Flash Cards

to accompany

A First Course in Linear Algebra

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Definition SSLE System of Simultaneous Linear Equations **1**

A **system of simultaneous linear equations** is a collection of m equations in the variable quantities x_1, x_2, $x_3, \ldots, x_n$ of the form,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Definition ES Equivalent Systems **2**

Two systems of simultaneous linear equations are **equivalent** if their solution sets are equal.

Definition EO Equation Operations **3**

Given a system of simultaneous linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

1. Swap the locations of two equations in the list.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Theorem EOPSS Equation Operations Preserve Solution Sets **4**

Suppose we apply one of the three equation operations of Definition EO to the system of simultaneous linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
\vdots &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

Then the original system and the transformed system are equivalent systems.

Definition M Matrix

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns.

Definition AM Augmented Matrix

Suppose we have a system of m equations in the n variables x_1, x_2, $x_3, \ldots$, x_n written as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\
a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\
\vdots & & & & & \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & b_m
\end{bmatrix}$$

Definition RO Row Operations **7**

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entry in the same column of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

Definition REM Row-Equivalent Matrices **8**

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Theorem REMES Row-Equivalent Matrices represent Equivalent Systems **9**

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

Definition RREF Reduced Row-Echelon Form **10**

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero is below any row containing a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $i < s$, then $j < t$.

Definition ZRM Zero Row of a Matrix **11**

A row of a matrix where every entry is zero is called a **zero row**.

Definition LO Leading Ones **12**

For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a **leading 1**.

Definition PC Pivot Columns **13**

For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

Theorem REMEF Row-Equivalent Matrix in Echelon Form **14**

Suppose A is a matrix. Then there is a (unique!) matrix B so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

Definition CS Consistent System **15**

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Definition IDV Independent and Dependent Variables **16**

Suppose A is the augmented matrix of a system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the number of a column of B that contains the leading 1 for some row, and it is not the last column. Then the variable j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem RCLS Recognizing Consistency of a Linear System **17**

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n+1$ of B.

Theorem ICRN Inconsistent Systems, r and n **18**

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Theorem CSRN Consistent Systems, r and n **19**

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Theorem FVCS Free Variables for Consistent Systems **20**

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS Possible Solution Sets for Linear Systems **21**

A simultaneous system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions **22**

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Definition HS Homogeneous System

A system of linear equations is **homogeneous** if each equation has a 0 for its constant term. Such a system then has the form,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Theorem HSC Homogeneous Systems are Consistent

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Definition TSHSE Trivial Solution to Homogeneous Systems of Equations **25**

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ is called the **trivial solution**.

Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions **26**

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Definition CV Column Vector **27**

A **column vector** of **size** m is an ordered list of m numbers, which is written vertically, in order from top to bottom. At times, we will refer to a column vector as simply a **vector**.

Definition ZV Zero Vector **28**

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Definition CM Coefficient Matrix **29**

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \dots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \dots & a_{3n}\\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants **30**

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1\\ b_2\\ b_3\\ \vdots\\ b_m \end{bmatrix}$$

Definition SV Solution Vector **31**

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the **solution vector** is the column vector of size m

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix}$$

Definition NSM Null Space of a Matrix **32**

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition SQM Square Matrix **33**

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM Nonsingular Matrix **34**

Suppose A is a square matrix. And suppose the homogeneous linear system of equations $\mathcal{LS}(A,\,\mathbf{0})$ has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Definition IM Identity Matrix **35**

The $m \times m$ **identity matrix**, $I_m = (a_{ij})$ has $a_{ij} = 1$ whenever $i = j$, and $a_{ij} = 0$ whenever $i \neq j$.

Theorem NSRRI NonSingular matrices Row Reduce to the Identity matrix **36**

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NSTNS NonSingular matrices have Trivial Null Spaces **37**

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the trivial solution to the system $\mathcal{LS}(A, \mathbf{0})$, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Theorem NSMUS NonSingular Matrices and Unique Solutions **38**

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Theorem NSME1 NonSingular Matrix Equivalences, Round 1 **39**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the trivial solution, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

Definition VSCV Vector Space of Column Vectors **40**

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE Column Vector Equality

The vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that $u_i = v_i$ for all $1 \leq i \leq m$.

Definition CVA Column Vector Addition

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_m + v_m \end{bmatrix}.$$

Definition CVSM Column Vector Scalar Multiplication

Given the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of $\mathbf{u}$ by α is

$$\alpha\mathbf{u} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \\ \vdots \\ \alpha u_m \end{bmatrix}.$$

Theorem VSPCV Vector Space Properties of Column Vectors

Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC Additive Closure, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- **SCC Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.
- **CC Commutativity, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **AAC Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **ZC Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- **AIC Additive Inverses, Column Vectors** For each vector $\mathbf{u} \in \mathbb{C}^m$, there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- **SMAC Scalar Multiplication Associativity, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- **DVAC Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- **DSAC Distributivity across Scalar Addition, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- **OC One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Definition LCCV Linear Combination of Column Vectors **45**

Given n vectors $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n$ and n scalars $\alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations **46**

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n$. Then $\mathbf{x} = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\\ \alpha_n\end{bmatrix}$ is a solution to the linear system of equations $\mathcal{LS}(A,\, \mathbf{b})$ if and only if

$$\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3 + \cdots + \alpha_n\mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems **47**

Suppose that $[A \,|\, \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A,\,\mathbf{b})$ of m equations in n variables. Denote the vector of variables as $\mathbf{x} = (x_i)$. Let $B = (b_{ij})$ be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's having indices $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r},\, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$. Define vectors $\mathbf{c} = (c_i)$, $\mathbf{u}_j = (u_{ij})$, $1 \leq j \leq n-r$ of size n by

$$c_i = \begin{cases} 0 & \text{if } i \in F \\ b_{k,n+1} & \text{if } i \in D,\ i = d_k \end{cases}$$

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Then the set of solutions to the system of equations represented by the vector equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}$$

is equal to the set of solutions of $\mathcal{LS}(A,\,\mathbf{b})$.

Theorem RREFU Reduced Row-Echelon Form is Unique **48**

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Definition SSCV Span of a Set of Column Vectors **49**

Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t$. Symbolically,

$$\begin{aligned}\mathcal{S}p(S) &= \{\, \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \leq i \leq t\} \\ &= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \leq i \leq t \right\}\end{aligned}$$

Theorem SSNS Spanning Sets for Null Spaces **50**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ and $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{u}_j = (u_{ij})$, $1 \leq j \leq n - r$ of size n as

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F,\, i = f_j \\ 0 & \text{if } i \in F,\, i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D,\, i = d_k \end{cases}.$$

Then the null space of A is given by

$$\mathcal{N}(A) = \mathcal{S}p(\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_{n-r}\}).$$

Definition RLDCV Relation of Linear Dependence for Column Vectors **51**

Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LICV Linear Independence of Column Vectors **52**

The set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems **53**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$ has a unique solution.

Theorem LIVRN Linearly Independent Vectors, r and n **54**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Theorem MVSLD More Vectors than Size implies Linear Dependence **55**

Suppose that $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Theorem DLDS Dependency in Linearly Dependent Sets **56**

Suppose that $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \leq t \leq n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_{t-1},\,\mathbf{u}_{t+1},\,\ldots,\,\mathbf{u}_n$.

Theorem NSLIC NonSingular matrices have Linearly Independent Columns **57**

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NSME2 NonSingular Matrix Equivalences, Round 2 **58**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces **59**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n-r$ vectors $\mathbf{z}_j = (z_{ij})$, $1 \leq j \leq n-r$ of size n as

$$z_{ij} = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Define the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \mathcal{S}p(S)$.
2. S is a linearly independent set.

Definition CCCV Complex Conjugate of a Column Vector **60**

Suppose that

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

is a vector from $\mathbb{C}^m$. Then the conjugate of the vector is defined as

$$\overline{\mathbf{u}} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \\ \vdots \\ \overline{u}_m \end{bmatrix}$$

Theorem CRVA Conjugation Respects Vector Addition **61**

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x}+\mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Theorem CRSM Conjugation Respects Vector Scalar Multiplication **62**

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP Inner Product **63**

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\, \mathbf{v} \rangle = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3} + \cdots + u_m\overline{v_m} = \sum_{i=1}^{m} u_i\overline{v_i}$$

Theorem IPVA Inner Product and Vector Addition **64**

Suppose $\mathbf{u}\,\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1. $\langle \mathbf{u}+\mathbf{v},\, \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{w} \rangle + \langle \mathbf{v},\, \mathbf{w} \rangle$
2. $\langle \mathbf{u},\, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle + \langle \mathbf{u},\, \mathbf{w} \rangle$

Theorem IPSM Inner Product and Scalar Multiplication **65**

Suppose $\mathbf{u},\,\mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

1. $\langle \alpha\mathbf{u},\, \mathbf{v}\rangle = \alpha \langle \mathbf{u},\, \mathbf{v}\rangle$
2. $\langle \mathbf{u},\, \alpha\mathbf{v}\rangle = \overline{\alpha} \langle \mathbf{u},\, \mathbf{v}\rangle$

Theorem IPAC Inner Product is Anti-Commutative **66**

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{v}\rangle = \overline{\langle \mathbf{v},\, \mathbf{u}\rangle}$.

Definition NV Norm of a Vector **67**

The **norm** of the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

is the scalar quantity in $\mathbb{C}^m$

$$\|\mathbf{u}\| = \sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2}$$

Theorem IPN Inner Products and Norms **68**

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle\mathbf{u},\, \mathbf{u}\rangle$.

Theorem PIP Positive Inner Products **69**

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{u}\rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV Orthogonal Vectors **70**

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u},\, \mathbf{v}\rangle = 0$.

Definition OSV Orthogonal Set of Vectors **71**

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i,\, \mathbf{u}_j\rangle = 0$ whenever $i \neq j$.

Theorem OSLI Orthogonal Sets are Linearly Independent **72**

Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSPCV Gram-Schmidt Procedure, Column Vectors **73**

Suppose that $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2\rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_3\rangle}{\langle \mathbf{u}_3,\, \mathbf{u}_3\rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i,\, \mathbf{u}_{i-1}\rangle}{\langle \mathbf{u}_{i-1},\, \mathbf{u}_{i-1}\rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\mathcal{S}p(T) = \mathcal{S}p(S)$.

Definition ONS OrthoNormal Set **74**

Suppose $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

Definition VSM Vector Space of $m \times n$ Matrices **75**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality **76**

The $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

are **equal**, written $A = B$ provided $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition MA Matrix Addition **77**

Given the $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

define the **sum** of A and B to be $A + B = C = (c_{ij})$, where

$$c_{ij} = a_{ij} + b_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Definition MSM Matrix Scalar Multiplication **78**

Given the $m \times n$ matrix $A = (a_{ij})$ and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A by α is the matrix $\alpha A = C = (c_{ij})$, where

$$c_{ij} = \alpha a_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Theorem VSPM Vector Space Properties of Matrices **79**

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM Additive Closure, Matrices** If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- **SCM Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- **CM Commutativity, Matrices** If $A, B \in M_{mn}$, then $A + B = B + A$.
- **AAM Additive Associativity, Matrices** If $A, B, C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.
- **ZM Zero Vector, Matrices** There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- **AIM Additive Inverses, Matrices** For each matrix $A \in M_{mn}$, there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- **SMAM Scalar Multiplication Associativity, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.
- **DMAM Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.
- **DSAM Distributivity across Scalar Addition, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- **OM One, Matrices** If $A \in M_{mn}$, then $1A = A$.

Definition ZM Zero Matrix **80**

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m\times n} = (z_{ij})$ and defined by $z_{ij} = 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. Or, equivalently, $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM Transpose of a Matrix **81**

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,, \quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

Definition SYM Symmetric Matrix **82**

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS Symmetric Matrices are Square **83**

Suppose that A is a symmetric matrix. Then A is square.

Theorem TMA Transpose and Matrix Addition **84**

Suppose that A and B are $m \times n$ matrices. Then $(A+B)^t = A^t + B^t$.

Theorem TMSM Transpose and Matrix Scalar Multiplication **85**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Theorem TT Transpose of a Transpose **86**

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Definition CCM Complex Conjugate of a Matrix **87**

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

Theorem CRMA Conjugation Respects Matrix Addition **88**

Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication **89**

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Definition RM Range of a Matrix **90**

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n\}$. Then the **range** of A, written $\mathcal{R}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{R}(A) = \mathcal{S}p(\{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n\})$$

Theorem RCS Range and Consistent Systems **91**

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{R}(A)$ if and only if $\mathcal{LS}(A,\, \mathbf{b})$ is consistent.

Theorem BROC Basis of the Range with Original Columns **92**

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_3,\, \ldots,\, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ be the set of column indices where B has leading 1's. Let $S = \{\mathbf{A}_{d_1},\, \mathbf{A}_{d_2},\, \mathbf{A}_{d_3},\, \ldots,\, \mathbf{A}_{d_r}\}$. Then

1. $\mathcal{R}(A) = \mathcal{S}p(S)$.
2. S is a linearly independent set.

Theorem RNS Range as a Null Space **93**

Suppose that A is an $m \times n$ matrix. Create the $m \times (n+m)$ matrix M by placing the $m \times m$ identity matrix I_m to the right of the matrix A. Symbolically, $M = [A \mid I_m]$. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Suppose there are r leading 1's of N in the first n columns. If $r = m$, then $\mathcal{R}(A) = \mathbb{C}^m$. Otherwise, $r < m$ and let K be the $(m-r) \times m$ matrix formed from the entries of N in the last $m-r$ rows and last m columns. Then

1. K is in reduced row-echelon form.
2. K has no zero rows, or equivalently, K has $m-r$ leading 1's.
3. $\mathcal{R}(A) = \mathcal{N}(K)$.

Theorem RNSM Range of a NonSingular Matrix **94**

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{R}(A) = \mathbb{C}^n$.

Theorem NSME3 NonSingular Matrix Equivalences, Round 3 **95**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

Definition RSM Row Space of a Matrix **96**

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{RS}(A)$, is the range of A^t, i.e. $\mathcal{RS}(A) = \mathcal{R}(A^t)$.

Theorem REMRS Row-Equivalent Matrices have equal Row Spaces **97**

Suppose A and B are row-equivalent matrices. Then $\mathcal{RS}(A) = \mathcal{RS}(B)$.

Theorem BRS Basis for the Row Space **98**

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{RS}(A) = \mathcal{S}p(S)$.
2. S is a linearly independent set.

Theorem RMRST Range of a Matrix is Row Space of Transpose **99**

Suppose A is a matrix. Then $\mathcal{R}(A) = \mathcal{RS}(A^t)$.

Definition MVP Matrix-Vector Product **100**

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\ldots$, $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is

$$A\mathbf{u} = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\ldots|\mathbf{A}_n] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{A}_1 + u_2\mathbf{A}_2 + u_3\mathbf{A}_3 + \cdots + u_n\mathbf{A}_n$$

Theorem SLEMM Systems of Linear Equations as Matrix Multiplication **101**

Solutions to the linear system $\mathcal{LS}(A,\,\mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Definition MM Matrix Multiplication **102**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1,\,\mathbf{B}_2,\,\mathbf{B}_3,\,\ldots,\,\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Theorem EMP Entries of Matrix Products **103**

Suppose $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix. Then the entries of $AB = C = (c_{ij})$ are given by

$$[C]_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix **104**

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
2. $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

Theorem MMIM Matrix Multiplication and Identity Matrix **105**

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Theorem MMDAA Matrix Multiplication Distributes Across Addition **106**

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication **107**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Theorem MMA Matrix Multiplication is Associative **108**

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

Theorem MMIP Matrix Multiplication and Inner Products **109**

If we consider the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$

Theorem MMCC Matrix Multiplication and Complex Conjugation **110**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT Matrix Multiplication and Transposes **111**

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Theorem PSPHS Particular Solution Plus Homogeneous Solutions **112**

Suppose that $\mathbf{z}$ is one solution to the linear system of equations $\mathcal{LS}(A,\, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\, b)$ if and only if $\mathbf{y} = \mathbf{z} + \mathbf{w}$ for some vector $\mathbf{w} \in N(A)$.

Definition MI Matrix Inverse **113**

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Definition SUV Standard Unit Vectors **114**

Let $\mathbf{e}_i \in \mathbb{C}^m$ denote the column vector that is column i of the $m \times m$ identity matrix I_m. Then the set

$$\{\mathbf{e}_1,\,\mathbf{e}_2,\,\mathbf{e}_3,\,\ldots,\,\mathbf{e}_m\} = \{\,\mathbf{e}_i \mid 1 \leq i \leq m\}$$

is the set of **standard unit vectors** in $\mathbb{C}^m$.

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Theorem TTMI Two-by-Two Matrix Inverse **115**

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem CINSM Computing the Inverse of a NonSingular Matrix **116**

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let B be the matrix formed from the final n columns of N. Then $AB = I_n$.

Theorem MIU Matrix Inverse is Unique **117**

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Theorem SS Socks and Shoes **118**

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI Matrix Inverse of a Matrix Inverse **119**

Suppose A is an invertible matrix. Then $(A^{-1})^{-1} = A$ and A^{-1} is invertible.

Theorem MIT Matrix Inverse of a Transpose **120**

Suppose A is an invertible matrix. Then $(A^t)^{-1} = (A^{-1})^t$ and A^t is invertible.

Theorem MISM Matrix Inverse of a Scalar Multiple **121**

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Theorem PWSMS Product With a Singular Matrix is Singular **122**

Suppose that A or B are matrices of size n, and one, or both, is singular. Then their product, AB, is singular.

Theorem OSIS One-Sided Inverse is Sufficient **123**

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Theorem NSI NonSingularity is Invertibility **124**

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NSME4 NonSingular Matrix Equivalences, Round 4 **125**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.
7. A is invertible.

Theorem SNSCM Solution with NonSingular Coefficient Matrix **126**

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A,\, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Definition OM Orthogonal Matrices **127**

Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^t Q = I_n$. Then we say Q is **orthogonal**.

Theorem OMI Orthogonal Matrices are Invertible **128**

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Theorem COMOS Columns of Orthogonal Matrices are Orthonormal Sets **129**

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set.

Theorem OMPIP Orthogonal Matrices Preserve Inner Products **130**

Suppose that Q is an orthogonal matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\,Q\mathbf{v}\rangle = \langle \mathbf{u},\,\mathbf{v}\rangle \qquad \text{and} \qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|$$

Definition A Adjoint **131**

If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$.

Definition HM Hermitian Matrix **132**

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = \left(\overline{A}\right)^t$

Definition VS Vector Space **133**

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten requirements (better known as "axioms") are met.

1. **AC Additive Closure** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
2. **SC Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.
3. **C Commutativity** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4. **AA Additive Associativity** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
5. **Z Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
6. **AI Additive Inverses** For each vector $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
7. **SMA Scalar Multiplication Associativity** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
8. **DVA Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
9. **DSA Distributivity across Scalar Addition** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
10. **O One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU Zero Vector is Unique **134**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU Additive Inverses are Unique **135**

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Theorem ZSSM Zero Scalar in Scalar Multiplication **136**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Theorem ZVSM Zero Vector in Scalar Multiplication **137**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM Additive Inverses from Scalar Multiplication **138**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Theorem SMEZV Scalar Multiplication Equals the Zero Vector **139**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then if $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$ (or both).

Theorem VAC Vector Addition Cancellation **140**

Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CSSM Canceling Scalars in Scalar Multiplication **141**

Suppose V is a vector space, $\mathbf{u}, \mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CVSM Canceling Vectors in Scalar Multiplication **142**

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and $\alpha, \beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Definition S Subspace **143**

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Theorem TSS Testing Subsets for Subspaces **144**

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.
2. Whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
3. Whenever $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Definition TS Trivial Subspaces **145**

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS Null Space of a Matrix is a Subspace **146**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC Linear Combination **147**

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Definition SS Span of a Set **148**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\begin{aligned}\mathcal{S}p(S) &= \{\, \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \leq i \leq t\} \\ &= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \;\middle|\; \alpha_i \in \mathbb{C},\, 1 \leq i \leq t \right\}\end{aligned}$$

Theorem SSS Span of a Set is a Subspace **149**

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\} \subseteq V$, their span, $\mathcal{S}p(S)$, is a subspace.

Theorem RMS Range of a Matrix is a Subspace **150**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^m$.

Theorem RSMS Row Space of a Matrix is a Subspace **151**

Suppose that A is an $m \times n$ matrix. Then $\mathcal{RS}(A)$ is a subspace of $\mathbb{C}^n$.

Definition RLD Relation of Linear Dependence **152**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LI Linear Independence **153**

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Definition TSS To Span a Subspace **154**

Suppose V is a vector space and W is a subspace. A subset S of W is a **spanning set** for W if $\mathcal{S}p(S) = W$. In this case, we also say S **spans** W.

Definition B Basis **155**

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

Theorem SUVB Standard Unit Vectors are a Basis **156**

The set of standard unit vectors for $\mathbb{C}^m$, $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\, \mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Theorem CNSMB Columns of NonSingular Matrix are a Basis **157**

Suppose that A is a square matrix. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Theorem NSME5 NonSingular Matrix Equivalences, Round 5 **158**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.
7. A is invertible.
8. The columns of A are a basis for $\mathbb{C}^n$.

Theorem VRRB Vector Representation Relative to a Basis **159**

Suppose that V is a vector space with basis $B = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$ and that $\mathbf{w}$ is a vector in V. Then there exist *unique* scalars $a_1,\,a_2,\,a_3,\,\ldots,\,a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Definition D Dimension **160**

Suppose that V is a vector space and $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim\left(V\right) = t$. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD Spanning Sets and Linear Dependence **161**

Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t+1$ or more vectors from V is linearly dependent.

Theorem BIS Bases have Identical Sizes **162**

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM Dimension of $\mathbb{C}^m$ **163**

The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Theorem DP Dimension of P_n **164**

The dimension of P_n (Example VSP) is $n+1$.

Theorem DM Dimension of M_{mn} **165**

The dimension of M_{mn} (Example VSM) is mn.

Definition NOM Nullity Of a Matrix **166**

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM Rank Of a Matrix **167**

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the range of A, $r(A) = \dim(\mathcal{R}(A))$.

Theorem CRN Computing Rank and Nullity **168**

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Theorem RPNC Rank Plus Nullity is Columns **169**

Suppose that A is an $m \times n$ matrix. Then $r\left(A\right) + n\left(A\right) = n$.

Theorem RNNSM Rank and Nullity of a NonSingular Matrix **170**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. The rank of A is n, $r\left(A\right) = n$.
3. The nullity of A is zero, $n\left(A\right) = 0$.

Theorem NSME6 NonSingular Matrix Equivalences, Round 6 **171**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.
7. A is invertible.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.

Theorem ELIS Extending Linearly Independent Sets **172**

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \mathcal{S}p(S)$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks **173**

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.
2. If $m < t$, then S does not span V.
3. If $m = t$ and S is linearly independent, then S spans V.
4. If $m = t$ and S spans V, then S is linearly independent.

Theorem EDYES Equal Dimensions Yields Equal Subspaces **174**

Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Theorem RMRT Rank of a Matrix is the Rank of the Transpose **175**

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem COB Coordinates and Orthonormal Bases **176**

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Definition SM SubMatrix **177**

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A_{ij} is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM Determinant **178**

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If $A = \begin{bmatrix} a \end{bmatrix}$ is a 1×1 matrix, then $\det(A) = a$.

If $A = (a_{ij})$ is a matrix of size n with $n \geq 2$, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

Theorem DMST Determinant of Matrices of Size Two **179**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Definition MIM Minor In a Matrix **180**

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **minor** for A at location $i\,j$ is the determinant of the submatrix, $M_{A,ij} = \det(A_{ij})$.

Definition CIM Cofactor In a Matrix **181**

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **cofactor** for A at location $i\,j$ is the signed determinant of the submatrix, $C_{A,ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem DERC Determinant Expansion about Rows and Columns **182**

Suppose that $A = (a_{ij})$ is a square matrix of size n. Then

$$\det(A) = a_{i1}C_{A,i1} + a_{i2}C_{A,i2} + a_{i3}C_{A,i3} + \cdots + a_{in}C_{A,in} \qquad 1 \leq i \leq n$$

which is known as **expansion** about row i, and

$$\det(A) = a_{1j}C_{A,1j} + a_{2j}C_{A,2j} + a_{3j}C_{A,3j} + \cdots + a_{nj}C_{A,nj} \qquad 1 \leq j \leq n$$

which is known as **expansion** about column j.

Theorem DT Determinant of the Transpose **183**

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Theorem DRMM Determinant Respects Matrix Multiplication **184**

Suppose that A and B are square matrices of size n. Then $\det(AB) = \det(A)\det(B)$.

Theorem SMZD Singular Matrices have Zero Determinants **185**

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NSME7 NonSingular Matrix Equivalences, Round 7 **186**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.
7. A is invertible.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.

Definition EEM Eigenvalues and Eigenvectors of a Matrix **187**

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector from $\mathbb{C}^n$, and λ is a scalar from $\mathbb{C}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.

Theorem EMHE Every Matrix Has an Eigenvalue **188**

Suppose A is a square matrix. Then A has at least one eigenvalue.

Definition CP Characteristic Polynomial **189**

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials **190**

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM Eigenspace of a Matrix **191**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $E_A(\lambda)$, is the set of all the eigenvectors of A for λ, with the addition of the zero vector.

Theorem EMS Eigenspace for a Matrix is a Subspace **192**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS Eigenspace of a Matrix is a Null Space **193**

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Definition AME Algebraic Multiplicity of an Eigenvalue **194**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME Geometric Multiplicity of an Eigenvalue **195**

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent **196**

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE Singular Matrices have Zero Eigenvalues **197**

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NSME8 NonSingular Matrix Equivalences, Round 8 **198**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.
5. The columns of A are a linearly independent set.
6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.
7. A is invertible.
8. The columns of A are a basis for $\mathbb{C}^n$.
9. The rank of A is n, $r\left(A\right) = n$.
10. The nullity of A is zero, $n\left(A\right) = 0$.
11. The determinant of A is nonzero, $\det\left(A\right) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix **199**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP Eigenvalues Of Matrix Powers **200**

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Theorem EPM Eigenvalues of the Polynomial of a Matrix **201**

Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Theorem EIM Eigenvalues of the Inverse of a Matrix **202**

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM Eigenvalues of the Transpose of a Matrix **203**

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs **204**

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP Degree of the Characteristic Polynomial **205**

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Theorem NEM Number of Eigenvalues of a Matrix **206**

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Theorem ME Multiplicities of an Eigenvalue **207**

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix **208**

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Theorem HMRE Hermitian Matrices have Real Eigenvalues **209**

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Theorem HMOE Hermitian Matrices have Orthogonal Eigenvectors **210**

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM Similar Matrices **211**

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER Similarity is an Equivalence Relation **212**

Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)
2. If A is similar to B, then B is similar to A. (Symmetric)
3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Theorem SMEE Similar Matrices have Equal Eigenvalues **213**

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.

Definition DIM Diagonal Matrix **214**

Suppose that $A = (a_{ij})$ is a square matrix. Then A is a **diagonal matrix** if $a_{ij} = 0$ whenever $i \neq j$.

Definition DZM Diagonalizable Matrix **215**

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC Diagonalization Characterization **216**

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMLE Diagonalizable Matrices have Large Eigenspaces **217**

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DED Distinct Eigenvalues implies Diagonalizable **218**

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Definition LT Linear Transformation **220**

A **linear transformation**, $T\colon U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ Linear Transformations Take Zero to Zero **221**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Theorem MBLT Matrices Build Linear Transformations **222**

Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV Matrix of a Linear Transformation, Column Vectors **223**

Suppose that $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations **224**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, $\ldots$, a_t are scalars from $\mathbb{C}$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t)$$

Theorem LTDB Linear Transformation Defined on a Basis **225**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ be the scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T\left(\mathbf{w}\right) = a_1 T\left(\mathbf{u}_1\right) + a_2 T\left(\mathbf{u}_2\right) + a_3 T\left(\mathbf{u}_3\right) + \cdots + a_n T\left(\mathbf{u}_n\right)$$

Definition PI Pre-Image **226**

Suppose that $T\colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \left\{\, \mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v} \right\}$$

Definition LTA Linear Transformation Addition **227**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT Sum of Linear Transformations is a Linear Transformation **228**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Definition LTSM Linear Transformation Scalar Multiplication **229**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation **230**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \mapsto V$ is a linear transformation.

Theorem VSLT Vector Space of Linear Transformations **231**

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\operatorname{LT}(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC Linear Transformation Composition **232**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT Composition of Linear Transformations is a Linear Transformation **233**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a linear transformation.

Definition ILT Injective Linear Transformation **234**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition NSLT Null Space of a Linear Transformation **235**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **null space** of T is the set $\mathcal{N}(T) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0}\}$

Theorem NSLTS Null Space of a Linear Transformation is a Subspace **236**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the null space of T, $\mathcal{N}(T)$, is a subspace of U.

Theorem NSPI Null Space and Pre-Image **237**

Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{\, \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{N}(T) \} = \mathbf{u} + \mathcal{N}(T)$$

Theorem NSILT Null Space of an Injective Linear Transformation **238**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the null space of T is trivial, $\mathcal{N}(T) = \{\mathbf{0}\}$.

Theorem ILTLI Injective Linear Transformations and Linear Independence **239**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases **240**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Theorem ILTD Injective Linear Transformations and Dimension **241**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Theorem CILTI Composition of Injective Linear Transformations is Injective **242**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is an injective linear transformation.

Definition SLT Surjective Linear Transformation **243**

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Definition RLT Range of a Linear Transformation **244**

Suppose $T: U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{\, T(\mathbf{u}) \mid \mathbf{u} \in U \}$$

Theorem RLTS Range of a Linear Transformation is a Subspace **245**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT Range of a Surjective Linear Transformation **246**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Theorem SSRLT Spanning Set for Range of a Linear Transformation **247**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ spans U. Then $R = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_t)\}$ spans $\mathcal{R}(T)$.

Theorem RPI Range and Pre-Image **248**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

Theorem SLTB Surjective Linear Transformations and Bases **249**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$ is a spanning set for V.

Theorem SLTD Surjective Linear Transformations and Dimension **250**

Suppose that $T\colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective **251**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation **252**

The **identity linear transformation** on the vector space W is defined as

$$I_W\colon W \mapsto W, \qquad I_W\left(\mathbf{w}\right) = \mathbf{w}$$

Definition IVLT Invertible Linear Transformations **253**

Suppose that $T\colon U \mapsto V$ is a linear transformation. If there is a function $S\colon V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation **254**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}\colon V \mapsto U$ is a linear transformation.

Theorem IILT Inverse of an Invertible Linear Transformation **255**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Theorem ILTIS Invertible Linear Transformations are Injective and Surjective **256**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT Composition of Invertible Linear Transformations **257**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T) : U \mapsto W$ is an invertible linear transformation.

Theorem ICLT Inverse of a Composition of Linear Transformations **258**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS Isomorphic Vector Spaces **259**

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension **260**

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Definition ROLT Rank Of a Linear Transformation **261**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,

$$r(T) = \dim(\mathcal{R}(T))$$

Definition NOLT Nullity Of a Linear Transformation **262**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n(T)$, is the dimension of the null space of T,

$$n(T) = \dim(\mathcal{N}(T))$$

Theorem ROSLT Rank Of a Surjective Linear Transformation **263**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r\left(T\right) = \dim\left(V\right)$, if and only if T is surjective.

Theorem NOILT Nullity Of an Injective Linear Transformation **264**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n\left(T\right) = 0$, if and only if T is injective.

Theorem RPNDD Rank Plus Nullity is Domain Dimension **265**

Suppose that $T\colon U \mapsto V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Definition VR Vector Representation **266**

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\}$. Define a function $\rho_B\colon V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1,\,a_2,\,a_3,\,\ldots,\,a_n$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$\rho_B(\mathbf{w}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Theorem VRLT Vector Representation is a Linear Transformation **267**

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI Vector Representation is Injective **268**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS Vector Representation is Surjective **269**

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT Vector Representation is an Invertible Linear Transformation **270**

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces **271**

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces **272**

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI Coordinatization and Linear Independence **273**

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS Coordinatization and Spanning Sets **274**

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \mathcal{S}p(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\})$ if and only if $\rho_B(\mathbf{u}) \in \mathcal{S}p(\{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\})$.

Definition MR Matrix Representation **275**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. The the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M^T_{B,C} = \left[\rho_C\left(T\left(\mathbf{u}_1\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_2\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_3\right)\right)\middle| \ldots \middle|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\right]$$

Theorem FTMR Fundamental Theorem of Matrix Representation **276**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

Theorem MRSLT Matrix Representation of a Sum of Linear Transformations **277**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M^{T+S}_{B,C} = M^{T}_{B,C} + M^{S}_{B,C}$$

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation **278**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M^{\alpha T}_{B,C} = \alpha M^{T}_{B,C}$$

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations **279**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M^{S\circ T}_{B,D} = M^{S}_{C,D} M^{T}_{B,C}$$

Theorem INS Isomorphic Null Spaces **280**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the null space of T is isomorphic to the null space of $M^{T}_{B,C}$,

$$\mathcal{N}(T) \cong \mathcal{N}\left(M^{T}_{B,C}\right)$$

Theorem IR Isomorphic Ranges

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the range of $M^{T}_{B,C}$,

$$\mathcal{R}(T) \cong \mathcal{R}\left(M^{T}_{B,C}\right)$$

Theorem IMR Invertible Matrix Representations

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M^{T}_{B,C}$ is an invertible matrix, and

$$M^{T^{-1}}_{C,B} = \left(M^{T}_{B,C}\right)^{-1}$$

Definition EELT Eigenvalue and Eigenvector of a Linear Transformation **283**

Suppose that $T\colon V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM Change-of-Basis Matrix **284**

Suppose that V is a vector space, and $I_V\colon V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$\begin{aligned}
C_{B,C} &= M^{I_V}_{B,C} \\
&= \left[\rho_C\left(I_V\left(\mathbf{v}_1\right)\right)\middle|\, \rho_C\left(I_V\left(\mathbf{v}_2\right)\right)\middle|\, \rho_C\left(I_V\left(\mathbf{v}_3\right)\right)\middle| \ldots \middle|\rho_C\left(I_V\left(\mathbf{v}_n\right)\right)\right] \\
&= \left[\rho_C\left(\mathbf{u}_1\right)\middle|\, \rho_C\left(\mathbf{u}_2\right)\middle|\, \rho_C\left(\mathbf{u}_3\right)\middle| \ldots \middle|\rho_C\left(\mathbf{u}_n\right)\right]
\end{aligned}$$

Theorem CB Change-of-Basis **285**

Suppose that $\mathbf{u}$ is a vector in the vector space V and B and C are bases of V. Then

$$C_{B,C}\rho_B(\mathbf{v}) = \rho_C(\mathbf{v})$$

Theorem ICBM Inverse of Change-of-Basis Matrix **286**

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB Matrix Representation and Change of Basis **287**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M^T_{B,D} = C_{E,D} M^T_{C,E} C_{B,C}$$

Theorem SCB Similarity and Change of Basis **288**

Suppose that $T\colon V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M^T_{B,B} = C^{-1}_{B,C} M^T_{C,C} C_{B,C}$$

Theorem EER Eigenvalues, Eigenvectors, Representations **289**

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ.