Flash Cards

to accompany

A First Course in Linear Algebra

by Robert A. Beezer Department of Mathematics and Computer Science University of Puget Sound

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Definition SLE System of Linear Equations

1

A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .

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Definition ES Equivalent Systems

 $\mathbf{2}$

Two systems of linear equations are **equivalent** if their solution sets are equal.

Definition EO Equation Operations

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Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

- 1. Swap the locations of two equations in the list.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M Matrix 5

An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation A will refer to the complex number in row A and column A and column A and column A the notation A is an account A and column A and column A the notation A is an account A and column A the notation A is an account A the notation A and column A the notation A is an account A the notation A and column A the notation A is an account A the notation A is a column A to A the notation A is a column A the notation A the notation A is a column A to A the notation A the notation A is a column A to A the notation A the nota

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Definition AM Augmented Matrix

Suppose we have a system of m equations in the n variables $x_1, x_2, x_3, \ldots, x_n$ written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

Definition RO Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

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Definition REM Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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Definition RREF Reduced Row-Echelon Form

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A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

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Definition ZRM Zero Row of a Matrix

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A row of a matrix where every entry is zero is called a **zero row**.

Definition LO Leading Ones 11 For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a leading 1.

Definition PC Pivot Columns

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For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

Theorem	REMEF	Row-Equivalent Matrix in Echelon	Form

Suppose A is a matrix. Then there is a matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

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Definition RR Row-Reducing

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To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

Definition	\mathbf{CS}	Consistent	System

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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Definition IDV Independent and Dependent Variables

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Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem	RCLS	Recognizing	Consistency	of a Linear	System
I IICOI CIII	ICCLD	TUCCOSITIZITIS	Combibuction	or a Lincar	Dybucin

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n+1 of B.

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Theorem ISRN Inconsistent Systems, r and n

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Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

Theorem	CSRN	Consistent	Systems.	r	and	n

Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

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Suppose A is the augmented matrix of a consistent system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n-r free variables.

Theorem PSSLS Possible Solution Sets for Linear Systems	21
A system of linear equations has no solutions, a unique solution or infinitely many solut	ions.
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Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions	22
Suppose a consistent system of linear equations has m equations in n variables. If $n > n$ the system has infinitely many solutions.	i, then

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Definition HS Homogeneous System

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A system of linear equations is $\mathbf{homogeneous}$ if each equation has a 0 for its constant term. Such a system then has the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0$$

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Theorem HSC Homogeneous Systems are Consistent

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Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE Trivial Solution to Homogeneous Systems of Equations					
11	ous system of linear equations has n variables. alled the trivial solution .	The solution $x_1 =$	= 0,		

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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the **entry** or **component** that is number i in the list that is the vector \mathbf{v} we write $[\mathbf{v}]_i$.

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Definition ZV Zero Vector

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The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Definition SV Solution Vector

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For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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Definition NSM Null Space of a Matrix

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The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition	SOM	Square	Matrix

A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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Definition NM Nonsingular Matrix

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Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

The $m \times m$ identity matrix, I_m is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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Theorem NSRRI NonSingular matrices Row Reduce to the Identity matrix

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Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NSTNS NonSingular matrices have Trivial Null Spaces		37
Suppose that A is a square matrix. Then A is nonsingular if and only $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{0\}$.	if the	null space of A ,
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Theorem NSMUS NonSingular Matrices and Unique Solutions

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Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} .

Theorem NSME1 NonSingular Matrix Equivalences, Round 1

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .

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Definition VSCV Vector Space of Column Vectors

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The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} .

Definition	\mathbf{CVE}	Column	Vector	Equality
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The vectors \mathbf{u} and \mathbf{v} are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \le i \le m$$

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Definition CVA Column Vector Addition

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Given the vectors ${\bf u}$ and ${\bf v}$ the ${\bf sum}$ of ${\bf u}$ and ${\bf v}$ is the vector ${\bf u}+{\bf v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \le i \le m$$

Definition	CVSM	Column	Vector	Scalar	Multiplication

Given the vector \mathbf{u} and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of \mathbf{u} by α , $\alpha \mathbf{u}$ is defined by

$$[\alpha \mathbf{u}]_i = \alpha [\mathbf{u}]_i \qquad 1 \le i \le m$$

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Theorem VSPCV Vector Space Properties of Column Vectors 44 Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **ZC Zero Vector, Column Vectors** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One, Column vectors if $\mathbf{u} \in \mathbb{C}^n$, then $\mathbf{u} = \mathbf{u}$.

Definition LCCV Linear Combination of Column Vectors

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Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

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Theorem SLSLC Solutions to Linear Systems are Linear Combinations

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Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$\left[\mathbf{x}\right]_{1}\mathbf{A}_{1}+\left[\mathbf{x}\right]_{2}\mathbf{A}_{2}+\left[\mathbf{x}\right]_{3}\mathbf{A}_{3}+\cdots+\left[\mathbf{x}\right]_{n}\mathbf{A}_{n}=\mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems

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Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$\begin{split} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} \end{split}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \right\}$$

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Theorem PSPHS Particular Solution Plus Homogeneous Solutions

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Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then **y** is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Theorem RREFU Reduced Row-Echelon Form is Unique

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Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

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Definition SSCV Span of a Set of Column Vectors

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Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, \ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle.$$

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Definition RLDCV Relation of Linear Dependence for Column Vectors

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Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.

Definition LICV Linear Independence of Column Vec	ctors
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The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems

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Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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Theorem MVSLD More Vectors than Size implies Linear Dependence

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Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that n > m. Then S is a linearly dependent set.

Theorem NSLIC NonSingular matrices have Linearly Independent Columns

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Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

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Theorem NSME2 NonSingular Matrix Equivalences, Round 2

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces

59

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -[B]_{k,f_i} & \text{if } i \in D, \ i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

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Theorem DLDS Dependency in Linearly Dependent Sets

60

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

Theorem BS Basis of a Span

61

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

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Definition CCCV Complex Conjugate of a Column Vector

62

Suppose that ${\bf u}$ is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\bf u}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

Theorem	CRVA	Conjugation Respe	ects Vector Addition
T IICOI CIII	CIUVII	Conjugation respe	cus vector reduition

Suppose **x** and **y** are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

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Theorem CRSM Conjugation Respects Vector Scalar Multiplication

64

Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP Inner Product

65

Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[\mathbf{u} \right]_1 \overline{\left[\mathbf{v} \right]_1} + \left[\mathbf{u} \right]_2 \overline{\left[\mathbf{v} \right]_2} + \left[\mathbf{u} \right]_3 \overline{\left[\mathbf{v} \right]_3} + \dots + \left[\mathbf{u} \right]_m \overline{\left[\mathbf{v} \right]_m} = \sum_{i=1}^m \left[\mathbf{u} \right]_i \overline{\left[\mathbf{v} \right]_i}$$

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Theorem IPVA Inner Product and Vector Addition

66

Suppose $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1.
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

2.
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

Theorem IPSM Inner Product and Scalar Multiplication

67

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

- 1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
- 2. $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \, \langle \mathbf{u}, \, \mathbf{v} \rangle$

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Theorem IPAC Inner Product is Anti-Commutative

68

Suppose that **u** and **v** are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

Definition NV Norm of a Vector

69

The **norm** of the vector \mathbf{u} is the scalar quantity in \mathbb{C}

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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Theorem IPN Inner Products and Norms

70

Suppose that \mathbf{u} is a vector in \mathbb{C}^m . Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.



Suppose that **u** is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

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Definition OV Orthogonal Vectors

72

A pair of vectors, \mathbf{u} and \mathbf{v} , from \mathbb{C}^m are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Definition	osv	Orthogonal Set of Vector	s

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

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Theorem OSLI Orthogonal Sets are Linearly Independent

74

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

75

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

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Definition ONS OrthoNormal Set

76

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an **orthonormal** set of vectors.

Definition VSM Vector Space of $m \times n$ Matrices

77

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

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Definition ME Matrix Equality

78

The $m \times n$ matrices A and B are **equal**, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA Matrix Addition

79

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written A + B, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

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Definition MSM Matrix Scalar Multiplication

80

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

Theorem VSPM Vector Space Properties of Matrices

8

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- **ZM Zero Vector, Matrices** There is a matrix, \mathcal{O} , called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices If α , $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One, Matrices if $A \in M_{mn}$, then A = A.

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Definition ZM Zero Matrix

82

The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM Transpose of a Matrix

83

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

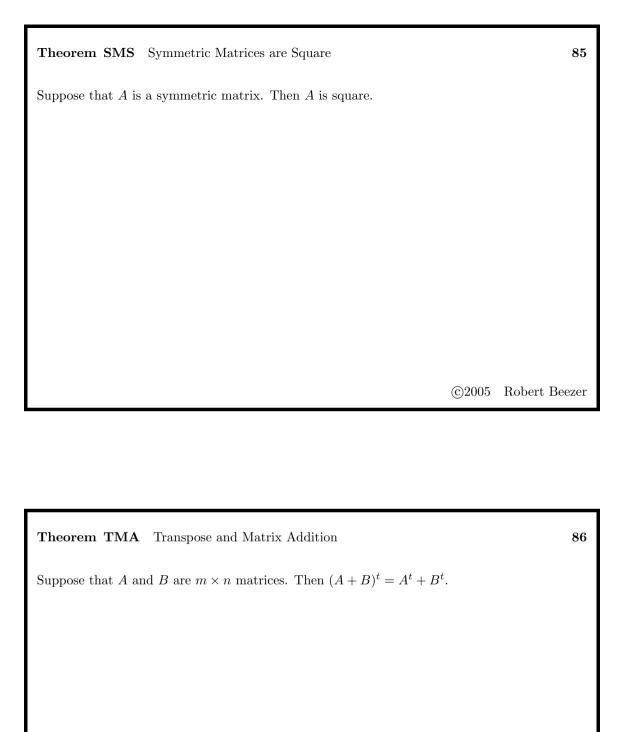
$$\left[A^t\right]_{ij} = [A]_{ji}\,, \quad 1 \leq i \leq n, \, 1 \leq j \leq m.$$

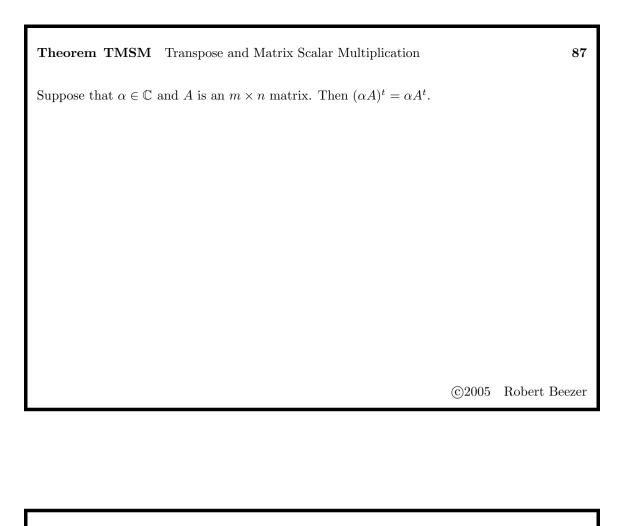
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Definition SYM Symmetric Matrix

84

The matrix A is **symmetric** if $A = A^t$.





Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

Definition CCM Complex Conjugate of a Matrix

89

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written \overline{A} is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

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Theorem CRMA Conjugation Respects Matrix Addition

90

Suppose that A and B are $m\times n$ matrices. Then $\overline{A+B}=\overline{A}+\overline{B}.$

Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

91

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.

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Theorem MCT Matrix Conjugation and Transposes

92

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.

Definition MVP Matrix-Vector Product

93

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$

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Theorem SLEMM Systems of Linear Equations as Matrix Multiplication

94

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.

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Theorem EMMVP Equal Matrices and Matrix-Vector Products

95

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then A = B.

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Definition MM Matrix Multiplication

96

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

Theorem EMP Entries of Matrix Products

97

Suppose A is an $m \times n$ matrix and B =is an $n \times p$ matrix. Then for $1 \le i \le m, \ 1 \le j \le p,$ the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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Theorem MMZM Matrix Multiplication and the Zero Matrix

98

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
- $2. \quad \mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$

Theorem	MMIM	Matrix	Multi	plication	and	Identity	Matrix
I IICOI CIII	TATTATTAT	MULLIA	TVIUIUI	piicautoii	and	TGC11010,y	TVICULIA

Suppose A is an $m \times n$ matrix. Then

- 1. $AI_n = A$
- $2. \quad I_m A = A$

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Theorem MMDAA Matrix Multiplication Distributes Across Addition

100

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

- $1. \quad A(B+C) = AB + AC$
- $2. \quad (B+C)D = BD + CD$

Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication

101

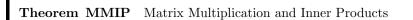
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

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Theorem MMA Matrix Multiplication is Associative

102

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then A(BD) = (AB)D.



If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

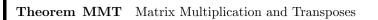
$$\langle \mathbf{u},\,\mathbf{v}
angle = \mathbf{u}^t \overline{\mathbf{v}}$$

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Theorem MMCC Matrix Multiplication and Complex Conjugation

104

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.



Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

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Definition MI Matrix Inverse

106

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Definition SUV Standard Unit Vectors

107

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m . Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j \mid 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

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Theorem TTMI Two-by-Two Matrix Inverse

108

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem	CINSM	Computing	the	Inverse	of a	Non	Singular	Matrix
THEOLEIN	CILIDIVI	Computing	ULIC	THIVCISC	or a	TAOH	Dingulai	Mania

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

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Theorem MIU Matrix Inverse is Unique

110

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

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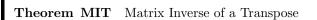
Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

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Theorem MIMI Matrix Inverse of a Matrix Inverse

112

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.



Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

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Theorem MISM Matrix Inverse of a Scalar Multiple

114

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem NPNT Nonsingular Product has Nonsingular Terms	115
Suppose that A and B are square matrices of size n and the product AB is nonsingular. A and B are both nonsingular.	Then
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	110

Theorem OSIS One-Sided Inverse is Sufficient

116

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

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Theorem	NSI	NonSingularity	is Invertibility

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

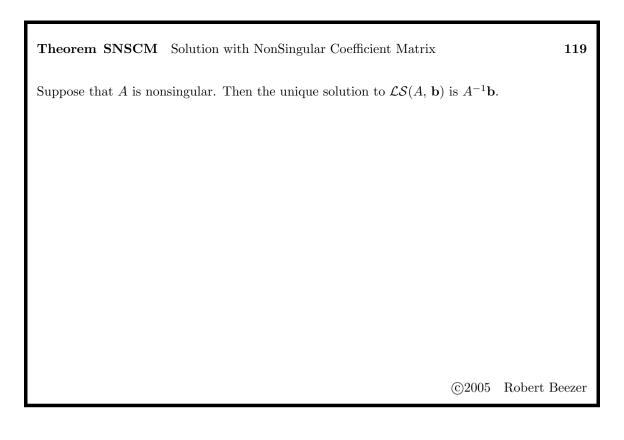
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Theorem NSME3 NonSingular Matrix Equivalences, Round 3

118

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

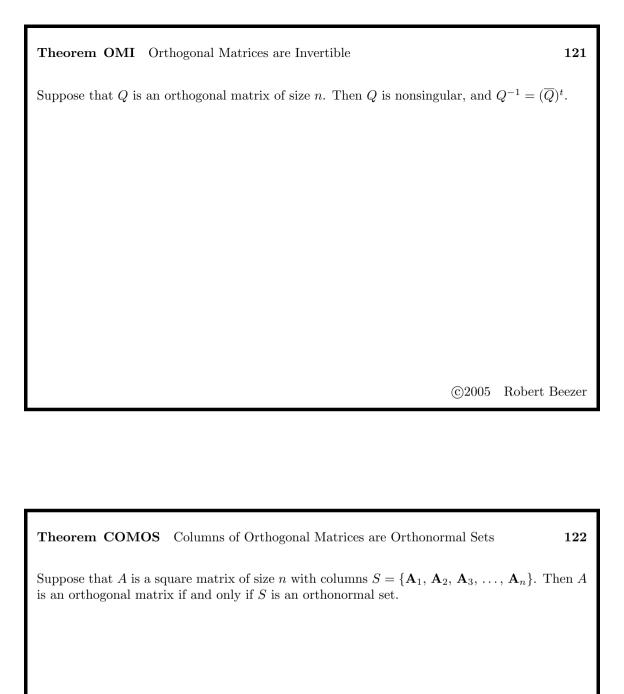


Definition OM Orthogonal Matrices

120

Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^tQ=I_n$. Then we say Q is **orthogonal**.

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Theorem OMPIP Orthogonal Matrices Preserve Inner Products

123

Suppose that Q is an orthogonal matrix of size n and ${\bf u}$ and ${\bf v}$ are two vectors from \mathbb{C}^n . Then

$$\langle Q\mathbf{u}, \, Q\mathbf{v} \rangle = \langle \mathbf{u}, \, \mathbf{v} \rangle$$

$$\|Q\mathbf{v}\| = \|\mathbf{v}\|$$

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Definition A Adjoint

124

If A is a square matrix, then its **adjoint** is $A^{H} = (\overline{A})^{t}$.



The square matrix A is **Hermitian** (or **self-adjoint**) if $A = (\overline{A})^t$

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Definition CSM Column Space of a Matrix

126

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$

Theorem CSCS Column Spaces and Consistent Syst	m tems
--	--------

Suppose A is an $m \times n$ matrix and **b** is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

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Theorem BCS Basis of the Column Space

128

Suppose that A is an $m \times n$ matrix with columns \mathbf{A}_1 , \mathbf{A}_2 , \mathbf{A}_3 , ..., \mathbf{A}_n , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

- 1. T is a linearly independent set.
- 2. $C(A) = \langle T \rangle$.

Theorem CSNSM Column Space of a NonSingular Matrix

129

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

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Theorem NSME4 NonSingular Matrix Equivalences, Round 4

130

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.



Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

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Theorem REMRS Row-Equivalent Matrices have equal Row Spaces

132

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Theorem BRS Basis for the Row Space

133

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

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Theorem CSRST Column Space, Row Space, Transpose

134

Suppose A is a matrix. Then $C(A) = \mathcal{R}(A^t)$.

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

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Definition EEF Extended Echelon Form

136

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

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Theorem FS Four Subsets

138

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS Vector Space

139

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- AC Additive Closure If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, **0**, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- One if $\mathbf{u} \in V$, then $\mathbf{u} = \mathbf{u}$.

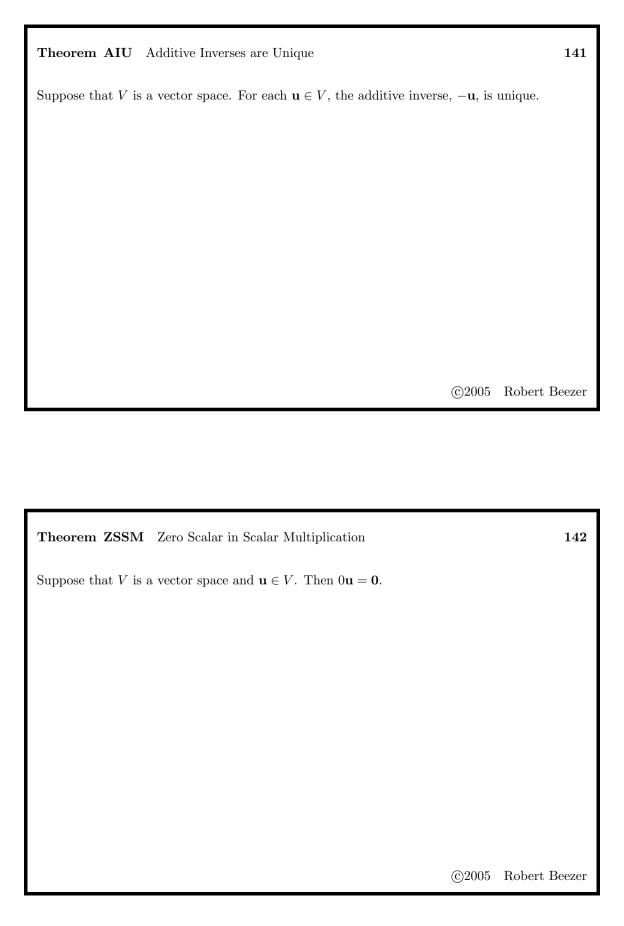
The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

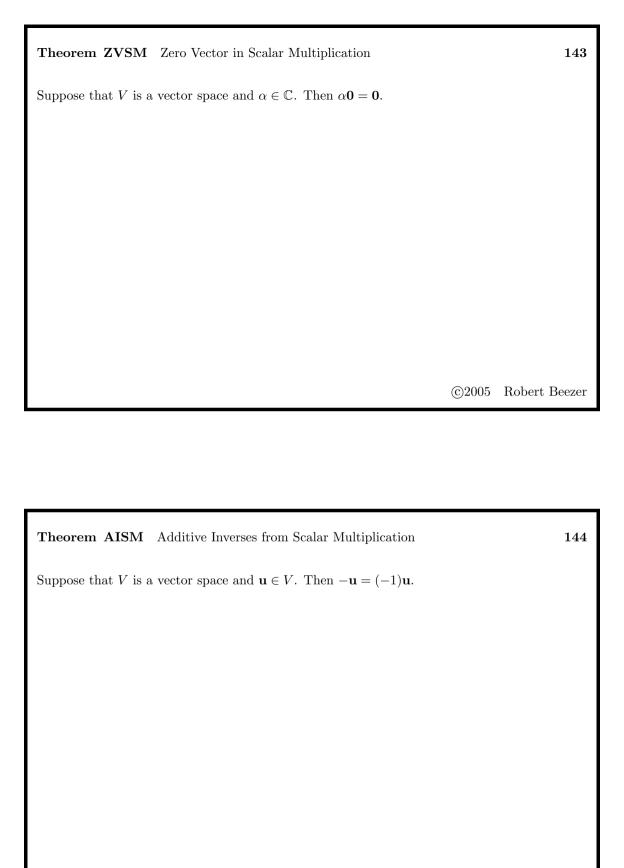
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Theorem ZVU Zero Vector is Unique

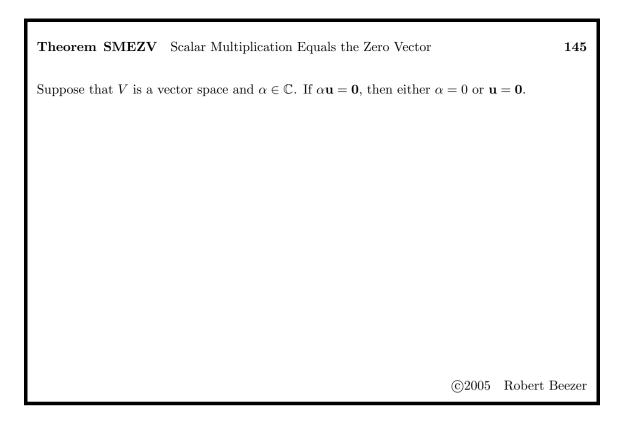
140

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.





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Theorem VAC Vector Addition Cancellation

146

Suppose that V is a vector space, and \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

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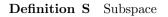
Suppose V is a vector space, $\mathbf{u}, \mathbf{v} \in V$ and α is a nonzero scalar from \mathbb{C} . If $\alpha \mathbf{u} = \alpha \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

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Theorem CVSM Canceling Vectors in Scalar Multiplication

148

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and α , $\beta \in \mathbb{C}$. If $\alpha \mathbf{u} = \beta \mathbf{u}$, then $\alpha = \beta$.



Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

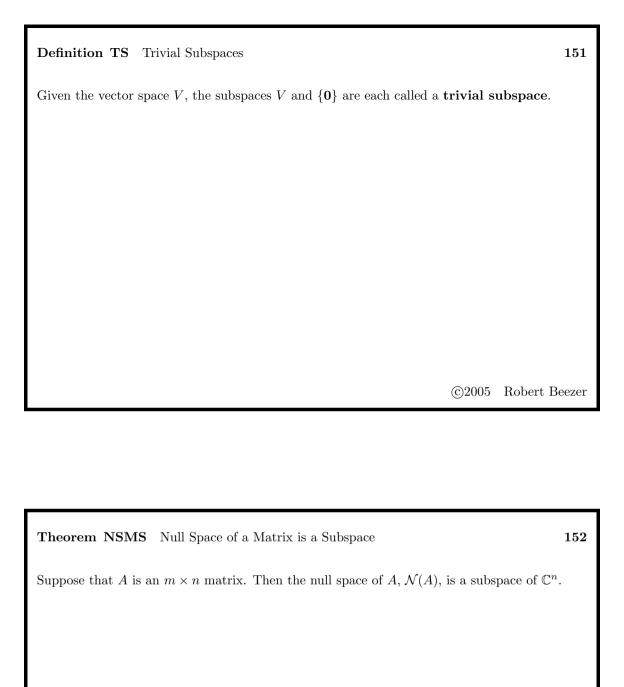
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Theorem TSS Testing Subsets for Subspaces

150

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.



Definition LC Linear Combination

153

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

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Definition SS Span of a Set

154

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

Theorem SSS Span of a Set is a Subspace

155

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

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Theorem CSMS Column Space of a Matrix is a Subspace

156

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .

Theorem RSMS Row Space of a Matrix is a Subspace		157
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .		
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Theorem LNSMS Left Null Space of a Matrix is a Subspace		158
Theorem LNSMS Left Null Space of a Matrix is a Subspace Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .		158
		158
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		158



Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

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Theorem SUVB Standard Unit Vectors are a Basis

160

The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .

Theorem CNSMB Columns of NonSingular Matrix are a Basis

161

Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

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Theorem NSME5 NonSingular Matrix Equivalences, Round 5

162

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

_				
Theorem	COB	Coordinates	and Orthone	rmal Bases

Suppose that $B = \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$$

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Definition D Dimension

164

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

Theorem S	SSLD	Spanning	Sets and	Linear	Dependence

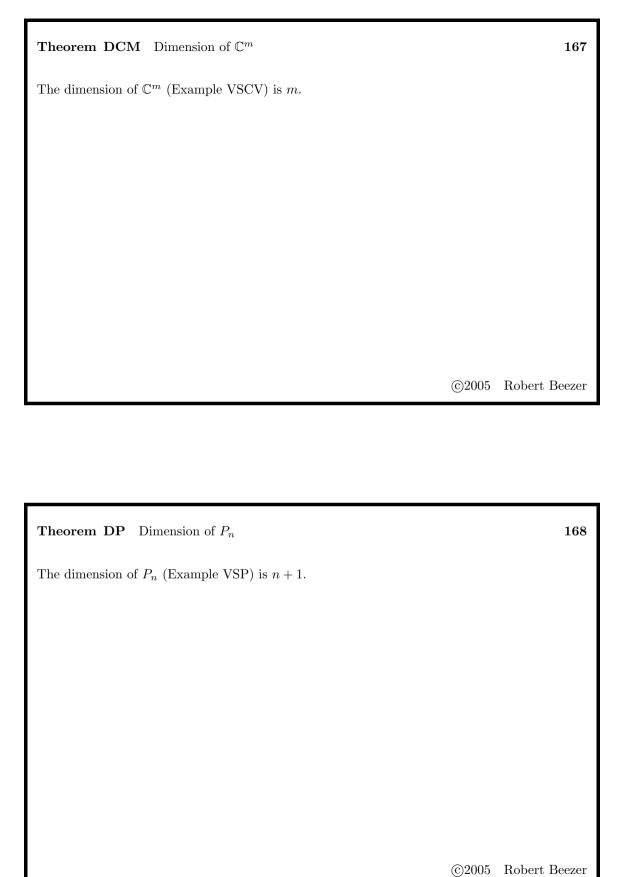
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of t+1 or more vectors from V is linearly dependent.

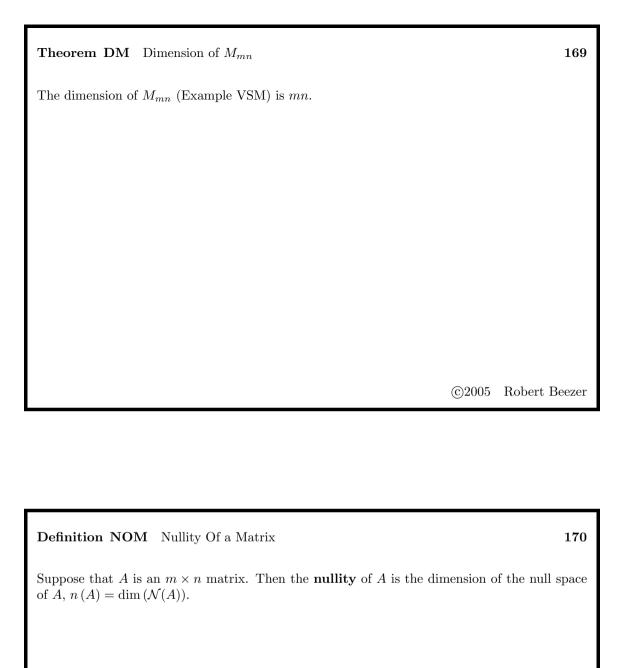
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Theorem BIS Bases have Identical Sizes

166

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.





Definition ROM Rank Of a Matrix

171

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

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Theorem CRN Computing Rank and Nullity

172

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

Theorem RPNC Rank Plus Nullity is Columns

173

Suppose that A is an $m \times n$ matrix. Then r(A) + n(A) = n.

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${\bf Theorem~RNNSM}~~{\rm Rank~and~Nullity~of~a~NonSingular~Matrix}$

174

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.

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Theorem ELIS Extending Linearly Independent Sets

176

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks

177

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

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Theorem EDYES Equal Dimensions Yields Equal Subspaces

178

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then U = V.

Theorem RMRT Rank of a Matrix is the Rank of the Transpose

179

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

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Theorem DFS Dimensions of Four Subspaces

180

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. $\dim (\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \end{cases}$$

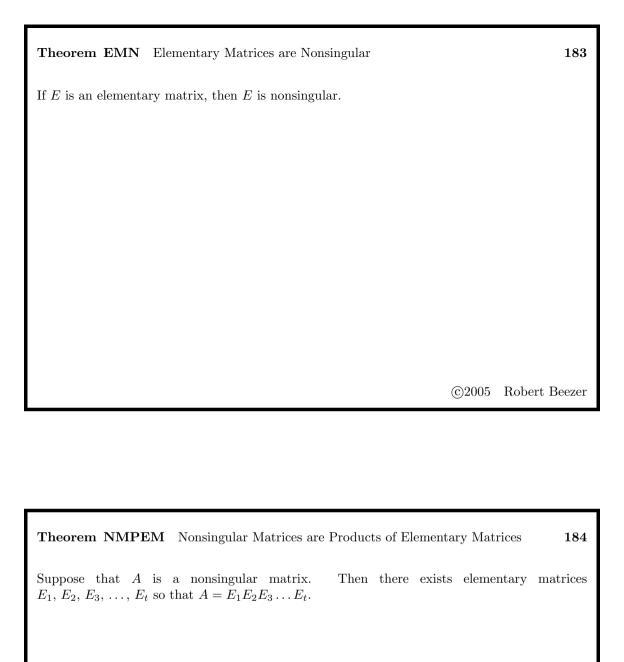
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Theorem EMDRO Elementary Matrices Do Row Operations

182

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.



Definition	\mathbf{SM}	SubMatrix

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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Definition DM Determinant of a Matrix

186

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

If A is a 1×1 matrix, then $det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det\left(A\right) = [A]_{11} \det\left(A\left(1|1\right)\right) - [A]_{12} \det\left(A\left(1|2\right)\right) + [A]_{13} \det\left(A\left(1|3\right)\right) - \dots + (-1)^{n+1} [A]_{1n} \det\left(A\left(1|r\right)\right)$$

Theorem DMST Determinant of Matrices of Size Two

187

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det{(A)} = ad - bc$

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Theorem DER Determinant Expansion about Rows

188

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A \, (i|1))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A \, (i|2))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A \, (i|3))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A \, (i|n))} \qquad 1 \leq i \leq n \end{split}$$

which is known as **expansion** about row i.

Theorem DT Determinant of the Transpose

189

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

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Theorem DEC Determinant Expansion about Columns

190

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

which is known as **expansion** about column j.

Theorem DZRC	Determinent	with Zana	Dores on	Calman
Theorem DZRC	Determinant	with Zero	Kow or	Column

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det{(A)}=0$.

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Theorem DRCS Determinant for Row or Column Swap

192

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM Determinant for Row or Column Multiples		193
Suppose that A is a square matrix. Let B be the square matrix obtain a single row by the scalar α , or by multiplying a single column by the $\alpha \det(A)$.		by multiplying
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Theorem DERC Determinant with Equal Rows or Columns		194
Suppose that A is a square matrix with two equal rows, or two equal co	lumns. Th	en $\det\left(A\right) = 0$.

Г

Theorem DRCMA Determinant for Row or Column Multiples and Addition

195

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

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Theorem DIM Determinant of the Identity Matrix

196

For every $n \geq 1$, $\det(I_n) = 1$.

Theorem DEM Determinants of Elementary Matrices

197

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$

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Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication

198

Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det\left(EA\right) = \det\left(E\right) \det\left(A\right)$$

Theorem SMZD Singular Matrices have Zero Determinants

199

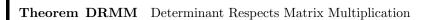
Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

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Theorem NSME7 NonSingular Matrix Equivalences, Round 7 Suppose that A is a square matrix of size n. The following are equivalent.

200

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det{(A)} \neq 0$.



Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A) \det(B)$.

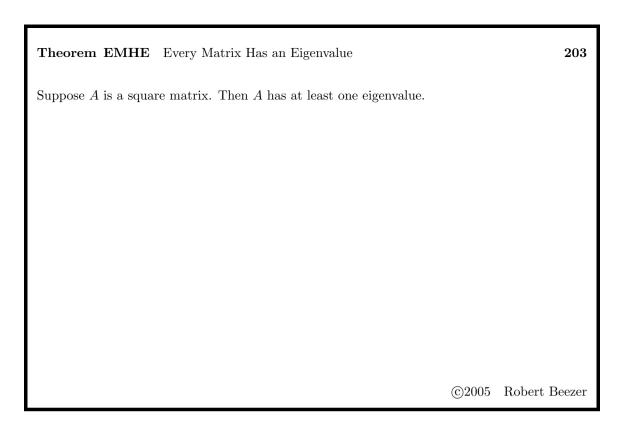
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Definition EEM Eigenvalues and Eigenvectors of a Matrix

202

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x}=\lambda\mathbf{x}$$



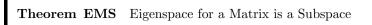
Definition CP Characteristic Polynomial

204

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_{A}(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

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Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space \mathbb{C}^n .

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Theorem EMNS Eigenspace of a Matrix is a Null Space

208

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Definition AME	Algebraic Multiplicity of an Eigenvalue	209
	quare matrix and λ is an eigenvalue of A . Then the algebrai ighest power of $(x - \lambda)$ that divides the characteristic poly	
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Definition GME	Geometric Multiplicity of an Eigenvalue	210
	square matrix and λ is an eigenvalue of A . Then the geo is the dimension of the eigenspace $E_A(\lambda)$.	metric multi-

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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 211
Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.
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Theorem SMZE Singular Matrices have Zero Eigenvalues 212

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

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Theorem NSME8 NonSingular Matrix Equivalences, Round 8 Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

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Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix

214

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA .

Theorem EOMP	Eigenvalues Of Matrix Powers
--------------	------------------------------

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \ge 0$ is an integer. Then λ^s is an eigenvalue of A^s .

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Theorem EPM Eigenvalues of the Polynomial of a Matrix

216

Suppose A is a square matrix and λ is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix q(A).

Theorem EIM Eigenvalues of the Inverse of a Matrix		217
Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. of the matrix A^{-1} .	Then $\frac{1}{\lambda}$	is an eigenvalue
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Theorem ETM Eigenvalues of the Transpose of a Matrix

218

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t .

Theorem ERMCP	Eigenvalues of Real Matrices come in Conjugate Pairs	219
	matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue $\overline{\lambda}$.	ılue

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Theorem DCP Degree of the Characteristic Polynomial

220

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_{A}\left(x\right)$, has degree n.

Theorem NEM Number of Eigenvalues of a Matrix

221

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

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Theorem ME Multiplicities of an Eigenvalue

222

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix		223
Suppose that A is a square matrix of size n . Then A cannot have more values.	ore than r	a distinct eigen-
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Theorem HMRE Hermitian Matrices have Real Eigenvalues		224
Theorem HMRE Hermitian Matrices have Real Eigenvalues Suppose that A is a Hermitian matrix and λ is an eigenvalue of A . The suppose that A is a Hermitian matrix and λ is an eigenvalue of A .	Then $\lambda \in \mathbb{F}$	
	Then $\lambda \in \mathbb{R}$	
	Then $\lambda \in \mathbb{R}$	
	Then $\lambda \in \mathbb{R}$	

Theorem	HMOE	Hammaitian	Matricas	harra	Outle a mare al	Eimanara at ana
1 neorem	HMOL	Hermitian	Matrices	nave	Ortnogonal	Eigenvectors

Suppose that A is a Hermitian matrix and \mathbf{x} and \mathbf{y} are two eigenvectors of A for different eigenvalues. Then \mathbf{x} and \mathbf{y} are orthogonal vectors.

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Definition SIM Similar Matrices

226

Suppose A and B are two square matrices of size n. Then A and B are similar if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER Similarity is an Equivalence Relation

227

Suppose A, B and C are square matrices of size n. Then

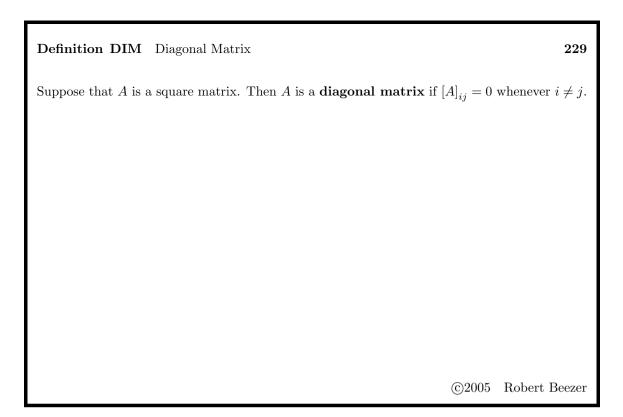
- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE Similar Matrices have Equal Eigenvalues

228

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.



Definition DZM Diagonalizable Matrix

230

Suppose A is a square matrix. Then A is ${f diagonalizable}$ if A is similar to a diagonal matrix.

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Theorem	\mathbf{DC}	Diagonalization	Characterization
T IICOI CIII	\mathbf{L}	Diagonandanon	CHALACUCHIZAUIOH

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

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Theorem DMLE Diagonalizable Matrices have Large Eigenspaces

232

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED Distinct Eigenvalues implies Diagonalizable	233
Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonal or A is diagonal or A is diagonal or A .	nalizable.
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Definition LT Linear Transformation

235

A linear transformation, $T \colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

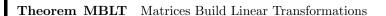
- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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Theorem LTTZZ Linear Transformations Take Zero to Zero

236

Suppose $T\colon U\mapsto V$ is a linear transformation. Then $T\left(\mathbf{0}\right)=\mathbf{0}.$



Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors

238

Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations

239

Suppose that $T\colon U\mapsto V$ is a linear transformation, $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t$ are vectors from U and $a_1,\,a_2,\,a_3,\,\ldots,\,a_t$ are scalars from $\mathbb C$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$

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Theorem LTDB Linear Transformation Defined on a Basis

240

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U and \mathbf{w} is a vector from U. Let $a_1, a_2, a_3, \dots, a_n$ be the scalars from $\mathbb C$ such that

$$\mathbf{w} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_n \mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$$

Definition PI Pre-Image

241

Suppose that $T: U \mapsto V$ is a linear transformation. For each \mathbf{v} , define the **pre-image** of \mathbf{v} to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v} \}$$

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Definition LTA Linear Transformation Addition

242

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T+S\colon U\mapsto V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT Sum of Linear Transformations is a Linear Transformation

243

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are two linear transformations with the same domain and codomain. Then $T+S\colon U\mapsto V$ is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication

244

Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T: U \mapsto V$ whose outputs are defined by

$$\left(\alpha T\right)\left(\mathbf{u}\right)=\alpha T\left(\mathbf{u}\right)$$

Theorem MLTLT M	ultiple of a Linear Transfo	ormation is a Linea	r Transfor	mation 245
Suppose that $T: U \mapsto V$ transformation.	is a linear transformation	and $\alpha \in \mathbb{C}$. Then	(αT) : U	$\rightarrow V$ is a linear
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Theorem VSLT Vector Space of Linear Transformations

246

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, LT (U, V) is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T) \colon U \mapsto W$ whose outputs are defined by

$$\left(S\circ T\right)\left(\mathbf{u}\right)=S\left(T\left(\mathbf{u}\right)\right)$$

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Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 248

Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are linear transformations. Then $(S\circ T)\colon U\mapsto W$ is a linear transformation.

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Definition ILT Injective Linear Transformation

249

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

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Definition KLT Kernel of a Linear Transformation

250

Suppose $T\colon U\mapsto V$ is a linear transformation. Then the \mathbf{kernel} of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$$

Theorem KLTS Kernel of a Linear Transformation is a Subspace

251

Suppose that $T:U\mapsto V$ is a linear transformation. Then the kernel of $T,\,\mathcal{K}(T),$ is a subspace of U.

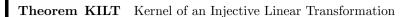
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Theorem KPI Kernel and Pre-Image

252

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$



Suppose that $T: U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}.$

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Theorem ILTLI Injective Linear Transformations and Linear Independence

254

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem	ПТВ	Injective	Linear	Transformations	and F	Bases
THEOLEIN	$1\mathbf{L}\mathbf{L}\mathbf{L}$	IIII	Lincar	Transiorina dions	and L	ascs

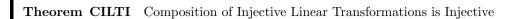
Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

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Theorem ILTD Injective Linear Transformations and Dimension

256

Suppose that $T \colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.



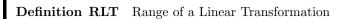
Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are injective linear transformations. Then $(S\circ T)\colon U\mapsto W$ is an injective linear transformation.

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Definition SLT Surjective Linear Transformation

258

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.



Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

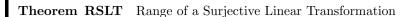
$$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

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Theorem RLTS Range of a Linear Transformation is a Subspace

260

Suppose that $T:U\mapsto V$ is a linear transformation. Then the range of $T,\,\mathcal{R}(T),$ is a subspace of V.



Suppose that $T \colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

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Theorem SSRLT Spanning Set for Range of a Linear Transformation

262

Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.



Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

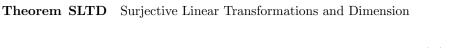
$$\mathbf{v} \in \mathcal{R}(T)$$
 if and only if $T^{-1}(\mathbf{v}) \neq \emptyset$

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Theorem SLTB Surjective Linear Transformations and Bases

264

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.



Suppose that $T \colon U \mapsto V$ is a surjective linear transformation. Then $\dim (U) \ge \dim (V)$.

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265

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 266

Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are surjective linear transformations. Then $(S\circ T)\colon U\mapsto W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation

267

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

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Definition IVLT Invertible Linear Transformations

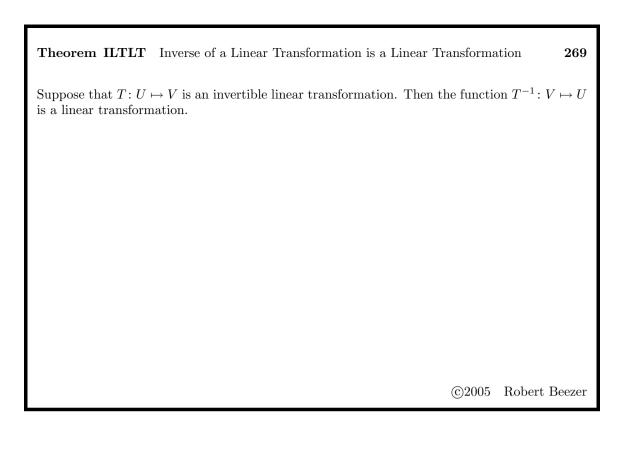
268

Suppose that $T\colon U\mapsto V$ is a linear transformation. If there is a function $S\colon V\mapsto U$ such that

$$S \circ T = I_U$$

$$T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.



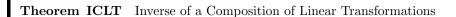
Theorem IILT Inverse of an Invertible Linear Transformation

270

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective 271
Suppose $T: U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.
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Theorem CIVLT Composition of Invertible Linear Transformations 272
Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are invertible linear transformations. Then the composition, $(S\circ T)\colon U\mapsto W$ is an invertible linear transformation.



Suppose that $T\colon U\mapsto V$ and $S\colon V\mapsto W$ are invertible linear transformations. Then $S\circ T$ is invertible and $(S\circ T)^{-1}=T^{-1}\circ S^{-1}$.

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Definition IVS Isomorphic Vector Spaces

274

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED	Isomorphic Vector Spaces have Equal Dimension	

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

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Definition ROLT Rank Of a Linear Transformation

276

Suppose that $T:U\mapsto V$ is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

$$r\left(T\right)=\dim\left(\mathcal{R}(T)\right)$$

Definition	NOLT	Nullity	Of a	Linear	Transformat	ion
	TIOLI	1 (ulli v	O_{1} α	Lincar	TI alibioi lila o	1011

Suppose that $T:U\mapsto V$ is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

$$n(T) = \dim (\mathcal{K}(T))$$

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Theorem ROSLT Rank Of a Surjective Linear Transformation

278

Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT Nullity Of an Injective Linear Transformation

279

Suppose that $T\colon U\mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n\left(T\right)=0$, if and only if T is injective.

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Theorem RPNDD Rank Plus Nullity is Domain Dimension

280

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

$$r\left(T\right)+n\left(T\right)=\dim\left(U\right)$$

Definition VR Vector Representation

281

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \dots, a_n$ so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

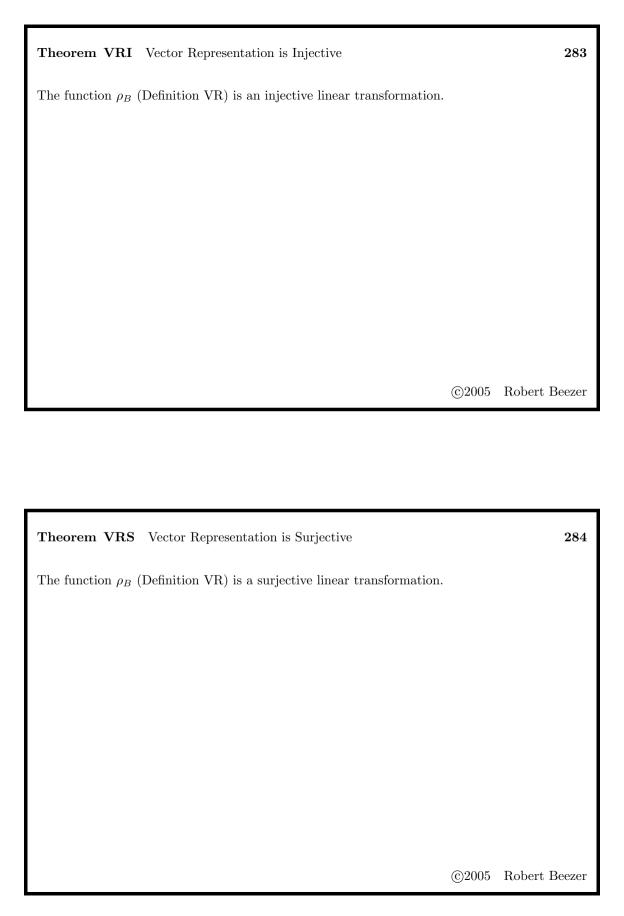
$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i \qquad 1 \le i \le n$$

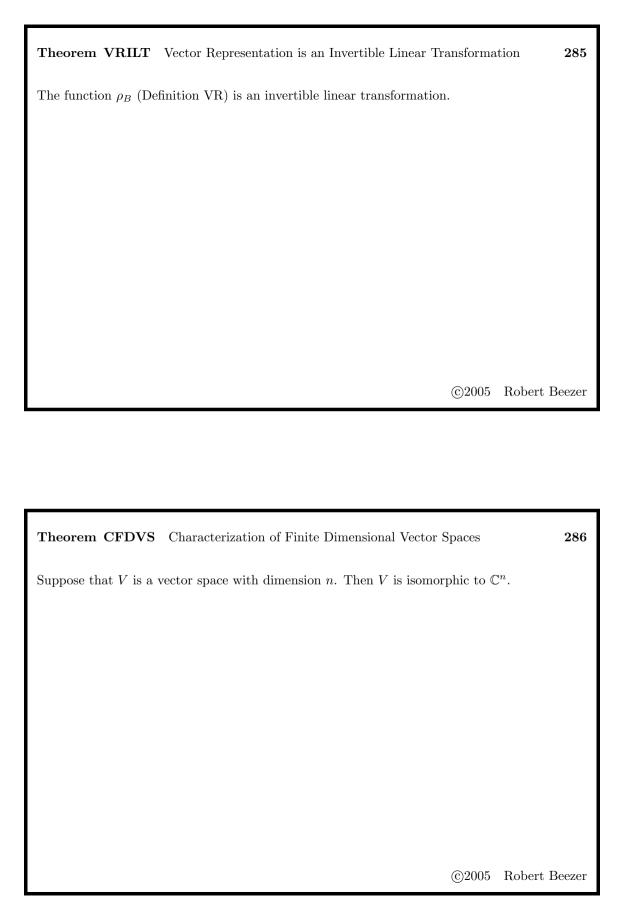
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Theorem VRLT Vector Representation is a Linear Transformation

282

The function ρ_B (Definition VR) is a linear transformation.





T1	TEDVC	T1:	- f T::4-	D:1	174 C	
1 neorem	$1\mathbf{F}\mathbf{D}\mathbf{V}\mathbf{S}$	Isomorphism	or runte	Dimensional	vector 5	paces

287

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

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Theorem CLI Coordinatization and Linear Independence

288

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n .

Theorem CSS Coordinatization and Spanning Sets

289

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$.

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Definition MR Matrix Representation

290

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\left. \rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right)\right| \left. \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right)\right| \left. \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right)\right| \dots \left| \rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$$

Theorem FTMR Fundamental Theorem of Matrix Representation

291

Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations

292

Suppose that $T\colon U\mapsto V$ and $S\colon U\mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 293

Suppose that $T\colon U\mapsto V$ is a linear transformation, $\alpha\in\mathbb{C},\,B$ is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 294

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

Theorem KNSI Kernel and Null Space Isomorphism

295

Suppose that $T \colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

296

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

Theorem IMR Invertible Matrix Representations

297

Suppose that $T\colon U\mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation

298

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation

300

Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

Definition CBM Change-of-Basis Matrix

301

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$

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Theorem CB Change-of-Basis

302

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_{C}\left(\mathbf{v}\right)=C_{B,C}\rho_{B}\left(\mathbf{v}\right)$$

Theorem ICBM Inverse of Change-of-Basis Matrix

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Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

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Theorem MRCB Matrix Representation and Change of Basis

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Suppose that $T\colon U\mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

Theorem SCB Similarity and Change of Basis

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Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

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Theorem EER Eigenvalues, Eigenvectors, Representations

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Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .