beezer	
	1
Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces	288
Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorand only if $\dim(U) = \dim(V)$. ©2005, 2006 Robert 1	



289

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n . ©2005, 2006 Robert Beezer

Theorem CSS Coordinatization and Spanning Sets

290

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$. ©2005, 2006 Robert

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Definition MR Matrix Representation

291

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\left. \rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right)\right| \right. \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right)\right| \left. \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right)\right| \ldots \left| \rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$$

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Theorem FTMR Fundamental Theorem of Matrix Representation

292

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_{C}\left(T\left(\mathbf{u}\right)\right)=M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

Theorem MRSLT M	Matrix Representation	of a Sum of Linear	Transformations293
-----------------	-----------------------	--------------------	--------------------

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^{T} + M_{B,C}^{S}$$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 294

Suppose that $T \colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 295

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

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Theorem KNSI Kernel and Null Space Isomorphism

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Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}\big(M_{B,C}^T\big)$$

Theorem RCSI Range and Column Space Isomorphism

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Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

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Theorem IMR Invertible Matrix Representations

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Suppose that $T\colon U\mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

${\bf Theorem~NSME9~~Non Singular~Matrix~Equivalences,~Round~9}$

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Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Suppose that $T\colon V\mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v}\in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v})=\lambda\mathbf{v}$.

Definition CBM Change-of-Basis Matrix

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Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$

Theorem	CB	Change-of-Basis

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$

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Theorem ICBM Inverse of Change-of-Basis Matrix

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Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB	Matrix	Representation	and	Change	of	Basis

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Suppose that $T \colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

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Theorem SCB Similarity and Change of Basis

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Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

Theorem	EER.	Eigenvalues.	Eigenvectors.	Representations
T IICOI CIII		Ligori varaco,	Ligorive Colors,	recht eperiodic

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Suppose that $T\colon V\mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v}\in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B\left(\mathbf{v}\right)$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .