Flash Cards

to accompany

A First Course in Linear Algebra

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Definition SSLE System of Simultaneous Linear Equations

A **system of simultaneous linear equations** is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Definition ES Equivalent Systems

Two systems of simultaneous linear equations are **equivalent** if their solution sets are equal.

Definition EO   Equation Operations

Given a system of simultaneous linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

1. Swap the locations of two equations in the list.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Theorem EOPSS   Equation Operations Preserve Solution Sets

Suppose we apply one of the three equation operations of Definition EO to the system of simultaneous linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots \quad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.$$

Then the original system and the transformed system are equivalent systems.

Definition M Matrix   **5**

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns.

Definition AM Augmented Matrix   **6**

Suppose we have a system of m equations in the n variables $x_1, x_2, x_3, \ldots, x_n$ written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Definition RO   Row Operations   **7**

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.

2. Multiply each entry of a single row by a nonzero quantity.

3. Multiply each entry of one row by some quantity, and add these values to the entry in the same column of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

Definition REM   Row-Equivalent Matrices   **8**

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero is below any row containing a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $i < s$, then $j < t$.

Definition ZRM   Zero Row of a Matrix

A row of a matrix where every entry is zero is called a **zero row**.

Definition LO   Leading Ones

For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a **leading 1**.

Definition PC Pivot Columns

For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

Theorem REMEF Row-Equivalent Matrix in Echelon Form

Suppose A is a matrix. Then there is a (unique!) matrix B so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

Definition CS   Consistent System **15**

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Definition IDV   Independent and Dependent Variables **16**

Suppose A is the augmented matrix of a system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Theorem CSRN   Consistent Systems, r and n

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Theorem FVCS   Free Variables for Consistent Systems

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS   Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI   Consistent, More Variables than Equations, Infinite solutions

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

A system of linear equations is **homogeneous** if each equation has a 0 for its constant term. Such a system then has the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE   Trivial Solution to Homogeneous Systems of Equations   **25**

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0,\ldots, x_n = 0$ is called the **trivial solution**.

Theorem HMVEI   Homogeneous, More Variables than Equations, Infinite solutions   **26**

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

A **column vector** of **size** m is an ordered list of m numbers, which is written vertically, in order from top to bottom. At times, we will refer to a column vector as simply a **vector**.

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Definition CM Coefficient Matrix

For a system of linear equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

the **coefficient matrix** is the $m \times n$ matrix

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\
\vdots & & & & \\
a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn}
\end{bmatrix}
$$

Definition VOC Vector of Constants

For a system of linear equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

the **vector of constants** is the column vector of size m

$$
\mathbf{b} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_m
\end{bmatrix}
$$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size m

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix}$$

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition SQM Square Matrix

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM Nonsingular Matrix

Suppose A is a square matrix. And suppose the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Definition IM Identity Matrix

The $m \times m$ **identity matrix**, $I_m = (a_{ij})$ has $a_{ij} = 1$ whenever $i = j$, and $a_{ij} = 0$ whenever $i \neq j$.

Theorem NSRRI NonSingular matrices Row Reduce to the Identity matrix

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the trivial solution to the system $\mathcal{LS}(A,\,\mathbf{0})$, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Theorem NSME1   NonSingular Matrix Equivalences, Round 1   **39**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the trivial solution, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

Definition VSCV   Vector Space of Column Vectors   **40**

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE   Column Vector Equality   **41**

The vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that $u_i = v_i$ for all $1 \le i \le m$.

Definition CVA   Column Vector Addition   **42**

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_m + v_m \end{bmatrix}.$$

Definition CVSM Column Vector Scalar Multiplication Given the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of $\mathbf{u}$ by α is

$$\alpha\mathbf{u} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \\ \vdots \\ \alpha u_m \end{bmatrix}.$$

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Theorem VSPCV Vector Space Properties of Column Vectors Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC   Additive Closure, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- **SCC   Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.

- **CC   Commutativity, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **AAC   Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **ZC   Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.

- **AIC   Additive Inverses, Column Vectors** For each vector $\mathbf{u} \in \mathbb{C}^m$, there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **SMAC   Scalar Multiplication Associativity, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- **DVAC   Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- **DSAC   Distributivity across Scalar Addition, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- **OC   One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

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Definition LCCV   Linear Combination of Column Vectors   45

Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_n$ and n scalars α_1, α_2, α_3, ..., α_n, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Theorem SLSLC   Solutions to Linear Systems are Linear Combinations   46

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$. Then $\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$

is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3 + \cdots + \alpha_n\mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS   Vector Form of Solutions to Linear Systems   **47**

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Denote the vector of variables as $\mathbf{x} = (x_i)$. Let $B = (b_{ij})$ be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's having indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c} = (c_i)$, $\mathbf{u}_j = (u_{ij})$, $1 \le j \le n - r$ of size n by

$$c_i = \begin{cases} 0 & \text{if } i \in F \\ b_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases}$$

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} .$$

Then the set of solutions to the system of equations represented by the vector equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \cdots + x_{f_{n-r}} \mathbf{u}_{n-r}$$

is equal to the set of solutions of $\mathcal{LS}(A, \mathbf{b})$.

Theorem RREFU   Reduced Row-Echelon Form is Unique   **48**

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Definition SSCV   Span of a Set of Column Vectors

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\mathcal{S}p(S) = \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \, 1 \le i \le t\}$$
$$= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C}, \, 1 \le i \le t \right\}$$

Theorem SSNS   Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{u}_j = (u_{ij})$, $1 \le j \le n - r$ of size n as

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \ne f_j \\ -b_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}.$$

Then the null space of A is given by

$$\mathcal{N}(A) = \mathcal{S}p(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}\}).$$

Definition RLDCV   Relation of Linear Dependence for Column Vectors   **51**

Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$, an equation of the form

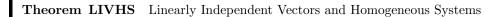$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0,\ 1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LICV   Linear Independence of Column Vectors   **52**

The set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Theorem LIVHS   Linearly Independent Vectors and Homogeneous Systems   **53**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Theorem LIVRN   Linearly Independent Vectors, r and n **54**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Theorem MVSLD   More Vectors than Size implies Linear Dependence

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Theorem NSLIC   NonSingular matrices have Linearly Independent Columns

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A form a linearly independent set.

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j = (z_{ij})$, $1 \leq j \leq n - r$ of size n as

$$
z_{ij} = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \ . \\ -b_{k,f_j} & \text{if } i \in D, \ i = d_k \end{cases}
$$

Define the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Definition CCCV Complex Conjugate of a Column Vector

Suppose that

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

is a vector from $\mathbb{C}^m$. Then the conjugate of the vector is defined as

$$\overline{\mathbf{u}} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \\ \vdots \\ \overline{u}_m \end{bmatrix}$$

Theorem CRVA Conjugation Respects Vector Addition

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Theorem CRSM Conjugation Respects Vector Scalar Multiplication

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha}\, \overline{\mathbf{x}}$$

Definition IP Inner Product

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\, \mathbf{v} \rangle = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3} + \cdots + u_m\overline{v_m} = \sum_{i=1}^{m} u_i\overline{v_i}$$

Theorem IPVA Inner Product and Vector Addition

Suppose $\mathbf{u}\,\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

$$1. \quad \langle \mathbf{u} + \mathbf{v},\, \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{w} \rangle + \langle \mathbf{v},\, \mathbf{w} \rangle$$
$$2. \quad \langle \mathbf{u},\, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle + \langle \mathbf{u},\, \mathbf{w} \rangle$$

Theorem IPSM Inner Product and Scalar Multiplication

Suppose $\mathbf{u},\, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

$$1. \quad \langle \alpha\mathbf{u},\, \mathbf{v} \rangle = \alpha \langle \mathbf{u},\, \mathbf{v} \rangle$$
$$2. \quad \langle \mathbf{u},\, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u},\, \mathbf{v} \rangle$$

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.
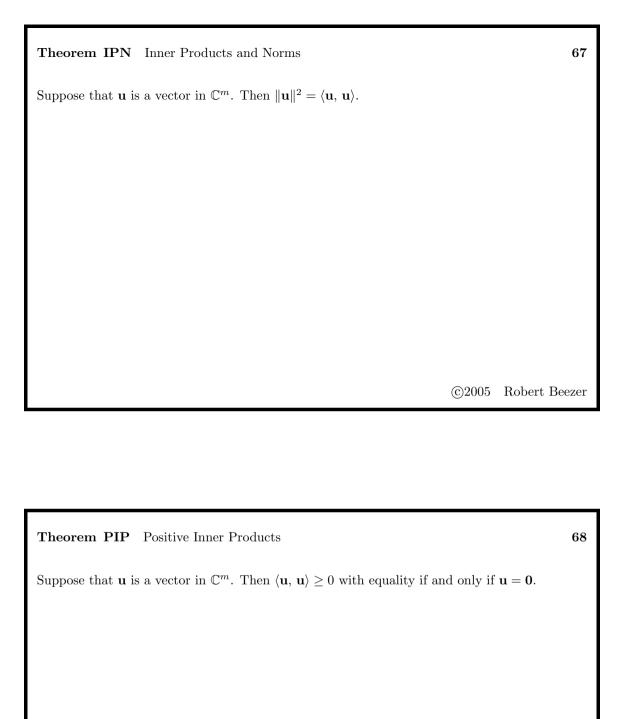

The **norm** of the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

is the scalar quantity in $\mathbb{C}^m$

$$\|\mathbf{u}\| = \sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2}$$

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV   Orthogonal Vectors

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Definition OSV   Orthogonal Set of Vectors

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \le i \le p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3,\, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1},\, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\mathcal{S}p(T) = \mathcal{S}p(S)$.

Definition ONS   OrthoNormal Set   **73**

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

Definition VSM   Vector Space of $m \times n$ Matrices   **74**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality

The $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

are **equal**, written $A = B$ provided $a_{ij} = b_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA Matrix Addition

Given the $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

define the **sum** of A and B to be $A + B = C = (c_{ij})$, where

$$c_{ij} = a_{ij} + b_{ij}, \quad 1 \le i \le m,\, 1 \le j \le n.$$

Given the $m \times n$ matrix $A = (a_{ij})$ and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A by α is the matrix $\alpha A = C = (c_{ij})$, where

$$c_{ij} = \alpha a_{ij}, \quad 1 \le i \le m, \, 1 \le j \le n.$$

Theorem VSPM   Vector Space Properties of Matrices   **78**
Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM   Additive Closure, Matrices** If A, $B \in M_{mn}$, then $A + B \in M_{mn}$.

- **SCM   Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

- **CM   Commutativity, Matrices** If A, $B \in M_{mn}$, then $A + B = B + A$.

- **AAM   Additive Associativity, Matrices** If A, B, $C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

- **ZM   Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.

- **AIM   Additive Inverses, Matrices** For each matrix $A \in M_{mn}$, there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.

- **SMAM   Scalar Multiplication Associativity, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.

- **DMAM   Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and A, $B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

- **DSAM   Distributivity across Scalar Addition, Matrices** If α, $\beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

- **OM   One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM Zero Matrix

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n} = (z_{ij})$ and defined by $z_{ij} = 0$ for all $1 \le i \le m$, $1 \le j \le n$. Or, equivalently, $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$.

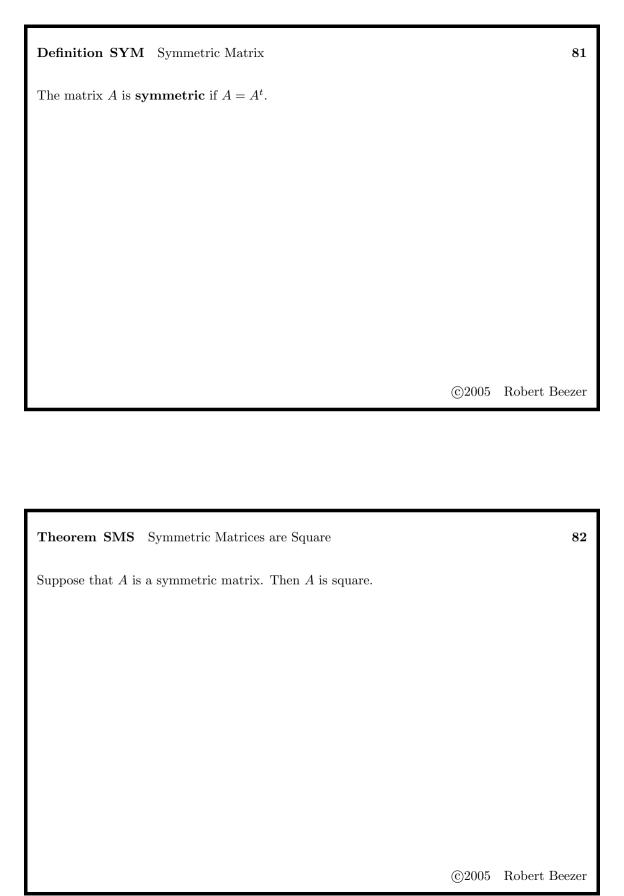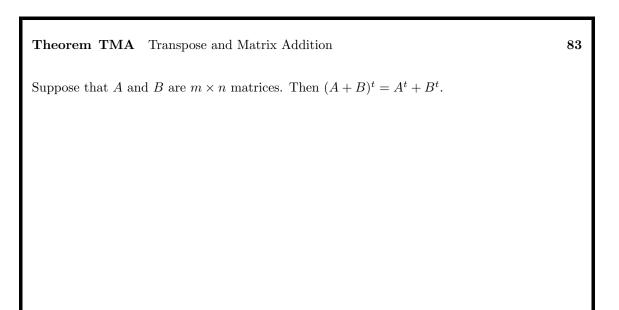**Definition TM** Transpose of a Matrix

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

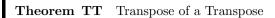$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \le i \le n, 1 \le j \le m.$$

Definition SYM Symmetric Matrix

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS Symmetric Matrices are Square

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA Transpose and Matrix Addition

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Theorem TMSM Transpose and Matrix Scalar Multiplication

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Theorem TT Transpose of a Transpose

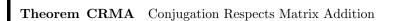Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Definition CCM Complex Conjugate of a Matrix

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{\left[A\right]_{ij}}$$

Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Definition RM Range of a Matrix

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the **range** of A, written $\mathcal{R}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{R}(A) = \mathcal{S}p(\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\})$$

Theorem RCS Range and Consistent Systems

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{R}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $S = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. $\mathcal{R}(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Suppose that A is an $m \times n$ matrix. Create the $m \times (n + m)$ matrix M by placing the $m \times m$ identity matrix I_m to the right of the matrix A. Symbolically, $M = [A \,|\, I_m]$. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Suppose there are r leading 1's of N in the first n columns. If $r = m$, then $\mathcal{R}(A) = \mathbb{C}^m$. Otherwise, $r < m$ and let K be the $(m - r) \times m$ matrix formed from the entries of N in the last $m - r$ rows and last m columns. Then

1. K is in reduced row-echelon form.

2. K has no zero rows, or equivalently, K has $m - r$ leading 1's.

3. $\mathcal{R}(A) = \mathcal{N}(K)$.

Theorem RNSM   Range of a NonSingular Matrix   **93**

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{R}(A) = \mathbb{C}^n$.

Theorem NSME3   NonSingular Matrix Equivalences, Round 3   **94**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

Definition RSM   Row Space of a Matrix

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the range of A^t, i.e. $\mathcal{R}(A) = \mathcal{R}(A^t)$.

Theorem REMRS   Row-Equivalent Matrices have equal Row Spaces

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Suppose A is a matrix. Then $\mathcal{R}(A) = \mathcal{R}(A^t)$.

Definition MVP Matrix-Vector Product

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is

$$A\mathbf{u} = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\dots|\mathbf{A}_n] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{A}_1 + u_2\mathbf{A}_2 + u_3\mathbf{A}_3 + \cdots + u_n\mathbf{A}_n$$
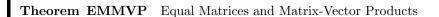
Theorem SLEMM Systems of Linear Equations as Matrix Multiplication

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, ..., $\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Suppose $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix. Then the entries of $AB = C = (c_{ij})$ are given by

$$[C]_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Theorem MMIM Matter Multiplication and Identity Matrix

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Theorem MMDAA Matrix Multiplication Distributes Across Addition

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

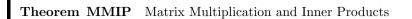**Theorem MMSMM**   Matrix Multiplication and Scalar Matrix Multiplication   **107**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA   Matrix Multiplication is Associative   **108**

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

If we consider the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then
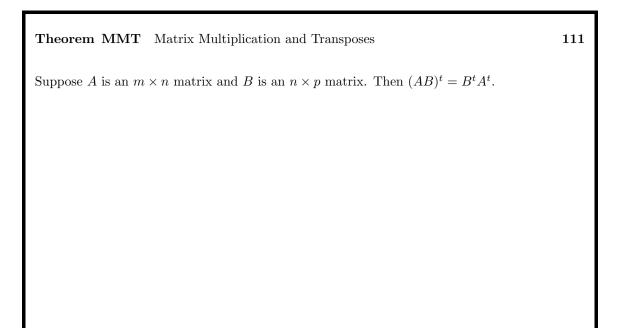
$$\langle \mathbf{u},\, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT   Matrix Multiplication and Transposes   **111**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Theorem PSPHS   Particular Solution Plus Homogeneous Solutions   **112**

Suppose that $\mathbf{z}$ is one solution to the linear system of equations $\mathcal{LS}(A,\, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\, b)$ if and only if $\mathbf{y} = \mathbf{z} + \mathbf{w}$ for some vector $\mathbf{w} \in N(A)$.

Definition MI   Matrix Inverse   **113**

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Definition SUV   Standard Unit Vectors   **114**

Let $\mathbf{e}_i \in \mathbb{C}^m$ denote the column vector that is column i of the $m \times m$ identity matrix I_m. Then the set
$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$$
is the set of **standard unit vectors** in $\mathbb{C}^m$.

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

Theorem MIU   Matrix Inverse is Unique Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS   Socks and Shoes Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI Matrix Inverse of a Matrix Inverse   **119**

Suppose A is an invertible matrix. Then $(A^{-1})^{-1} = A$ and A^{-1} is invertible.

Theorem MIT Matrix Inverse of a Transpose   **120**

Suppose A is an invertible matrix. Then $(A^t)^{-1} = (A^{-1})^t$ and A^t is invertible.

Theorem MISM   Matrix Inverse of a Scalar Multiple   121

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and αA is invertible.

Theorem PWSMS   Product With a Singular Matrix is Singular   122

Suppose that A or B are matrices of size n, and one, or both, is singular. Then their product, AB, is singular.

Theorem OSIS   One-Sided Inverse is Sufficient   **123**

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Theorem NSI   NonSingularity is Invertibility   **124**

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NSME3   NonSingular Matrix Equivalences, Round 3   **125**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

Theorem SNSCM   Solution with NonSingular Coefficient Matrix   **126**

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Definition OM   Orthogonal Matrices   **127**

Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^t Q = I_n$. Then we say Q is **orthogonal**.

Theorem OMI   Orthogonal Matrices are Invertible   **128**

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set.

Suppose that Q is an orthogonal matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\, Q\mathbf{v}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle \qquad \text{and} \qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|$$

Definition A Adjoint 131

If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$.

Definition HM Hermitian Matrix 132

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = \left(\overline{A}\right)^t$

Definition VS Vector Space   **133**

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten requirements (better known as "axioms") are met.

1. **AC   Additive Closure** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

2. **SC   Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

3. **C   Commutativity** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. **AA   Additive Associativity** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

5. **Z   Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

6. **AI   Additive Inverses** For each vector $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

7. **SMA   Scalar Multiplication Associativity** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

8. **DVA   Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

9. **DSA   Distributivity across Scalar Addition** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

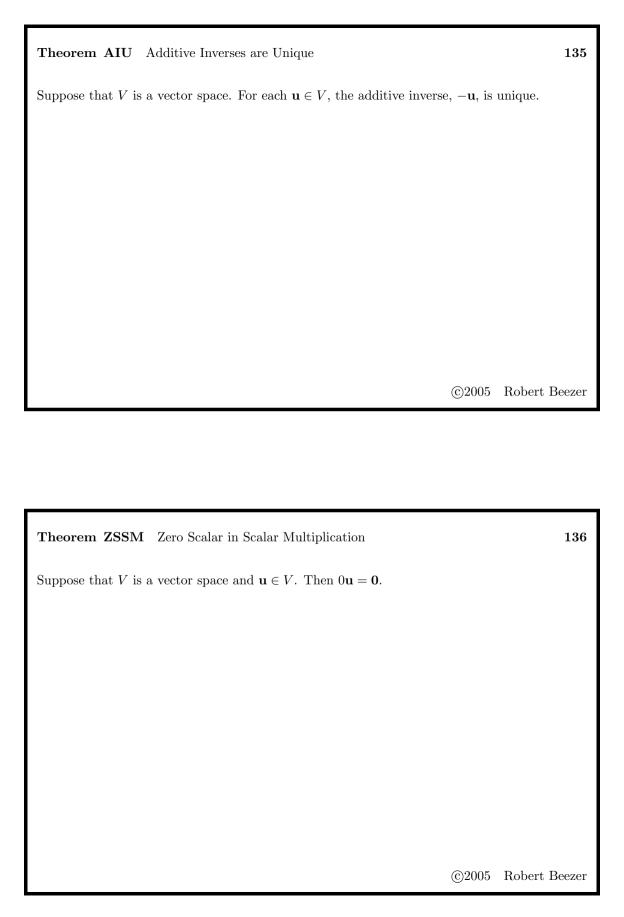10. **O   One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU   Zero Vector is Unique   **134**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU   Additive Inverses are Unique   **135**

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Theorem ZSSM   Zero Scalar in Scalar Multiplication   **136**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Theorem ZVSM   Zero Vector in Scalar Multiplication   **137**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM   Additive Inverses from Scalar Multiplication   **138**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Theorem SMEZV Scalar Multiplication Equals the Zero Vector

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then if $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$ (or both).

Theorem VAC Vector Addition Cancellation

Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.
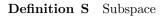
Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and α, $\beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Definition S   Subspace   **143**

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS   Testing Subsets for Subspaces   **144**

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.

2. Whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.

3. Whenever $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

Definition TS   Trivial Subspaces   **145**

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS   Null Space of a Matrix is a Subspace   **146**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC   Linear Combination   **147**

Suppose that V is a vector space.   Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

Definition SS   Span of a Set   **148**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\begin{aligned}
\mathcal{S}p(S) &= \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_t \mathbf{u}_t \,\middle|\, \alpha_i \in \mathbb{C}, \, 1 \le i \le t \right\} \\
&= \left\{ \sum_{i=1}^{t} \alpha_i \mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C}, \, 1 \le i \le t \right\}
\end{aligned}$$

Theorem SSS   Span of a Set is a Subspace   **149**

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\} \subseteq V$, their span, $\mathcal{S}p(S)$, is a subspace.

Theorem RMS   Range of a Matrix is a Subspace   **150**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$, an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is a **trivial relation of linear dependence** on S.

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose V is a vector space and W is a subspace. A subset S of W is a **spanning set** for W if $\mathcal{S}p(S) = W$. In this case, we also say S **spans** W.

Definition B   Basis   **155**

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem SUVB   Standard Unit Vectors are a Basis   **156**

The set of standard unit vectors for $\mathbb{C}^m$, $B = \{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \ldots,\, \mathbf{e}_m\} = \{\, \mathbf{e}_i \mid 1 \leq i \leq m \,\}$ is a basis for the vector space $\mathbb{C}^m$.

Suppose that A is a square matrix. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

Theorem VRRB Vector Representation Relative to a Basis **159**

Suppose that V is a vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ and that $\mathbf{w}$ is a vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Definition D Dimension **160**

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.
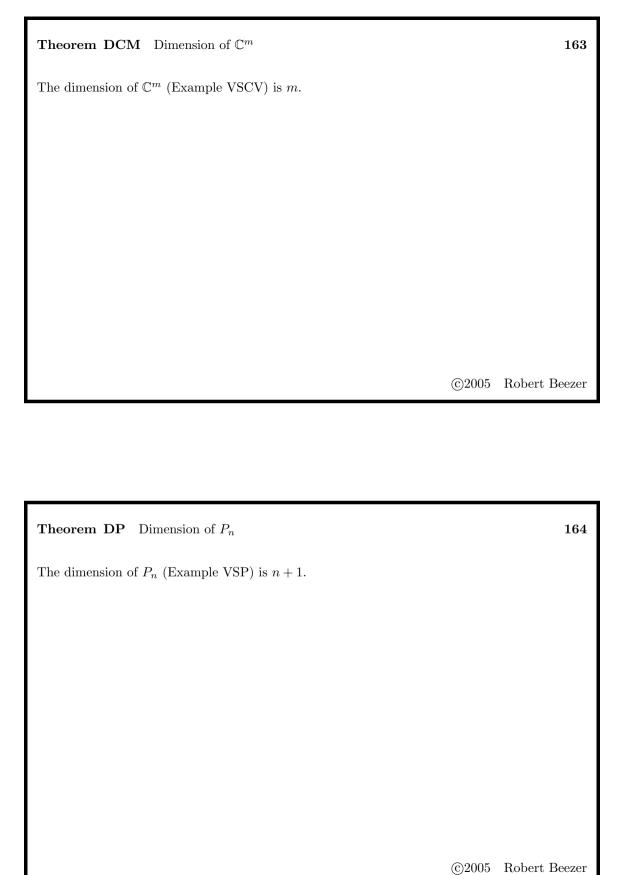
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.
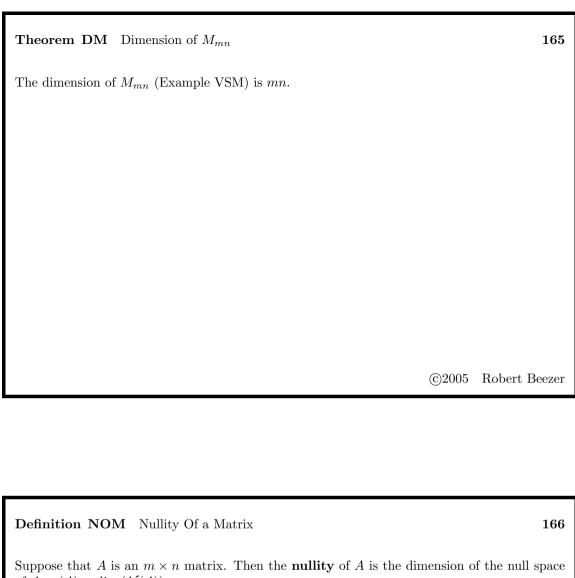
Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM Dimension of $\mathbb{C}^m$ **163**

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP Dimension of P_n **164**

The dimension of P_n (Example VSP) is $n + 1$.

The dimension of M_{mn} (Example VSM) is mn.

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM   Rank Of a Matrix

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Theorem CRN   Computing Rank and Nullity

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Theorem RPNC   Rank Plus Nullity is Columns   **169**

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Theorem RNNSM   Rank and Nullity of a NonSingular Matrix   **170**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. The rank of A is n, $r(A) = n$.

3. The nullity of A is zero, $n(A) = 0$.

Theorem NSME6 NonSingular Matrix Equivalences, Round 6 **171**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

Theorem ELIS Extending Linearly Independent Sets **172**

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \mathcal{S}p(S)$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.

2. If $m < t$, then S does not span V.

3. If $m = t$ and S is linearly independent, then S spans V.

4. If $m = t$ and S spans V, then S is linearly independent.

Theorem EDYES Equal Dimensions Yields Equal Subspaces

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Theorem RMRT   Rank of a Matrix is the Rank of the Transpose

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem COB   Coordinates and Orthonormal Bases

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,
$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A_{ij} is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If $A = \begin{bmatrix} a \end{bmatrix}$ is a 1×1 matrix, then $\det(A) = a$.

If $A = (a_{ij})$ is a matrix of size n with $n \geq 2$, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **minor** for A at location ij is the determinant of the submatrix, $M_{A,ij} = \det(A_{ij})$.

Definition CIM   Cofactor In a Matrix   **181**

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **cofactor** for A at location ij is the signed determinant of the submatrix, $C_{A,ij} = (-1)^{i+j} \det(A_{ij})$.

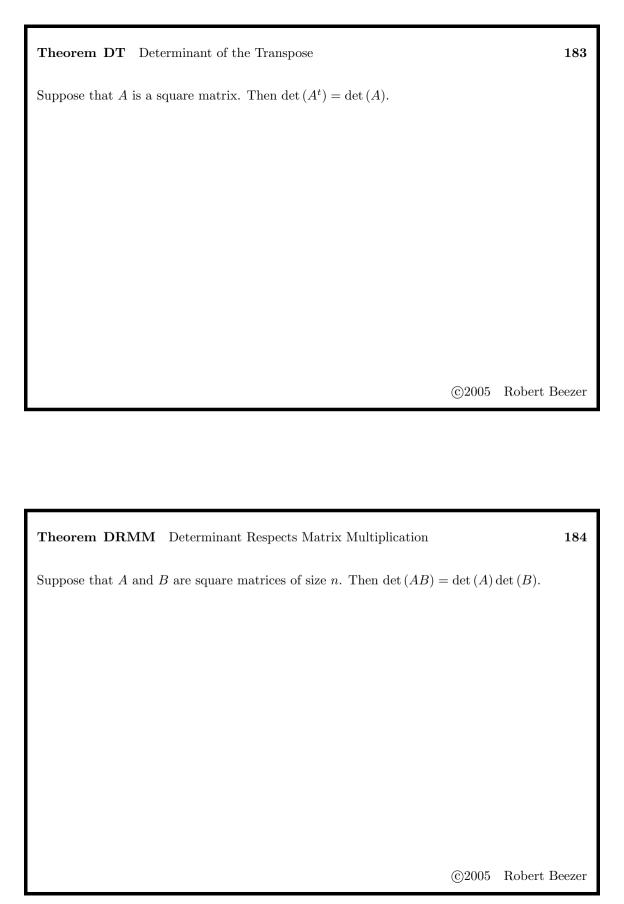**Theorem DERC**   Determinant Expansion about Rows and Columns   **182**

Suppose that $A = (a_{ij})$ is a square matrix of size n. Then

$$\det(A) = a_{i1}C_{A,i1} + a_{i2}C_{A,i2} + a_{i3}C_{A,i3} + \cdots + a_{in}C_{A,in} \qquad 1 \le i \le n$$

which is known as **expansion** about row i, and

$$\det(A) = a_{1j}C_{A,1j} + a_{2j}C_{A,2j} + a_{3j}C_{A,3j} + \cdots + a_{nj}C_{A,nj} \qquad 1 \le j \le n$$

which is known as **expansion** about column j.

Suppose that A is a square matrix. Then $\det\left(A^t\right) = \det\left(A\right)$.

Suppose that A and B are square matrices of size n. Then $\det\left(AB\right) = \det\left(A\right)\det\left(B\right)$.

Theorem SMZD   Singular Matrices have Zero Determinants

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NSME7   NonSingular Matrix Equivalences, Round 7
Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

Definition EEM   Eigenvalues and Eigenvectors of a Matrix

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector from $\mathbb{C}^n$, and λ is a scalar from $\mathbb{C}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.

Theorem EMHE   Every Matrix Has an Eigenvalue

Suppose A is a square matrix. Then A has at least one eigenvalue.

Definition CP   Characteristic Polynomial   **189**

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by
$$p_A(x) = \det(A - xI_n)$$

Theorem EMRCP   Eigenvalues of a Matrix are Roots of Characteristic Polynomials   **190**

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM   Eigenspace of a Matrix   **191**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $E_A(\lambda)$, is the set of all the eigenvectors of A for λ, with the addition of the zero vector.

Theorem EMS   Eigenspace for a Matrix is a Subspace   **192**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS   Eigenspace of a Matrix is a Null Space   **193**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then
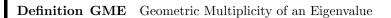
$$E_A\left(\lambda\right) = \mathcal{N}(A - \lambda I_n)$$

Definition AME   Algebraic Multiplicity of an Eigenvalue   **194**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A\left(\lambda\right)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A\left(x\right)$.

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$.

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. A is invertible.

7. The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix **199**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP Eigenvalues Of Matrix Powers **200**

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Theorem EPM   Eigenvalues of the Polynomial of a Matrix   **201**

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

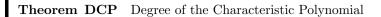**Theorem EIM**   Eigenvalues of the Inverse of a Matrix   **202**

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM   Eigenvalues of the Transpose of a Matrix   **203**

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP   Eigenvalues of Real Matrices come in Conjugate Pairs   **204**

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP   Degree of the Characteristic Polynomial

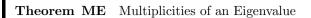Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Theorem NEM   Number of Eigenvalues of a Matrix

Suppose that A is a square matrix of size n with distinct eigenvalues λ_1, λ_2, λ_3, $\ldots$, λ_k. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Theorem ME   Multiplicities of an Eigenvalue

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Theorem MNEM   Maximum Number of Eigenvalues of a Matrix

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM   Similar Matrices Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

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Theorem SER   Similarity is an Equivalence Relation Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)

2. If A is similar to B, then B is similar to A. (Symmetric)

3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE   Similar Matrices have Equal Eigenvalues

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.

Definition DIM   Diagonal Matrix

Suppose that $A = (a_{ij})$ is a square matrix. Then A is a **diagonal matrix** if $a_{ij} = 0$ whenever $i \neq j$.

Definition DZM   Diagonalizable Matrix   **215**

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC   Diagonalization Characterization   **216**

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMLE Diagonalizable Matrices have Large Eigenspaces **217**

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED Distinct Eigenvalues implies Diagonalizable **218**

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Definition LT   Linear Transformation   **220**

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T\left(\mathbf{u}_1 + \mathbf{u}_2\right) = T\left(\mathbf{u}_1\right) + T\left(\mathbf{u}_2\right)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$

2. $T\left(\alpha\mathbf{u}\right) = \alpha T\left(\mathbf{u}\right)$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

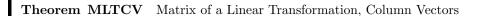**Theorem LTTZZ**   Linear Transformations Take Zero to Zero   **221**

Suppose $T: U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Theorem MBLT   Matrices Build Linear Transformations   **222**

Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV   Matrix of a Linear Transformation, Column Vectors   **223**

Suppose that $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC   Linear Transformations and Linear Combinations   **224**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, ..., a_t are scalars from $\mathbb{C}$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \cdots + a_t T(\mathbf{u}_t)$$

Theorem LTDB   Linear Transformation Defined on a Basis   **225**

Suppose that $T \colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let $a_1, a_2, a_3, \ldots, a_n$ be the scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \cdots + a_n T(\mathbf{u}_n)$$

Definition PI   Pre-Image   **226**

Suppose that $T \colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{\,\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\,\}$$

Definition LTA   Linear Transformation Addition   **227**

Suppose that $T \colon U \mapsto V$ and $S \colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S \colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT   Sum of Linear Transformations is a Linear Transformation   **228**

Suppose that $T \colon U \mapsto V$ and $S \colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S \colon U \mapsto V$ is a linear transformation.

Definition LTSM   Linear Transformation Scalar Multiplication **229**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT   Multiple of a Linear Transformation is a Linear Transformation **230**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \mapsto V$ is a linear transformation.

Theorem VSLT   Vector Space of Linear Transformations   **231**

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\text{LT}(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC   Linear Transformation Composition   **232**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT   Composition of Linear Transformations is a Linear Transformation **233**

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are linear transformations. Then $(S \circ T) \colon U \mapsto W$ is a linear transformation.

Definition ILT   Injective Linear Transformation   **234**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition KLT   Kernel of a Linear Transformation   **235**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set $\mathcal{K}(T) = \{\,\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0}\,\}$

Theorem KLTS   Kernel of a Linear Transformation is a Subspace   **236**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Theorem KPI   Kernel and Pre-Image   **237**

Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then
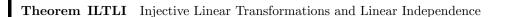$$T^{-1}(\mathbf{v}) = \left\{\, \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T) \,\right\} = \mathbf{u} + \mathcal{K}(T)$$

Theorem KILT   Kernel of an Injective Linear Transformation   **238**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.
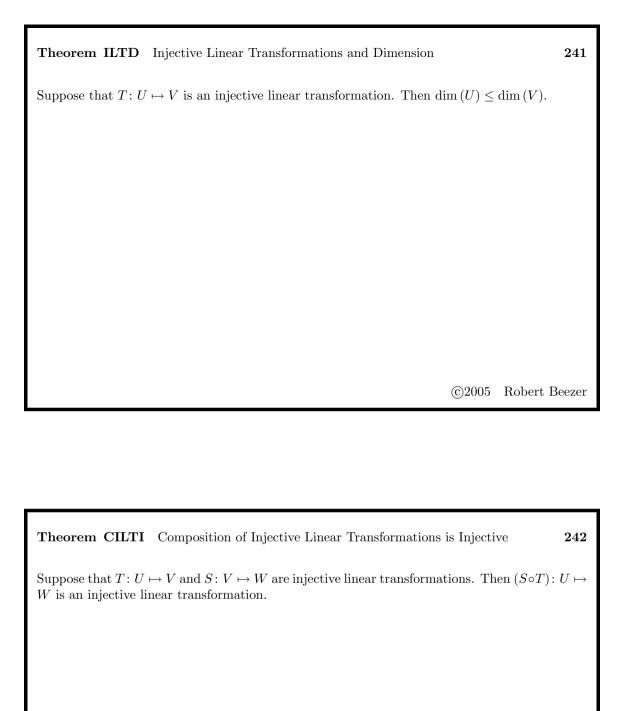
Theorem ILTLI Injective Linear Transformations and Linear Independence **239**

Suppose that $T \colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases **240**

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Theorem ILTD   Injective Linear Transformations and Dimension   **241**

Suppose that $T \colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Theorem CILTI   Composition of Injective Linear Transformations is Injective   **242**

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are injective linear transformations. Then $(S \circ T) \colon U \mapsto W$ is an injective linear transformation.

Definition SLT   Surjective Linear Transformation

Suppose $T : U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.
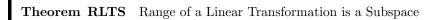

Definition RLT   Range of a Linear Transformation

Suppose $T : U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{\, T(\mathbf{u}) \,|\, \mathbf{u} \in U \,\}$$

Theorem RLTS   Range of a Linear Transformation is a Subspace   **245**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT   Range of a Surjective Linear Transformation   **246**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT   Spanning Set for Range of a Linear Transformation   **247**

Suppose that $T \colon U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.

Theorem RPI   Range and Pre-Image   **248**

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a spanning set for V.

Suppose that $T \colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS   Composition of Surjective Linear Transformations is Surjective   **251**

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T) \colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT   Identity Linear Transformation   **252**

The **identity linear transformation** on the vector space W is defined as

$$I_W \colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

Definition IVLT Invertible Linear Transformations

Suppose that $T\colon U \mapsto V$ is a linear transformation. If there is a function $S\colon V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}\colon V \mapsto U$ is a linear transformation.

Theorem IILT   Inverse of an Invertible Linear Transformation   **255**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Theorem ILTIS   Invertible Linear Transformations are Injective and Surjective   **256**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT   Composition of Invertible Linear Transformations   **257**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T)\colon U \mapsto W$ is an invertible linear transformation.

Theorem ICLT   Inverse of a Composition of Linear Transformations   **258**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Definition ROLT   Rank Of a Linear Transformation

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the **rank** of T, $r\,(T)$, is the dimension of the range of T,
$$r\,(T) = \dim\left(\mathcal{R}(T)\right)$$

Definition NOLT   Nullity Of a Linear Transformation

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n\,(T)$, is the dimension of the kernel of T,
$$n\,(T) = \dim\left(\mathcal{K}(T)\right)$$

Theorem ROSLT   Rank Of a Surjective Linear Transformation   **263**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT   Nullity Of an Injective Linear Transformation   **264**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

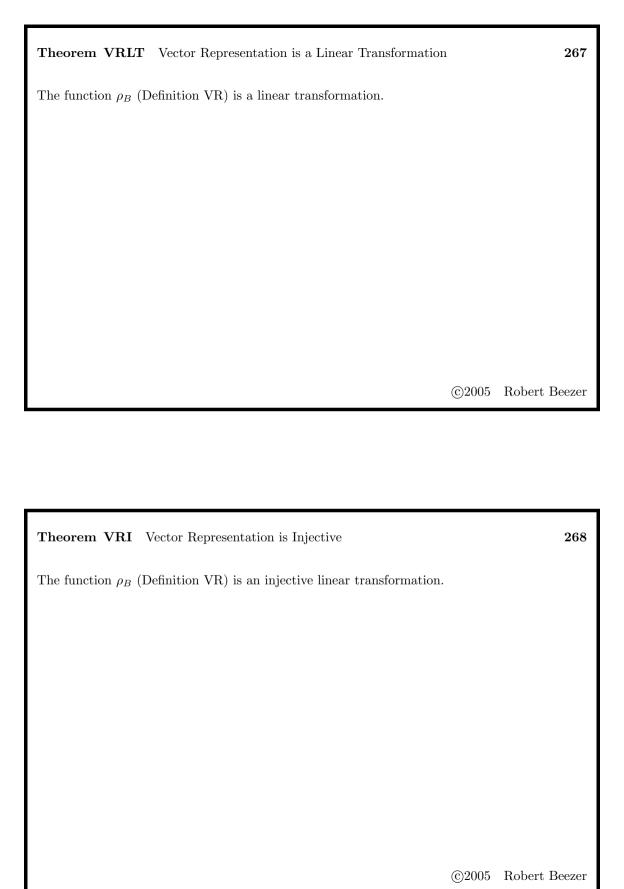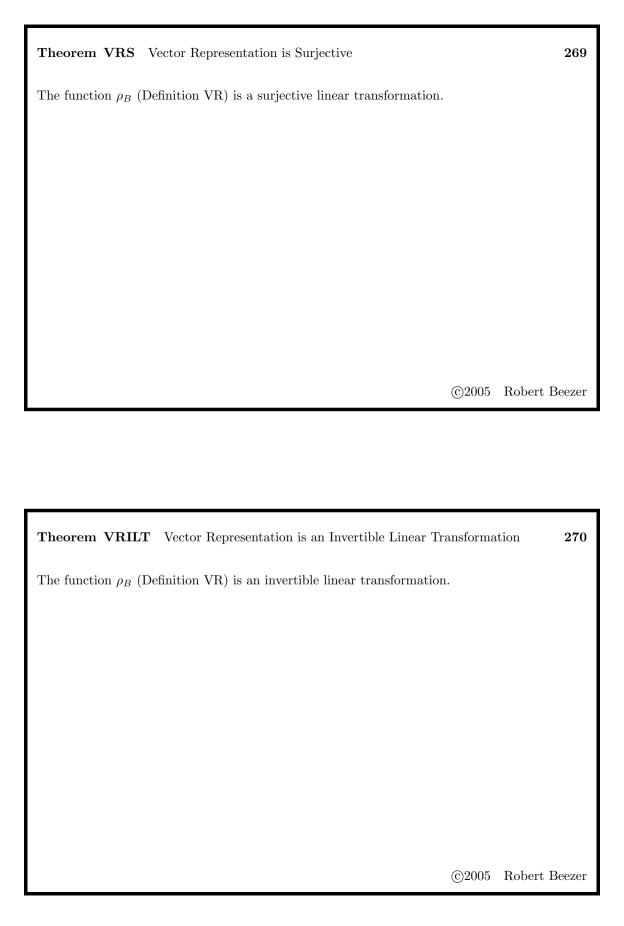$$r\left(T\right) + n\left(T\right) = \dim\left(U\right)$$

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$. Define a function $\rho_B\colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ so that

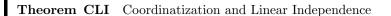$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$\rho_B\left(\mathbf{w}\right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Theorem VRLT Vector Representation is a Linear Transformation **267**

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI Vector Representation is Injective **268**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS   Vector Representation is Surjective   **269**

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT   Vector Representation is an Invertible Linear Transformation   **270**

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS   Characterization of Finite Dimensional Vector Spaces   **271**

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS   Isomorphism of Finite Dimensional Vector Spaces   **272**

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI   Coordinatization and Linear Independence   **273**

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B\left(\mathbf{u}_1\right), \rho_B\left(\mathbf{u}_2\right), \rho_B\left(\mathbf{u}_3\right), \ldots, \rho_B\left(\mathbf{u}_k\right)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS   Coordinatization and Spanning Sets   **274**

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \mathcal{Sp}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\})$ if and only if $\rho_B\left(\mathbf{u}\right) \in \mathcal{Sp}(\{\rho_B\left(\mathbf{u}_1\right), \rho_B\left(\mathbf{u}_2\right), \rho_B\left(\mathbf{u}_3\right), \ldots, \rho_B\left(\mathbf{u}_k\right)\})$.

Definition MR   Matrix Representation Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. The the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = [\,\rho_C\left(T\left(\mathbf{u}_1\right)\right)|\ \rho_C\left(T\left(\mathbf{u}_2\right)\right)|\ \rho_C\left(T\left(\mathbf{u}_3\right)\right)|\ldots|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\,]$$

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Theorem FTMR   Fundamental Theorem of Matrix Representation Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

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Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^{T} + M_{B,C}^{S}$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^{T}$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations **279**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^{S} M_{B,C}^{T}$$

Theorem KNSI Kernel and Null Space Isomorphism   **280**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^{T}$,

$$\mathcal{K}(T) \cong \mathcal{N}\!\left(M_{B,C}^{T}\right)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}\left(M_{B,C}^T\right)$$

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

Definition EELT   Eigenvalue and Eigenvector of a Linear Transformation   **283**

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM   Change-of-Basis Matrix   **284**

Suppose that V is a vector space, and $I_V\colon V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M_{B,C}^{I_V} \\
&= [\rho_C(I_V(\mathbf{v}_1))|\ \rho_C(I_V(\mathbf{v}_2))|\ \rho_C(I_V(\mathbf{v}_3))|\ldots|\rho_C(I_V(\mathbf{v}_n))] \\
&= [\rho_C(\mathbf{v}_1)|\ \rho_C(\mathbf{v}_2)|\ \rho_C(\mathbf{v}_3)|\ldots|\rho_C(\mathbf{v}_n)]
\end{aligned}
$$

Theorem CB   Change-of-Basis   **285**

Suppose that $\mathbf{u}$ is a vector in the vector space V and B and C are bases of V. Then

$$C_{B,C}\rho_B\left(\mathbf{v}\right) = \rho_C\left(\mathbf{v}\right)$$

Theorem ICBM   Inverse of Change-of-Basis Matrix   **286**

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB   Matrix Representation and Change of Basis   **287**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then
$$M_{B,D}^{T} = C_{E,D} M_{C,E}^{T} C_{B,C}$$

Theorem SCB   Similarity and Change of Basis   **288**

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then
$$M_{B,B}^{T} = C_{B,C}^{-1} M_{C,C}^{T} C_{B,C}$$

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B\left(\mathbf{v}\right)$ is an eigenvector of $M_{B,B}^{T}$ for the eigenvalue λ.