Flash Cards

to accompany

A First Course in Linear Algebra

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A **system of linear equations** is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Two systems of linear equations are **equivalent** if their solution sets are equal.

Definition EO Equation Operations **3**

Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

1. Swap the locations of two equations in the list.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Theorem EOPSS Equation Operations Preserve Solution Sets **4**

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Definition M Matrix

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet $(A, B, C, \dots)$ to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Definition AM Augmented Matrix

Suppose we have a system of m equations in the n variables $x_1, x_2, x_3, \dots, x_n$ written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

Definition RO Row Operations 7

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.

2. Multiply each entry of a single row by a nonzero quantity.

3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.

2. αR_i: Multiply row i by the nonzero scalar α.

3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

Definition REM Row-Equivalent Matrices 8

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

A row of a matrix where every entry is zero is called a **zero row**.

Definition LO   Leading Ones

For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a **leading 1**.

Definition PC   Pivot Columns

For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

Suppose A is a matrix. Then there is a matrix B so that

1. A and B are row-equivalent.

2. B is in reduced row-echelon form.

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

Definition CS   Consistent System

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Definition IDV   Independent and Dependent Variables

Suppose A is the augmented matrix of a system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS   Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI   Consistent, More Variables than Equations, Infinite solutions

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

A system of linear equations is **homogeneous** if each equation has a 0 for its constant term. Such a system then has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE   Trivial Solution to Homogeneous Systems of Equations   **25**

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ is called the **trivial solution**.

Theorem HMVEI   Homogeneous, More Variables than Equations, Infinite solutions   **26**

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Definition CV   Column Vector   **27**

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\underset{\sim}{u}$. To refer to the **entry** or **component** that is number i in the list that is the vector $\mathbf{v}$ we write $[\mathbf{v}]_i$.

Definition ZV   Zero Vector   **28**

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Definition IM   Identity Matrix

The $m \times m$ **identity matrix**, I_m is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem NSRRI   NonSingular matrices Row Reduce to the Identity matrix

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE   Column Vector Equality

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Definition CVA   Column Vector Addition

Given the vectors $\mathbf{u}$ and $\mathbf{v}$ the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m$$

Given the vector $\mathbf{u}$ and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of $\mathbf{u}$ by α, $\alpha\mathbf{u}$ is defined by

$$[\alpha\mathbf{u}]_i = \alpha\,[\mathbf{u}]_i \qquad\qquad 1 \le i \le m$$

Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC   Additive Closure, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- **SCC   Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.

- **CC   Commutativity, Column Vectors** If $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AAC   Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **ZC   Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.

- **AIC   Additive Inverses, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMAC   Scalar Multiplication Associativity, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVAC   Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSAC   Distributivity across Scalar Addition, Column Vectors** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **OC   One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Definition LCCV Linear Combination of Column Vectors

Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars α_1, α_2, α_3, ..., α_n, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$. Then $\mathbf{x}$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \ne f_j \\ -[B]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r} \,\middle|\, x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \right\}$$

Theorem PSPHS Particular Solution Plus Homogeneous Solutions

Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \,|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p\}$$
$$= \left\{\sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le p\right\}$$

Theorem SSNS   Spanning Sets for Null Spaces   **51**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \le j \le n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \ne f_j \\ -[B]_{k, f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\} \rangle.$$

Definition RLDCV   Relation of Linear Dependence for Column Vectors   **52**

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is the **trivial relation of linear dependence** on S.

Definition LICV   Linear Independence of Column Vectors

The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Theorem LIVHS   Linearly Independent Vectors and Homogeneous Systems

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Theorem NSLIC NonSingular matrices have Linearly Independent Columns **57**

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NSME2 NonSingular Matrix Equivalences, Round 2 **58**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \le j \le n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \ne f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Theorem DLDS Dependency in Linearly Dependent Sets

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \le t \le n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_n$.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \ldots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots \mathbf{v}_{d_r}\}$ is a linearly independent set.

2. $W = \langle T \rangle$.

Suppose that $\mathbf{u}$ is a vector from $\mathbb{C}^m$. Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \qquad\qquad 1 \le i \le m$$

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP   Inner Product

Given the vectors $\mathbf{u},\ \mathbf{v} \in \mathbb{C}^m$ the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\ \mathbf{v} \rangle = [\mathbf{u}]_1\,\overline{[\mathbf{v}]_1} + [\mathbf{u}]_2\,\overline{[\mathbf{v}]_2} + [\mathbf{u}]_3\,\overline{[\mathbf{v}]_3} + \cdots + [\mathbf{u}]_m\,\overline{[\mathbf{v}]_m} = \sum_{i=1}^{m} [\mathbf{u}]_i\,\overline{[\mathbf{v}]_i}$$

Theorem IPVA   Inner Product and Vector Addition

Suppose $\mathbf{u}\,\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1. $\langle \mathbf{u} + \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{u},\ \mathbf{w} \rangle + \langle \mathbf{v},\ \mathbf{w} \rangle$
2. $\langle \mathbf{u},\ \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u},\ \mathbf{v} \rangle + \langle \mathbf{u},\ \mathbf{w} \rangle$

Suppose $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

$$1. \quad \langle \alpha \mathbf{u}, \, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \, \mathbf{v} \rangle$$
$$2. \quad \langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \, \mathbf{v} \rangle$$

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \, \mathbf{u} \rangle}$.

Definition NV Norm of a Vector

The **norm** of the vector **u** is the scalar quantity in $\mathbb{C}$

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \cdots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^{m} |[\mathbf{u}]_i|^2}$$

Theorem IPN Inner Products and Norms

Suppose that **u** is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u},\, \mathbf{u} \rangle$.

Theorem PIP Positive Inner Products   **71**

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV Orthogonal Vectors   **72**

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Definition ONS OrthoNormal Set

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

Definition VSM   Vector Space of $m \times n$ Matrices   **77**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME   Matrix Equality   **78**

The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA   Matrix Addition

Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \qquad\qquad 1 \le i \le m,\, 1 \le j \le n$$

Definition MSM   Matrix Scalar Multiplication

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha\, [A]_{ij} \qquad\qquad 1 \le i \le m,\, 1 \le j \le n$$

Theorem VSPM Vector Space Properties of Matrices

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM   Additive Closure, Matrices** If A, $B \in M_{mn}$, then $A + B \in M_{mn}$.

- **SCM   Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

- **CM   Commutativity, Matrices** If A, $B \in M_{mn}$, then $A + B = B + A$.

- **AAM   Additive Associativity, Matrices** If A, B, $C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

- **ZM   Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.

- **AIM   Additive Inverses, Matrices** If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.

- **SMAM   Scalar Multiplication Associativity, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.

- **DMAM   Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and A, $B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

- **DSAM   Distributivity across Scalar Addition, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

- **OM   One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM Zero Matrix

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM   Transpose of a Matrix   83

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM   Symmetric Matrix   84

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS   Symmetric Matrices are Square   **85**

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA   Transpose and Matrix Addition   **86**

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Theorem TMSM   Transpose and Matrix Scalar Multiplication

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Theorem TT   Transpose of a Transpose

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Theorem CRMSM   Conjugation Respects Matrix Scalar Multiplication   **91**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Theorem MCT   Matrix Conjugation and Transposes   **92**

Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \, \mathbf{A}_1 + [\mathbf{u}]_2 \, \mathbf{A}_2 + [\mathbf{u}]_3 \, \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \, \mathbf{A}_n$$

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Theorem EMMVP Equal Matrices and Matrix-Vector Products **95**

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Definition MM Matrix Multiplication **96**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\ldots$, $\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\,[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p] = [A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p].$$

Suppose A is an $m \times n$ matrix and $B =$ is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1}[B]_{1j} + [A]_{i2}[B]_{2j} + [A]_{i3}[B]_{3j} + \cdots + [A]_{in}[B]_{nj} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}$$

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Theorem MMIM   Matrix Multiplication and Identity Matrix

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Theorem MMDAA   Matrix Multiplication Distributes Across Addition

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Theorem MMSMM   Matrix Multiplication and Scalar Matrix Multiplication   **101**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA   Matrix Multiplication is Associative   **102**

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

If we consider the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT   Matrix Multiplication and Transposes

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Definition MI   Matrix Inverse

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Let $\mathbf{e}_j \in \mathbb{C}^m$ denote the column vector that is column j of the $m \times m$ identity matrix I_m. Then the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \leq j \leq m\}$$

is the set of **standard unit vectors** in $\mathbb{C}^m$.

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS   Socks and Shoes

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI   Matrix Inverse of a Matrix Inverse

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Theorem MIT Matrix Inverse of a Transpose **113**

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Theorem MISM Matrix Inverse of a Scalar Multiple **114**

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem NPNT Nonsingular Product has Nonsingular Terms

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Theorem OSIS One-Sided Inverse is Sufficient

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Theorem NSI   NonSingularity is Invertibility

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NSME3   NonSingular Matrix Equivalences, Round 3

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A,\ \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^{t} Q = I_n$. Then we say Q is **orthogonal**.

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set.

Suppose that Q is an orthogonal matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\, Q\mathbf{v}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle \qquad \text{and} \qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|$$

If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$.

Definition HM   Hermitian Matrix

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = \left(\overline{A}\right)^t$

Definition CSM   Column Space of a Matrix

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,
$$\mathcal{C}(A) = \langle\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}\rangle$$

Theorem CSCS   Column Spaces and Consistent Systems

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Theorem BCS   Basis of the Column Space

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\ldots$, $\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.

2. $\mathcal{C}(A) = \langle T \rangle$.

Theorem CSNSM   Column Space of a NonSingular Matrix   **129**

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Theorem NSME4   NonSingular Matrix Equivalences, Round 4   **130**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Theorem BRS   Basis for the Row Space   **133**

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Theorem CSRST   Column Space, Row Space, Transpose   **134**

Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Definition LNS   Left Null Space   **135**

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Definition EEF   Extended Echelon Form   **136**

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom $m-r$ rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array} \right]$$

Theorem PEEF   Properties of Extended Echelon Form   **137**

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. J is nonsingular.

2. $B = JA$.

3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.

4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.

5. L is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

Theorem FS   Four Subsets   **138**

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m - r$ rows. Then

1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.

2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.

3. The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.

4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS   Vector Space   **139**

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten properties hold.

- **AC**   **Additive Closure** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

- **SC**   **Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

- **C**   **Commutativity** If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AA**   **Additive Associativity** If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Z**   **Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

- **AI**   **Additive Inverses** If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMA**   **Scalar Multiplication Associativity** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVA**   **Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}$, $\mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSA**   **Distributivity across Scalar Addition** If α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **O**   **One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU   Zero Vector is Unique   **140**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU   Additive Inverses are Unique

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Theorem ZSSM   Zero Scalar in Scalar Multiplication

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Theorem ZVSM Zero Vector in Scalar Multiplication **143**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM Additive Inverses from Scalar Multiplication **144**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Theorem SMEZV   Scalar Multiplication Equals the Zero Vector

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Theorem VAC   Vector Addition Cancellation

Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CSSM   Canceling Scalars in Scalar Multiplication

Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CVSM   Canceling Vectors in Scalar Multiplication

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and α, $\beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Definition S   Subspace   149

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS   Testing Subsets for Subspaces   150

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.

2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.

3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

Definition TS Trivial Subspaces   **151**

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS Null Space of a Matrix is a Subspace   **152**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC   Linear Combination   **153**

Suppose that V is a vector space.   Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_n$ and n scalars α_1, α_2, α_3, $\ldots$, α_n, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Definition SS   Span of a Set   **154**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}$$

$$= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}$$

Theorem SSS   Span of a Set is a Subspace   **155**

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Theorem CSMS   Column Space of a Matrix is a Subspace   **156**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Theorem RSMS Row Space of a Matrix is a Subspace

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Theorem LNSMS Left Null Space of a Matrix is a Subspace

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$.

Definition B   Basis   **159**

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem SUVB   Standard Unit Vectors are a Basis   **160**

The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Theorem CNSMB   Columns of NonSingular Matrix are a Basis   **161**

Suppose that A is a square matrix of size m. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Theorem NSME5   NonSingular Matrix Equivalences, Round 5   **162**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

Suppose that $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w},\, \mathbf{v}_1 \rangle\, \mathbf{v}_1 + \langle \mathbf{w},\, \mathbf{v}_2 \rangle\, \mathbf{v}_2 + \langle \mathbf{w},\, \mathbf{v}_3 \rangle\, \mathbf{v}_3 + \cdots + \langle \mathbf{w},\, \mathbf{v}_p \rangle\, \mathbf{v}_p$$

Suppose that V is a vector space and $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.
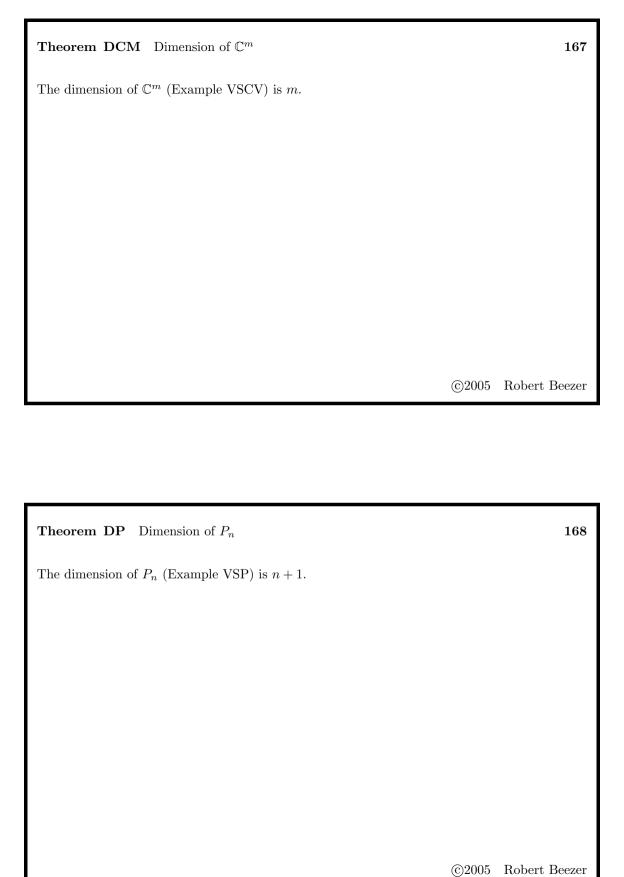
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM Dimension of $\mathbb{C}^m$

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP Dimension of P_n

The dimension of P_n (Example VSP) is $n + 1$.

The dimension of M_{mn} (Example VSM) is mn.

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM Rank Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r\,(A) = \dim\,(\mathcal{C}(A))$.

Theorem CRN Computing Rank and Nullity

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r\,(A) = r$ and $n\,(A) = n - r$.

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. The rank of A is n, $r(A) = n$.

3. The nullity of A is zero, $n(A) = 0$.

Theorem NSME6 NonSingular Matrix Equivalences, Round 6 **175**
Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

Theorem ELIS Extending Linearly Independent Sets **176**

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.

2. If $m < t$, then S does not span V.

3. If $m = t$ and S is linearly independent, then S spans V.

4. If $m = t$ and S spans V, then S is linearly independent.

Theorem EDYES Equal Dimensions Yields Equal Subspaces

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Theorem RMRT   Rank of a Matrix is the Rank of the Transpose

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem DFS   Dimensions of Four Subspaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim(\mathcal{N}(A)) = n - r$

2. $\dim(\mathcal{C}(A)) = r$

3. $\dim(\mathcal{R}(A)) = r$

4. $\dim(\mathcal{L}(A)) = m - r$

Definition ELEM Elementary Matrices

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$
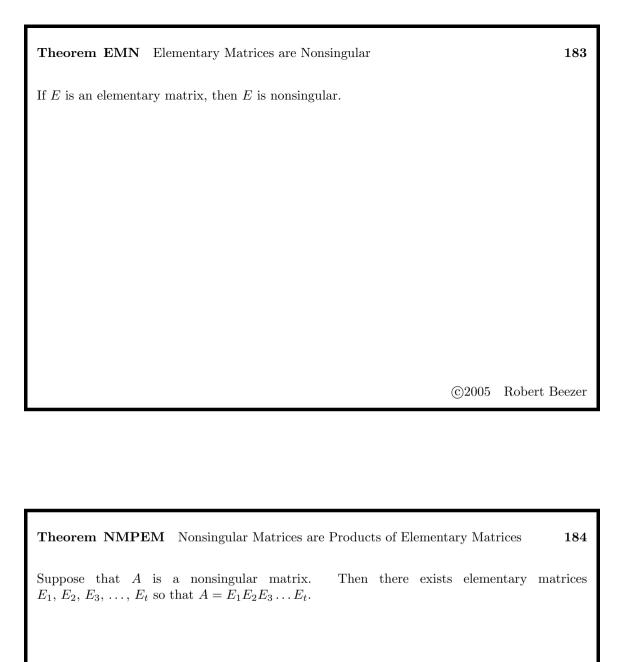

Theorem EMDRO Elementary Matrices Do Row Operations

Suppose that A is a matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO).

1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.

2. If the row operation multiplies row i by α, then $B = E_i(\alpha)A$.

3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha)A$.

Theorem EMN   Elementary Matrices are Nonsingular   **183**

If E is an elementary matrix, then E is nonsingular.

Theorem NMPEM   Nonsingular Matrices are Products of Elementary Matrices   **184**

Suppose that A is a nonsingular matrix.   Then there exists elementary matrices E_1, E_2, E_3, …, E_t so that $A = E_1 E_2 E_3 \ldots E_t$.

Definition SM SubMatrix

Suppose that A is an $m \times n$ matrix. Then the **submatrix** $A(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM Determinant of a Matrix

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11}\det(A(1|1)) - [A]_{12}\det(A(1|2)) + [A]_{13}\det(A(1|3)) - \cdots + (-1)^{n+1}[A]_{1n}\det(A(1|n))$$

Theorem DMST Determinant of Matrices of Size Two

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Theorem DER Determinant Expansion about Rows

Suppose that A is a square matrix of size n. Then

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2))$$
$$+ (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \qquad 1 \le i \le n$$

which is known as **expansion** about row i.

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Suppose that A is a square matrix of size n. Then

$$\det(A) = (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j))$$
$$+ (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j)) \qquad 1 \le j \le n$$

which is known as **expansion** about column j.

Theorem DRMM   Determinant Respects Matrix Multiplication   **191**

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

Theorem SMZD   Singular Matrices have Zero Determinants   **192**

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NSME7   NonSingular Matrix Equivalences, Round 7   **193**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.
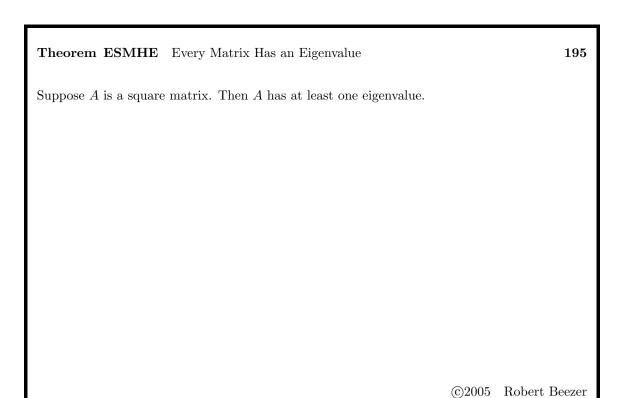
11. The determinant of A is nonzero, $\det(A) \neq 0$.

Definition EEM   Eigenvalues and Eigenvectors of a Matrix   **194**

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in $\mathbb{C}^n$, and λ is a scalar in $\mathbb{C}$. Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

Theorem ESMHE   Every Matrix Has an Eigenvalue   **195**

Suppose A is a square matrix. Then A has at least one eigenvalue.
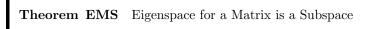
Definition CP   Characteristic Polynomial   **196**

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

Theorem EMRCP   Eigenvalues of a Matrix are Roots of Characteristic Polynomials   **197**

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM   Eigenspace of a Matrix   **198**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $E_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Theorem EMS   Eigenspace for a Matrix is a Subspace

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS   Eigenspace of a Matrix is a Null Space

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Definition AME   Algebraic Multiplicity of an Eigenvalue

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME   Geometric Multiplicity of an Eigenvalue

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$.

Theorem EDELI   Eigenvectors with Distinct Eigenvalues are Linearly Independent   **203**

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE   Singular Matrices have Zero Eigenvalues   **204**

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NSME8   NonSingular Matrix Equivalences, Round 8   **205**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM   Eigenvalues of a Scalar Multiple of a Matrix   **206**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP   Eigenvalues Of Matrix Powers

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Theorem EPM   Eigenvalues of the Polynomial of a Matrix

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Theorem EIM   Eigenvalues of the Inverse of a Matrix   209

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM   Eigenvalues of the Transpose of a Matrix   210

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP   Eigenvalues of Real Matrices come in Conjugate Pairs   **211**

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP   Degree of the Characteristic Polynomial   **212**

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Theorem NEM Number of Eigenvalues of a Matrix

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Theorem ME Multiplicities of an Eigenvalue

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$$

Theorem MNEM   Maximum Number of Eigenvalues of a Matrix   **215**

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Theorem HMRE   Hermitian Matrices have Real Eigenvalues   **216**

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Theorem HMOE   Hermitian Matrices have Orthogonal Eigenvectors   **217**

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM   Similar Matrices   **218**

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

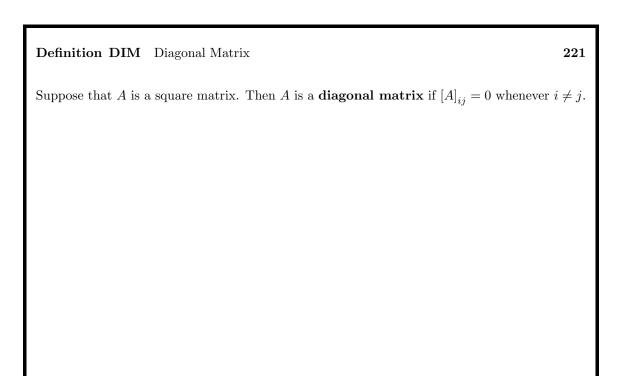Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)

2. If A is similar to B, then B is similar to A. (Symmetric)

3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.

Definition DIM   Diagonal Matrix   **221**

Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$.

Definition DZM   Diagonalizable Matrix   **222**

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC   Diagonalization Characterization   **223**

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMLE   Diagonalizable Matrices have Large Eigenspaces   **224**

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Definition LT   Linear Transformation   **227**

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1$, $\mathbf{u}_2 \in U$

2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ   Linear Transformations Take Zero to Zero   **228**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Suppose that A is an $m \times n$ matrix. Define a function $T \colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Suppose that $T \colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations **231**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, $\ldots$, a_t are scalars from $\mathbb{C}$. Then

$$T\left(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t\right) = a_1 T\left(\mathbf{u}_1\right) + a_2 T\left(\mathbf{u}_2\right) + a_3 T\left(\mathbf{u}_3\right) + \cdots + a_t T\left(\mathbf{u}_t\right)$$

Theorem LTDB Linear Transformation Defined on a Basis **232**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let a_1, a_2, a_3, $\ldots$, a_n be the scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T\left(\mathbf{w}\right) = a_1 T\left(\mathbf{u}_1\right) + a_2 T\left(\mathbf{u}_2\right) + a_3 T\left(\mathbf{u}_3\right) + \cdots + a_n T\left(\mathbf{u}_n\right)$$

Definition PI   Pre-Image   **233**

Suppose that $T\colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{\, \mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v} \,\right\}$$

Definition LTA   Linear Transformation Addition   **234**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$\left(T + S\right)\left(\mathbf{u}\right) = T\left(\mathbf{u}\right) + S\left(\mathbf{u}\right)$$

Theorem SLTLT   Sum of Linear Transformations is a Linear Transformation   **235**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Definition LTSM   Linear Transformation Scalar Multiplication   **236**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT   Multiple of a Linear Transformation is a Linear Transformation   **237**

Suppose that $T \colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T) \colon U \mapsto V$ is a linear transformation.

Theorem VSLT   Vector Space of Linear Transformations   **238**

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\text{LT}\,(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC   Linear Transformation Composition   **239**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT   Composition of Linear Transformations is a Linear Transformation   **240**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a linear transformation.

Definition ILT   Injective Linear Transformation   **241**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition KLT   Kernel of a Linear Transformation   **242**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \left\{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \right\}$$

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Suppose $T \colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then
$$T^{-1}(\mathbf{v}) = \{\, \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T) \,\} = \mathbf{u} + \mathcal{K}(T)$$
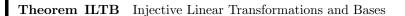
Theorem KILT   Kernel of an Injective Linear Transformation   **245**

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Theorem ILTLI   Injective Linear Transformations and Linear Independence   **246**

Suppose that $T \colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T\left(\mathbf{u}_1\right),\, T\left(\mathbf{u}_2\right),\, T\left(\mathbf{u}_3\right),\, \ldots,\, T\left(\mathbf{u}_m\right)\}$ is a linearly independent subset of V.

Theorem ILTD Injective Linear Transformations and Dimension

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim\left(U\right) \leq \dim\left(V\right)$.

Theorem CILTI   Composition of Injective Linear Transformations is Injective   **249**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is an injective linear transformation.

Definition SLT   Surjective Linear Transformation   **250**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Definition RLT   Range of a Linear Transformation

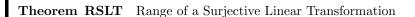Suppose $T \colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

Theorem RLTS   Range of a Linear Transformation is a Subspace

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.

Theorem RPI   Range and Pre-Image   **255**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}\left(\mathbf{v}\right) \neq \emptyset$$

Theorem SLTB   Surjective Linear Transformations and Bases   **256**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T\left(\mathbf{u}_1\right),\, T\left(\mathbf{u}_2\right),\, T\left(\mathbf{u}_3\right),\, \ldots,\, T\left(\mathbf{u}_m\right)\}$ is a spanning set for V.

Theorem SLTD   Surjective Linear Transformations and Dimension

Suppose that $T\colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS   Composition of Surjective Linear Transformations is Surjective

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT   Identity Linear Transformation

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

Definition IVLT   Invertible Linear Transformations

Suppose that $T \colon U \mapsto V$ is a linear transformation. If there is a function $S \colon V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT   Inverse of a Linear Transformation is a Linear Transformation   **261**

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1} \colon V \mapsto U$ is a linear transformation.

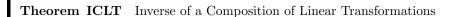**Theorem IILT**   Inverse of an Invertible Linear Transformation   **262**

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Theorem ILTIS   Invertible Linear Transformations are Injective and Surjective   **263**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT   Composition of Invertible Linear Transformations   **264**

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T) \colon U \mapsto W$ is an invertible linear transformation.

Theorem ICLT   Inverse of a Composition of Linear Transformations   **265**
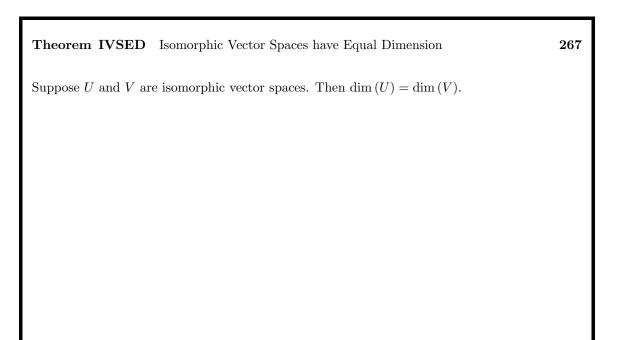
Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS   Isomorphic Vector Spaces   **266**

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED   Isomorphic Vector Spaces have Equal Dimension   **267**

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Definition ROLT   Rank Of a Linear Transformation   **268**

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,

$$r(T) = \dim(\mathcal{R}(T))$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n\left(T\right)$, is the dimension of the kernel of T,

$$n\left(T\right) = \dim\left(\mathcal{K}(T)\right)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r\left(T\right) = \dim\left(V\right)$, if and only if T is surjective.

Theorem NOILT   Nullity Of an Injective Linear Transformation

Suppose that $T \colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n\,(T) = 0$, if and only if T is injective.

Theorem RPNDD   Rank Plus Nullity is Domain Dimension

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

$$r\,(T) + n\,(T) = \dim\,(U)$$

Definition VR   Vector Representation

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \ldots, a_n$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

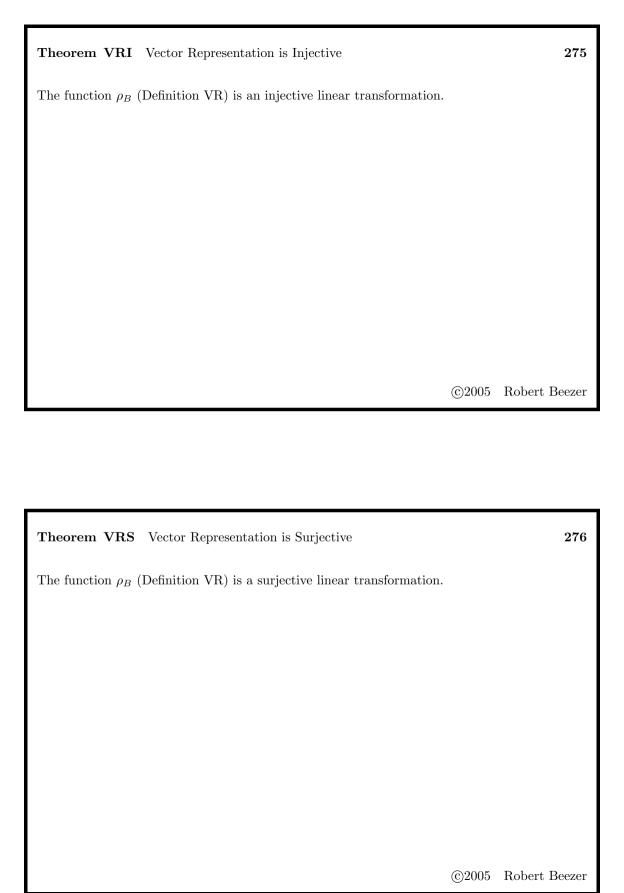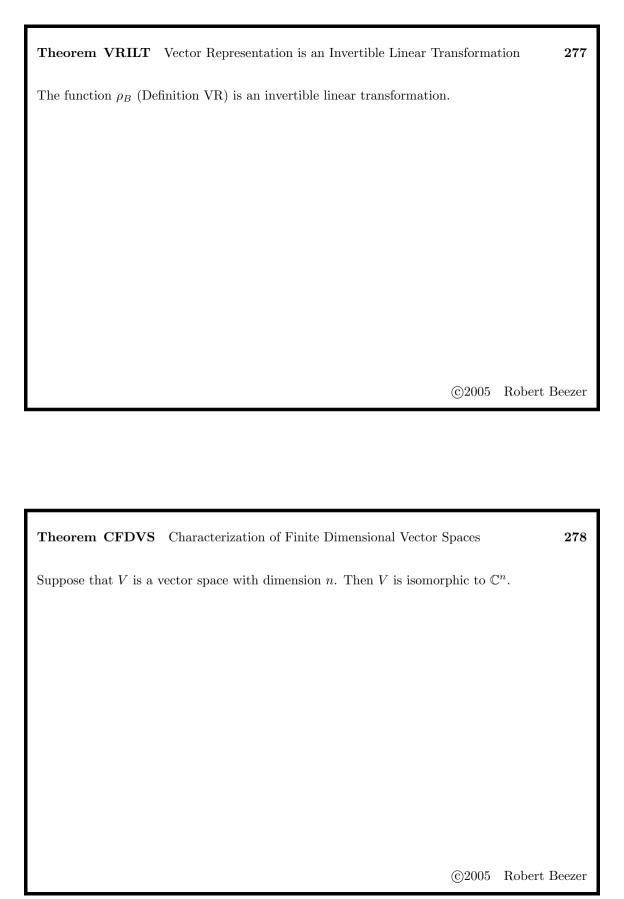$$[\rho_B\left(\mathbf{w}\right)]_i = a_i \qquad\qquad 1 \le i \le n$$

Theorem VRLT   Vector Representation is a Linear Transformation

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI   Vector Representation is Injective   **275**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS   Vector Representation is Surjective   **276**

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT Vector Representation is an Invertible Linear Transformation **277**

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces **278**

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

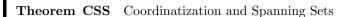**Theorem IFDVS**   Isomorphism of Finite Dimensional Vector Spaces   **279**

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI   Coordinatization and Linear Independence   **280**

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \langle\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}\rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle\{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}\rangle$.

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = \left[\, \rho_C(T(\mathbf{u}_1)) \,|\, \rho_C(T(\mathbf{u}_2)) \,|\, \rho_C(T(\mathbf{u}_3)) \,|\, \ldots \,|\, \rho_C(T(\mathbf{u}_n)) \,\right]$$

Theorem FTMR   Fundamental Theorem of Matrix Representation   **283**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^{T}$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^{T}\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M_{B,C}^{T}\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

Theorem MRSLT   Matrix Representation of a Sum of Linear Transformations   **284**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^{T} + M_{B,C}^{S}$$

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation **285**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^{T}$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations **286**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^{S} M_{B,C}^{T}$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}\big(M_{B,C}^T\big)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}\big(M_{B,C}^T\big)$$

Suppose that $T: U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^{T}$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^{T} \right)^{-1}$$

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Theorem NSME9   NonSingular Matrix Equivalences, Round 9 291

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

13. The linear transformation $T \colon \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Definition EELT   Eigenvalue and Eigenvector of a Linear Transformation

Suppose that $T \colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

Definition CBM   Change-of-Basis Matrix   **293**

Suppose that V is a vector space, and $I_V : V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M_{B,C}^{I_V} \\
&= [\rho_C(I_V(\mathbf{v}_1))|\ \rho_C(I_V(\mathbf{v}_2))|\ \rho_C(I_V(\mathbf{v}_3))|\ldots|\rho_C(I_V(\mathbf{v}_n))] \\
&= [\rho_C(\mathbf{v}_1)|\ \rho_C(\mathbf{v}_2)|\ \rho_C(\mathbf{v}_3)|\ldots|\rho_C(\mathbf{v}_n)]
\end{aligned}
$$

Theorem CB   Change-of-Basis   **294**

Suppose that $\mathbf{v}$ is a vector in the vector space V and B and C are bases of V. Then

$$
\rho_C(\mathbf{v}) = C_{B,C}\rho_B(\mathbf{v})
$$

Theorem ICBM Inverse of Change-of-Basis Matrix

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and
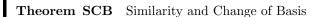$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB Matrix Representation and Change of Basis

Suppose that $T \colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then
$$M_{B,D}^{T} = C_{E,D} M_{C,E}^{T} C_{B,C}$$

Theorem SCB   Similarity and Change of Basis

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

Theorem EER   Eigenvalues, Eigenvectors, Representations

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ.