Flash Cards

to accompany

A First Course in Linear Algebra

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A **system of simultaneous linear equations** is a collection of m equations in the variable quantities $x_1,\ x_2,\ x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j are from the set of complex numbers, $\mathbb{C}$.

Two systems of simultaneous linear equations are **equivalent** if their solution sets are equal.

Definition EO   Equation Operations

Given a system of simultaneous linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

1. Swap the locations of two equations in the list.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Theorem EOPSS   Equation Operations Preserve Solution Sets

Suppose we apply one of the three equation operations of Definition EO to the system of simultaneous linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots \quad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.$$

Then the original system and the transformed system are equivalent systems.

Definition M   Matrix   **5**

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns.

Definition AM   Augmented Matrix   **6**

Suppose we have a system of m equations in the n variables $x_1,\, x_2,\, x_3, \ldots,\, x_n$ written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Definition RO Row Operations   **7**

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.

2. Multiply each entry of a single row by a nonzero quantity.

3. Multiply each entry of one row by some quantity, and add these values to the entry in the same column of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

Definition REM Row-Equivalent Matrices   **8**

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero is below any row containing a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $i < s$, then $j < t$.

A row of a matrix where every entry is zero is called a **zero row**.

For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a **leading 1**.

Definition PC Pivot Columns **13**

For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

Theorem REMEF Row-Equivalent Matrix in Echelon Form **14**

Suppose A is a matrix. Then there is a (unique!) matrix B so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Suppose A is the augmented matrix of a system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the number of a column of B that contains the leading 1 for some row, and it is not the last column. Then the variable j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Theorem RCLS   Recognizing Consistency of a Linear System   **17**

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B.

Theorem ICRN   Inconsistent Systems, r and n **18**

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Theorem CSRN   Consistent Systems, r and n

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Theorem FVCS   Free Variables for Consistent Systems

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Theorem PSSLS   Possible Solution Sets for Linear Systems

A simultaneous system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI   Consistent, More Variables than Equations, Infinite solutions

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Definition HS   Homogeneous System

A system of linear equations is **homogeneous** if each equation has a 0 for its constant term. Such a system then has the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

Theorem HSC   Homogeneous Systems are Consistent

Suppose that a system of linear equations is homogeneous. Then it is consistent.

Definition TSHSE   Trivial Solution to Homogeneous Systems of Equations   **25**

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0,\ldots, x_n = 0$ is called the **trivial solution**.

Theorem HMVEI   Homogeneous, More Variables than Equations, Infinite solutions   **26**

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Definition CV   Column Vector

A **column vector** of **size** m is an ordered list of m numbers, which is written vertically, in order from top to bottom. At times, we will refer to a column vector as simply a **vector**.

Definition ZV   Zero Vector

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Definition CM Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Definition SV   Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size m

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix}$$

Definition NSM   Null Space of a Matrix

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Definition SQM   Square Matrix

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Definition NM   Nonsingular Matrix

Suppose A is a square matrix. And suppose the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Definition IM   Identity Matrix   **35**

The $m \times m$ **identity matrix**, $I_m = (a_{ij})$ has $a_{ij} = 1$ whenever $i = j$, and $a_{ij} = 0$ whenever $i \neq j$.

Theorem NSRRI   NonSingular matrices Row Reduce to the Identity matrix   **36**

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NSTNS   NonSingular matrices have Trivial Null Spaces

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the trivial solution to the system $\mathcal{LS}(A, \mathbf{0})$, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Theorem NSMUS   NonSingular Matrices and Unique Solutions

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Theorem NSME1   NonSingular Matrix Equivalences, Round 1

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the trivial solution, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

Definition VSCM   Vector Space $\mathbb{C}^m$

The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$.

Definition CVE Column Vector Equality

The vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

are **equal**, written $\mathbf{u} = \mathbf{v}$ provided that $u_i = v_i$ for all $1 \le i \le m$.

Definition CVA Column Vector Addition

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_m + v_m \end{bmatrix}.$$

Definition CVSM Column Vector Scalar Multiplication

Given the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of $\mathbf{u}$ by α is

$$\alpha\mathbf{u} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \\ \vdots \\ \alpha u_m \end{bmatrix}.$$

Theorem VSPCM Vector Space Properties of $\mathbb{C}^m$
Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{C}^m$ and α and β are scalars. Then

1. $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ (Additive closure)

2. $\alpha\mathbf{u} \in \mathbb{C}^m$ (Scalar closure)

3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)

4. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (Associativity of vector addition)

5. There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$. (Additive identity)

6. For each vector $\mathbf{u} \in \mathbb{C}^m$, there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (Additive inverses)

7. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ (Associativity of scalar multiplication)

8. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (Distributivity across vector addition)

9. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (Distributivity across addition)

10. $1\mathbf{u} = \mathbf{u}$ (Scalar multiplication with 1)

Definition LCCV   Linear Combination of Column Vectors   **45**

Given n vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, ..., $\mathbf{u}_n$ and n scalars α_1, α_2, α_3, ..., α_n, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Theorem SLSLC   Solutions to Linear Systems are Linear Combinations   **46**

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$. Then $\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$

is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3 + \cdots + \alpha_n\mathbf{A}_n = \mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems   **47**

Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Denote the vector of variables as $\mathbf{x} = (x_i)$. Let $B = (b_{ij})$ be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's having indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c} = (c_i)$, $\mathbf{u}_j = (u_{ij})$, $1 \le j \le n-r$ of size n by

$$c_i = \begin{cases} 0 & \text{if } i \in F \\ b_{k,n+1} & \text{if } i \in D,\ i = d_k \end{cases}$$

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \ne f_j \\ -b_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Then the set of solutions to the system of equations represented by the vector equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}$$

is equal to the set of solutions of $\mathcal{LS}(A, \mathbf{b})$.

Theorem RREFU   Reduced Row-Echelon Form is Unique   **48**

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Definition SSCV   Span of a Set of Column Vectors

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$. Symbolically,

$$\mathcal{S}p(S) = \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \, 1 \leq i \leq t\}$$
$$= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \; \middle| \; \alpha_i \in \mathbb{C}, \, 1 \leq i \leq t \right\}$$

Theorem SSNS   Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{u}_j = (u_{ij})$, $1 \leq j \leq n - r$ of size n as

$$u_{ij} = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}.$$

Then the null space of A is given by

$$\mathcal{N}(A) = \mathcal{S}p(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}\}).$$

Definition RLDCV   Relation of Linear Dependence for Column Vectors   **51**

Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$
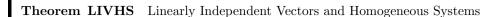
is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Definition LICV   Linear Independence of Column Vectors   **52**

The set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems **53**

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Theorem MVSLD More Vectors than Size implies Linear Dependence **54**

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is the set of vectors in $\mathbb{C}^m$, and that $n > m$. Then S is a linearly dependent set.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\}$ is the set of vectors in $\mathbb{C}^m$ that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if $n = r$.

Suppose that $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t, $1 \leq t \leq n$ such that $\mathbf{u_t}$ is a linear combination of the vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_{t-1},\ \mathbf{u}_{t+1},\ \ldots,\ \mathbf{u}_n$.

Theorem NSLIC NonSingular matrices have Linearly Independent Columns   **57**

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Theorem NSME2 NonSingular Matrix Equivalences, Round 2   **58**

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A form a linearly independent set.

Theorem BNS   Basis for Null Spaces   **59**

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors $\mathbf{z}_j = (z_{ij})$, $1 \le j \le n - r$ of size n as

$$
z_{ij} = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \ne f_j \\ -b_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases} .
$$

Define the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Definition CCV   Conjugate of a Column Vector   **60**

Suppose that

$$
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}
$$

is a vector from $\mathbb{C}^m$. Then the conjugate of the vector is defined as

$$
\overline{\mathbf{u}} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \\ \vdots \\ \overline{u}_m \end{bmatrix}
$$

Theorem CCRVA   Complex Conjugation Respects Vector Addition   **61**

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Theorem CCRSM   Complex Conjugation Respects Scalar Multiplication   **62**

Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Definition IP Inner Product

Given the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\, \mathbf{v} \rangle = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3} + \cdots + u_m\overline{v_m} = \sum_{i=1}^{m} u_i\overline{v_i}$$

Theorem IPVA Inner Product and Vector Addition

Suppose $\mathbf{u}\,\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

$$1. \quad \langle \mathbf{u} + \mathbf{v},\, \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{w} \rangle + \langle \mathbf{v},\, \mathbf{w} \rangle$$
$$2. \quad \langle \mathbf{u},\, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u},\, \mathbf{v} \rangle + \langle \mathbf{u},\, \mathbf{w} \rangle$$

Suppose $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

$$1. \quad \langle \alpha\mathbf{u},\, \mathbf{v} \rangle = \alpha \langle \mathbf{u},\, \mathbf{v} \rangle$$
$$2. \quad \langle \mathbf{u},\, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u},\, \mathbf{v} \rangle$$

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{C}^m$. Then $\langle \mathbf{u},\, \mathbf{v} \rangle = \overline{\langle \mathbf{v},\, \mathbf{u} \rangle}$.

Definition NV   Norm of a Vector

The **norm** of the vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}$$

is the scalar quantity in $\mathbb{C}^m$

$$\|\mathbf{u}\| = \sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2}$$

Theorem IPN   Inner Products and Norms

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \, \mathbf{u} \rangle$.

Theorem PIP   Positive Inner Products   **69**

Suppose that $\mathbf{u}$ is a vector in $\mathbb{C}^m$. Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV   Orthogonal Vectors   **70**

A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Definition OSV   Orthogonal Set of Vectors **71**

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then S is **orthogonal** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Theorem OSLI   Orthogonal Sets are Linearly Independent **72**

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSPCV   Gram-Schmidt Procedure, Column Vectors   **73**

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent set of vectors in $\mathbb{C}^m$. Define the vectors $\mathbf{u}_i$, $1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\mathcal{S}p(T) = \mathcal{S}p(S)$.

Definition ONS   OrthoNormal Set   **74**

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

Definition VSM Vector Space of $m \times n$ Matrices **75**

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality **76**

The $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad B = (b_{ij})$$

are **equal**, written $A = B$ provided $a_{ij} = b_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA   Matrix Addition

Given the $m \times n$ matrices

$$A = (a_{ij}) \qquad\qquad\qquad B = (b_{ij})$$

define the **sum** of A and B to be $A + B = C = (c_{ij})$, where

$$c_{ij} = a_{ij} + b_{ij}, \quad 1 \le i \le m,\, 1 \le j \le n.$$

Definition SMM   Scalar Matrix Multiplication

Given the $m \times n$ matrix $A = (a_{ij})$ and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A by α is the matrix $\alpha A = C = (c_{ij})$, where

$$c_{ij} = \alpha a_{ij}, \quad 1 \le i \le m,\, 1 \le j \le n.$$

Theorem VSPM   Vector Space Properties of M_{mn} **79**
Suppose that A, B and C are $m \times n$ matrices in M_{mn} and α and β are scalars. Then

1. $A + B \in M_{mn}$ (Additive closure)

2. $\alpha A \in M_{mn}$ (Scalar closure)

3. $A + B = B + A$ (Commutativity)

4. $A + (B + C) = (A + B) + C$ (Associativity of matrix addition)

5. There is a matrix, $\mathcal{O}$, called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$. (Additive identity)

6. For each matrix $A \in M_{mn}$, there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$. (Additive inverses)

7. $\alpha(\beta A) = (\alpha\beta)A$ (Associativity of scalar multiplication)

8. $\alpha(A + B) = \alpha A + \alpha B$ (Distributivity across matrix addition)

9. $(\alpha + \beta)A = \alpha A + \beta A$ (Distributivity across addition)

10. $1A = A$ (Scalar multiplication with 1)

Definition ZM   Zero Matrix   **80**

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n} = (z_{ij})$ and defined by $z_{ij} = 0$ for all $1 \le i \le m$, $1 \le j \le n$. Or, equivalently, $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$.

Definition TM   Transpose of a Matrix

Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}, \quad 1 \le i \le n, 1 \le j \le m.$$

Definition SYM   Symmetric Matrix

The matrix A is **symmetric** if $A = A^t$.

Theorem SMS Symmetric Matrices are Square

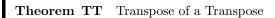Suppose that A is a symmetric matrix. Then A is square.

Theorem TASM Transposes, Addition, Scalar Multiplication

Suppose that A and B are $m \times n$ matrices. Then

1. $(A + B)^t = A^t + B^t$

2. $(\alpha A)^t = \alpha A^t$

Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

Definition RM   Range of a Matrix   **87**

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\}$. Then the **range** of A, written $\mathcal{R}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{R}(A) = \mathcal{S}p(\{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\})$$

Theorem RCS   Range and Consistent Systems   **88**

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{R}(A)$ if and only if $\mathcal{LS}(A,\ \mathbf{b})$ is consistent.

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\ldots$, $\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $S = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. $\mathcal{R}(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Suppose that A is an $m \times n$ matrix. Create the $m \times (n + m)$ matrix M by placing the $m \times m$ identity matrix I_m to the right of the matrix A. Symbolically, $M = [A \mid I_m]$. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Suppose there are r leading 1's of N in the first n columns. If $r = m$, then $\mathcal{R}(A) = \mathbb{C}^m$. Otherwise, $r < m$ and let K be the $(m - r) \times m$ matrix formed from the entries of N in the last $m - r$ rows and last m columns. Then

1. K is in reduced row-echelon form.

2. K has no zero rows, or equivalently, K has $m - r$ leading 1's.

3. $\mathcal{R}(A) = \mathcal{N}(K)$.

Theorem RNSM   Range of a NonSingular Matrix   **91**

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{R}(A) = \mathbb{C}^n$.

Theorem NSME3   NonSingular Matrix Equivalences, Round 3   **92**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

Definition RSM   Row Space of a Matrix

Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\nabla s(A)$, is the range of A^t, i.e. $\nabla s(A) = \mathcal{R}(A^t)$.

Theorem REMRS   Row-Equivalent Matrices have equal Row Spaces

Suppose A and B are row-equivalent matrices. Then $\nabla s(A) = \nabla s(B)$.

Theorem BRS   Basis for the Row Space   **95**

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\nabla s(A) = \mathcal{S}p(S)$.

2. S is a linearly independent set.

Theorem RMRST   Range of a Matrix is Row Space of Transpose   **96**

Suppose A is a matrix. Then $\mathcal{R}(A) = \nabla s(A^t)$.

Definition MVP   Matrix-Vector Product

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ..., $\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is

$$A\mathbf{u} = [\mathbf{A}_1 | \mathbf{A}_2 | \mathbf{A}_3 | \ldots | \mathbf{A}_n] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = u_1\mathbf{A}_1 + u_2\mathbf{A}_2 + u_3\mathbf{A}_3 + \cdots + u_n\mathbf{A}_n$$

Theorem SLEMM   Systems of Linear Equations as Matrix Multiplication

Solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ are the solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Definition MM Matrix Multiplication

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, ..., $\mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

Theorem EMP Entries of Matrix Products

Suppose $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix. Then the entries of $AB = C = (c_{ij})$ are given by

$$[C]_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}$$

Suppose A is an $m \times n$ matrix. Then
1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Suppose A is an $m \times n$ matrix. Then
1. $AI_n = A$
2. $I_m A = A$

Theorem MMDAA   Matrix Multiplication Distributes Across Addition

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
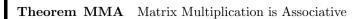1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Theorem MMSMM   Matrix Multiplication and Scalar Matrix Multiplication

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.
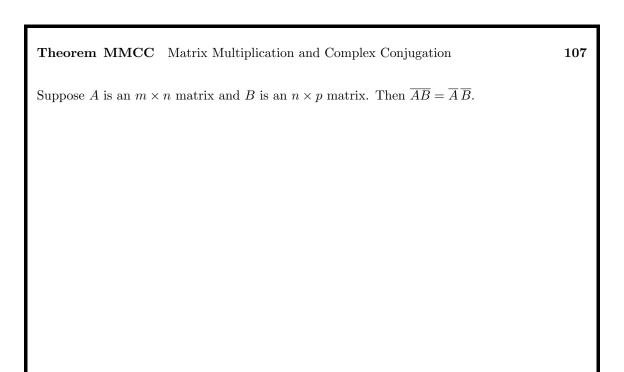
If we consider the vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$$
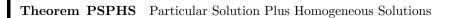
Theorem MMCC   Matrix Multiplication and Complex Conjugation

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\,\overline{B}$.

Theorem MMT   Matrix Multiplication and Transposes

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Suppose that $\mathbf{z}$ is one solution to the linear system of equations $\mathcal{L}S(A,\, b)$. Then $\mathbf{y}$ is a solution to $\mathcal{L}S(A,\, b)$ if and only if $\mathbf{y} = \mathbf{z} + \mathbf{w}$ for some vector $\mathbf{w} \in N(A)$.

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A, and we write $B = A^{-1}$.

Definition SUV Standard Unit Vectors

Let $\mathbf{e}_i \in \mathbb{C}^m$ denote the column vector that is column i of the $m \times m$ identity matrix I_m. Then the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \le i \le m\}$$

is the set of **standard unit vectors** in $\mathbb{C}^m$.

Theorem TTMI Two-by-Two Matrix Inverse

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem CINSM   Computing the Inverse of a NonSingular Matrix   **113**

Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let B be the matrix formed from the final n columns of N. Then $AB = I_n$.

Theorem MIU   Matrix Inverse is Unique   **114**

Suppose the square matrix A has an inverse. Then A^{-1} is unique.
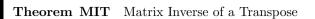
Theorem SS   Socks and Shoes

Suppose A and B are invertible matrices of size n. Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Theorem MIMI   Matrix Inverse of a Matrix Inverse

Suppose A is an invertible matrix. Then $(A^{-1})^{-1} = A$ and A^{-1} is invertible.

Theorem MIT   Matrix Inverse of a Transpose   117

Suppose A is an invertible matrix. Then $(A^t)^{-1} = (A^{-1})^t$ and A^t is invertible.

Theorem MISM   Matrix Inverse of a Scalar Multiple   118

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and αA is invertible.

Theorem PWSMS   Product With a Singular Matrix is Singular   **119**

Suppose that A or B are matrices of size n, and one, or both, is singular. Then their product, AB, is singular.

Theorem OSIS   One-Sided Inverse is Sufficient   **120**

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Theorem NSI NonSingularity is Invertibility

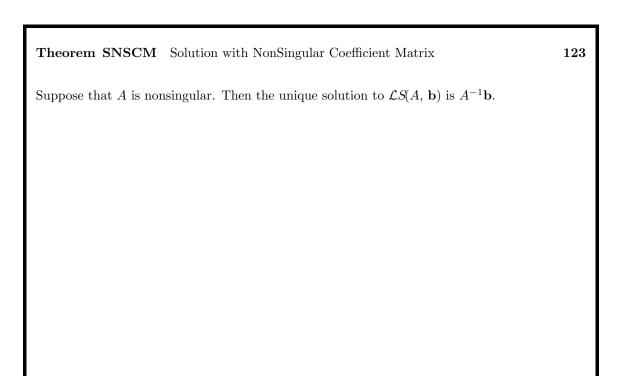Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NSME4 NonSingular Matrix Equivalences, Round 4

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

7. A is invertible.

Theorem SNSCM   Solution with NonSingular Coefficient Matrix   123

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Definition OM   Orthogonal Matrices   124
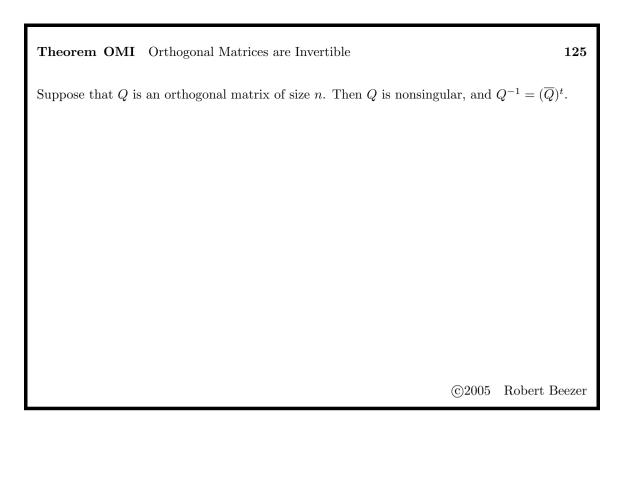
Suppose that Q is a square matrix of size n such that $\left(\overline{Q}\right)^t Q = I_n$. Then we say Q is **orthogonal**.

Theorem OMI   Orthogonal Matrices are Invertible   **125**

Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

Theorem COMOS   Columns of Orthogonal Matrices are Orthonormal Sets   **126**

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\}$. Then A is an orthogonal matrix if and only if S is an orthonormal set.

Suppose that Q is an orthogonal matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle Q\mathbf{u},\, Q\mathbf{v}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle \qquad\qquad \|Q\mathbf{v}\| = \|\mathbf{v}\|.$$

Definition A   Adjoint   **128**

If A is a square matrix, then its **adjoint** is $A^H = \left(\overline{A}\right)^t$.

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = \left(\overline{A}\right)^t$
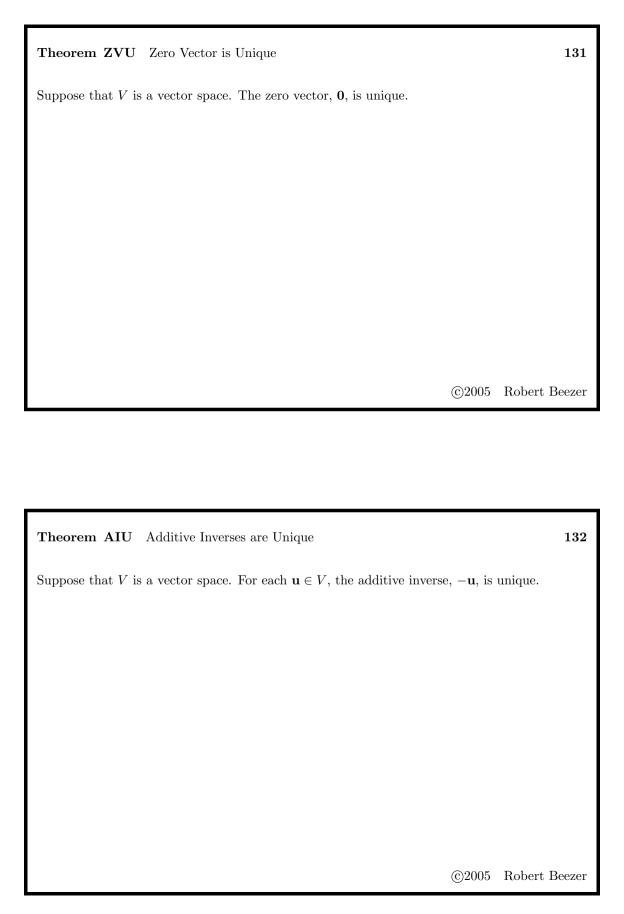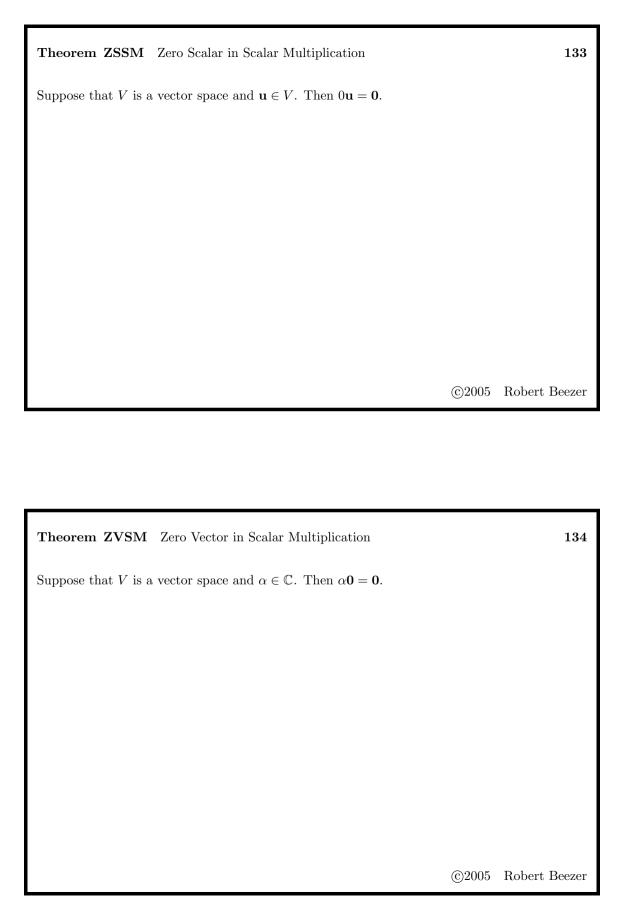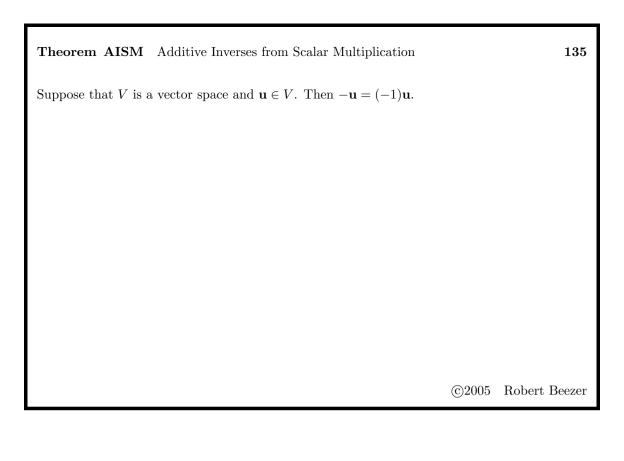
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** if the following ten requirements (better known as "axioms") are met. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and α, $\beta \in \mathbb{C}$.

1. $\mathbf{u} + \mathbf{v} \in V$ (Additive closure)

2. $\alpha\mathbf{u} \in V$ (Scalar closure)

3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)

4. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (Associativity of vector addition)

5. There is a vector, $\mathbf{0} \in V$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$. (Additive identity)

6. For each vector $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (Additive inverses)

7. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ (Associativity of scalar multiplication)

8. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (Distributivity across vector addition)

9. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (Distributivity across addition)

10. $1\mathbf{u} = \mathbf{u}$ (Scalar multiplication with 1)

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU   Zero Vector is Unique   **131**

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Theorem AIU   Additive Inverses are Unique   **132**

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Theorem ZSSM Zero Scalar in Scalar Multiplication

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Theorem ZVSM Zero Vector in Scalar Multiplication

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Theorem AISM   Additive Inverses from Scalar Multiplication   **135**

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.
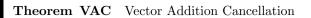
Theorem SMEZV   Scalar Multiplication Equals the Zero Vector   **136**

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then if $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.
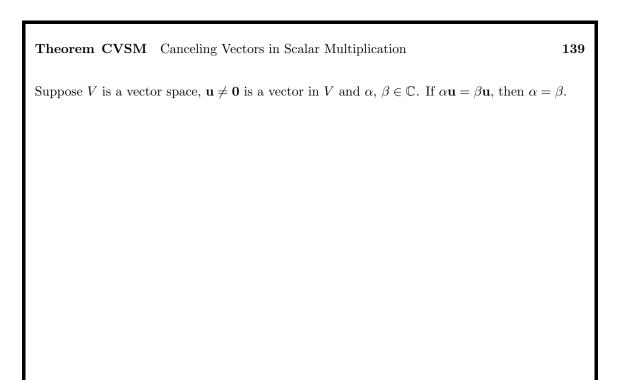
Theorem VAC   Vector Addition Cancellation   **137**

Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CSSM   Canceling Scalars in Scalar Multiplication   **138**

Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Theorem CVSM Canceling Vectors in Scalar Multiplication   **139**

Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and $\alpha,\ \beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Definition S Subspace   **140**

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V.

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met
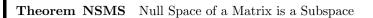
1. W is non-empty, $W \neq \emptyset$.

2. Whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.

3. Whenever $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Theorem NSMS Null Space of a Matrix is a Subspace   **143**

Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Definition LC Linear Combination   **144**

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

Definition SS Span of a Set   **145**

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \dots,\, \mathbf{u}_t\}$, their **span**, $\mathcal{S}p(S)$, is the set of all possible linear combinations of $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \dots,\, \mathbf{u}_t$. Symbolically,
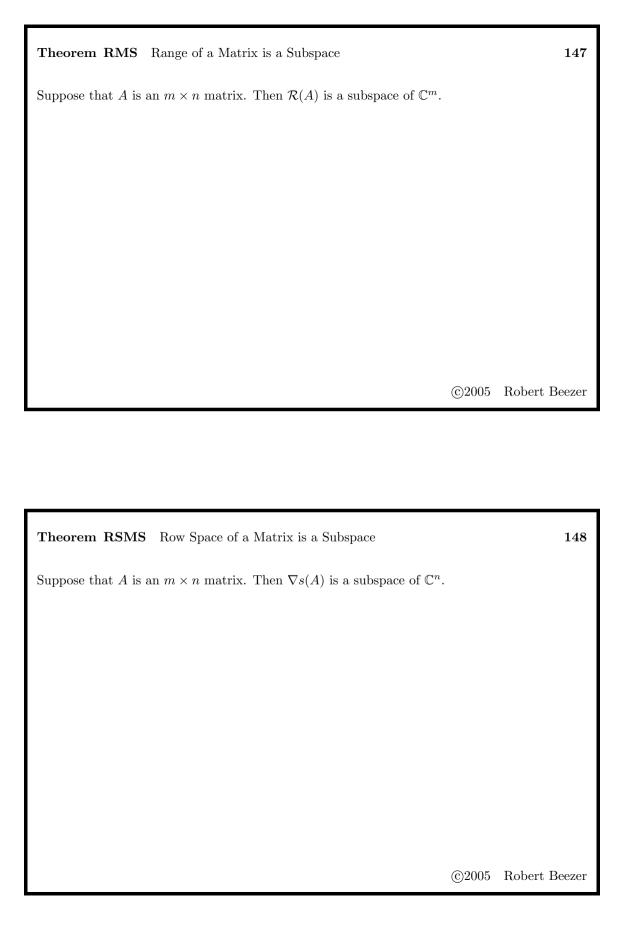
$$\mathcal{S}p(S) = \{\,\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \le i \le t\}$$

$$= \left\{ \sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \le i \le t \right\}$$

Theorem SSS Span of a Set is a Subspace   **146**

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \dots,\, \mathbf{u}_t\} \subseteq V$, their span, $\mathcal{S}p(S)$, is a subspace.

Theorem RMS   Range of a Matrix is a Subspace

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^m$.

Theorem RSMS   Row Space of a Matrix is a Subspace

Suppose that A is an $m \times n$ matrix. Then $\nabla s(A)$ is a subspace of $\mathbb{C}^n$.

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$
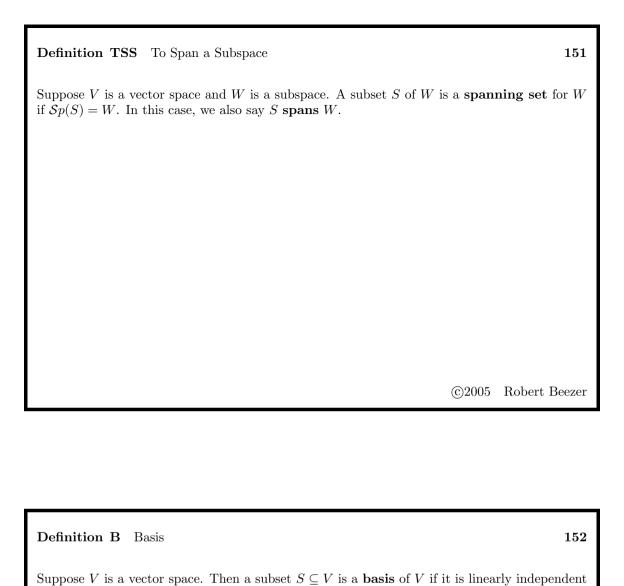
is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Definition TSS To Span a Subspace **151**

Suppose V is a vector space and W is a subspace. A subset S of W is a **spanning set** for W if $\mathcal{S}p(S) = W$. In this case, we also say S **spans** W.

Definition B Basis **152**

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V.

The set of standard unit vectors for $\mathbb{C}^m$, $B = \{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \ldots,\, \mathbf{e}_m\} = \{\, \mathbf{e}_i \mid 1 \leq i \leq m \,\}$ is a basis for the vector space $\mathbb{C}^m$.

Suppose that A is a square matrix. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Theorem NSME5 NonSingular Matrix Equivalences, Round 5

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

7. A is invertible.

8. The columns of A are a basis for $\mathbb{C}^n$.

Theorem VRRB Vector Representation Relative to a Basis

Suppose that V is a vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ and that $\mathbf{w}$ is a vector in V. Then there exist *unique* scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Definition D   Dimension   **157**

Suppose that V is a vector space and $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD   Spanning Sets and Linear Dependence   **158**

Suppose that $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.

Theorem BIS   Bases have Identical Sizes   **159**

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Theorem DCM   Dimension of $\mathbb{C}^m$ **160**

The dimension of $\mathbb{C}^m$ (Example VSCM) is m.

Theorem DP   Dimension of P_n **161**

The dimension of P_n (Example VSP) is $n + 1$.

Theorem DM   Dimension of M_{mn} **162**

The dimension of M_{mn} (Example VSM) is mn.

Definition NOM   Nullity Of a Matrix   **163**
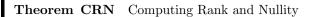
Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM   Rank Of a Matrix   **164**

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the range of A, $r(A) = \dim(\mathcal{R}(A))$.

Theorem CRN   Computing Rank and Nullity   **165**

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Theorem RPNC   Rank Plus Nullity is Columns   **166**

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Theorem RNNSM   Rank and Nullity of a NonSingular Matrix   **167**

Suppose that A is a square matrix of size n. The following are equivalent.
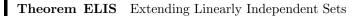
  1. A is nonsingular.

  2. The rank of A is n, $r(A) = n$.

  3. The nullity of A is zero, $n(A) = 0$.

Theorem NSME6   NonSingular Matrix Equivalences, Round 6   **168**

Suppose that A is a square matrix of size n. The following are equivalent.

  1. A is nonsingular.

  2. A row-reduces to the identity matrix.

  3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

  4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

  5. The columns of A are a linearly independent set.

  6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

  7. A is invertible.

  8. The columns of A are a basis for $\mathbb{C}^n$.

  9. The rank of A is n, $r(A) = n$.

  10. The nullity of A is zero, $n(A) = 0$.

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \notin \mathcal{S}p(S)$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ be a set of vectors from V. Then

1. If $m > t$, then S is linearly dependent.

2. If $m < t$, then S does not span V.

3. If $m = t$ and S is linearly independent, then S spans V.

4. If $m = t$ and S spans V, then S is linearly independent.

Theorem RMRT   Rank of a Matrix is the Rank of the Transpose

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Theorem COB   Coordinates and Orthonormal Bases

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,
$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Definition SM   SubMatrix   **173**

Suppose that A is an $m \times n$ matrix. Then the **submatrix** A_{ij} is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM   Determinant   **174**

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

If $A = \begin{bmatrix} a \end{bmatrix}$ is a 1×1 matrix, then $\det(A) = a$.

If $A = (a_{ij})$ is a matrix of size n with $n \geq 2$, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **minor** for A at location ij is the determinant of the submatrix, $M_{A,ij} = \det(A_{ij})$.

Definition CIM   Cofactor In a Matrix **177**

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j. Then the **cofactor** for A at location ij is the signed determinant of the submatrix, $C_{A,ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem DERC   Determinant Expansion about Rows and Columns **178**

Suppose that $A = (a_{ij})$ is a square matrix of size n. Then
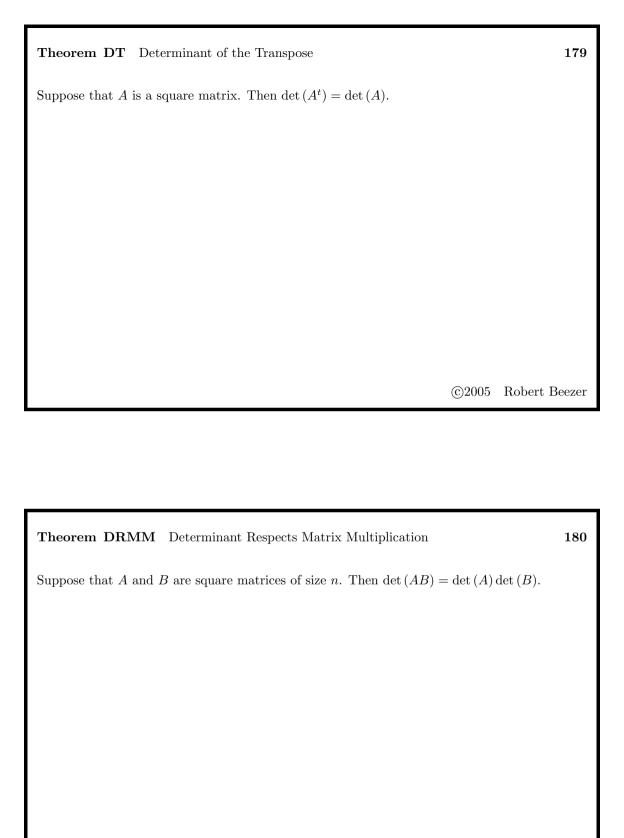
$$\det(A) = a_{i1}C_{A,i1} + a_{i2}C_{A,i2} + a_{i3}C_{A,i3} + \cdots + a_{in}C_{A,in} \qquad 1 \le i \le n$$

which is known as **expansion** about row i, and

$$\det(A) = a_{1j}C_{A,1j} + a_{2j}C_{A,2j} + a_{3j}C_{A,3j} + \cdots + a_{nj}C_{A,nj} \qquad 1 \le j \le n$$

which is known as **expansion** about column j.

Theorem DT   Determinant of the Transpose

Suppose that A is a square matrix. Then $\det\left(A^t\right) = \det\left(A\right)$.

Theorem DRMM   Determinant Respects Matrix Multiplication

Suppose that A and B are square matrices of size n. Then $\det\left(AB\right) = \det\left(A\right)\det\left(B\right)$.

Theorem SMZD   Singular Matrices have Zero Determinants

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Theorem NSME7   NonSingular Matrix Equivalences, Round 7
Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

7. A is invertible.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

Definition EEM   Eigenvalues and Eigenvectors of a Matrix

Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector from $\mathbb{C}^n$, and λ is a scalar from $\mathbb{C}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.

Theorem EMHE   Every Matrix Has an Eigenvalue

Suppose A is a square matrix. Then A has at least one eigenvalue.

Definition CP   Characteristic Polynomial   **185**

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by
$$p_A(x) = \det(A - xI_n)$$

Theorem EMRCP   Eigenvalues of a Matrix are Roots of Characteristic Polynomials   **186**

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace** of A for λ, $E_A(\lambda)$, is the set of all the eigenvectors of A for λ, with the addition of the zero vector.

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $E_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Theorem EMNS Eigenspace of a Matrix is a Null Space

Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$E_A\left(\lambda\right) = \mathcal{N}(A - \lambda I_n)$$
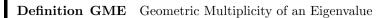
Definition AME Algebraic Multiplicity of an Eigenvalue

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A\left(\lambda\right)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A\left(x\right)$.

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$.

Suppose that A is a square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Theorem SMZE   Singular Matrices have Zero Eigenvalues   **193**

Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Theorem NSME8   NonSingular Matrix Equivalences, Round 8   **194**

Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.

4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of A are a linearly independent set.

6. The range of A is $\mathbb{C}^n$, $\mathcal{R}(A) = \mathbb{C}^n$.

7. A is invertible.

8. The columns of A are a basis for $\mathbb{C}^n$.

9. The rank of A is n, $r(A) = n$.

10. The nullity of A is zero, $n(A) = 0$.

11. The determinant of A is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of A.

Theorem ESMM   Eigenvalues of a Scalar Multiple of a Matrix   **195**

Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Theorem EOMP   Eigenvalues Of Matrix Powers   **196**

Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.
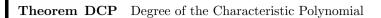
Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Theorem ETM Eigenvalues of the Transpose of a Matrix **199**

Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs **200**

Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Theorem DCP   Degree of the Characteristic Polynomial   **201**

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.
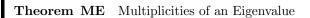
Theorem NEM   Number of Eigenvalues of a Matrix   **202**

Suppose that A is a square matrix of size n with distinct eigenvalues λ_1, λ_2, λ_3, ..., λ_k. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Theorem ME Multiplicities of an Eigenvalue

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Theorem MNEM Maximum Number of Eigenvalues of a Matrix

Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Theorem HMRE   Hermitian Matrices have Real Eigenvalues

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Theorem HMOE   Hermitian Matrices have Orthogonal Eigenvectors

Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Definition SIM   Similar Matrices

Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER   Similarity is an Equivalence Relation

Suppose A, B and C are square matrices of size n. Then

1. A is similar to A. (Reflexive)

2. If A is similar to B, then B is similar to A. (Symmetric)

3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

Theorem SMEE   Similar Matrices have Equal Eigenvalues   **209**

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is $p_A(x) = p_B(x)$.
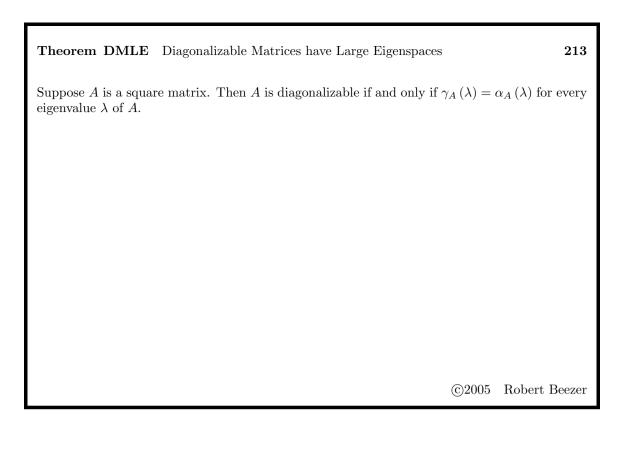

Definition DIM   Diagonal Matrix   **210**

Suppose that $A = (a_{ij})$ is a square matrix. Then A is a **diagonal matrix** if $a_{ij} = 0$ whenever $i \neq j$.

Definition DZM Diagonalizable Matrix   **211**

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Theorem DC Diagonalization Characterization   **212**

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Theorem DMLE   Diagonalizable Matrices have Large Eigenspaces

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.
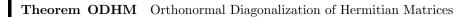
Theorem DED   Distinct Eigenvalues implies Diagonalizable

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Theorem ODHM   Orthonormal Diagonalization of Hermitian Matrices   **215**

Suppose that A is a Hermitian matrix of size n. Then A can be diagonalized by a similarity transformation using an orthonormal transformation.
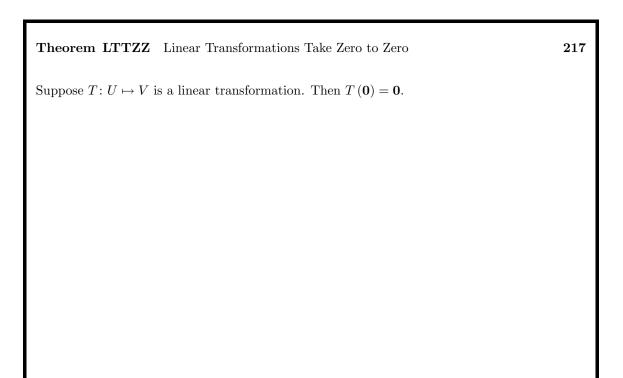
Definition LT   Linear Transformation   **216**

A **linear transformation**, $T : U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$

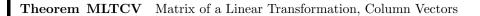2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Theorem LTTZZ   Linear Transformations Take Zero to Zero   **217**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Theorem MBLT   Matrices Build Linear Transformations   **218**

Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem MLTCV Matrix of a Linear Transformation, Column Vectors

Suppose that $T \colon \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations

Suppose that $T \colon U \mapsto V$ is a linear transformation, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\ldots$, $\mathbf{u}_t$ are vectors from U and a_1, a_2, a_3, $\ldots$, a_t are scalars from $\mathbb{C}$. Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \cdots + a_t T(\mathbf{u}_t)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U and $\mathbf{w}$ is a vector from U. Let $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ be scalars from $\mathbb{C}$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then

$$T\left(\mathbf{w}\right) = a_1 T\left(\mathbf{u}_1\right) + a_2 T\left(\mathbf{u}_2\right) + a_3 T\left(\mathbf{u}_3\right) + \cdots + a_n T\left(\mathbf{u}_n\right)$$

Suppose that $T\colon U \mapsto V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}\left(\mathbf{v}\right) = \{\,\mathbf{u} \in U \mid T\left(\mathbf{u}\right) = \mathbf{v}\,\}$$

Definition LTA   Linear Transformation Addition   **223**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Theorem SLTLT   Sum of Linear Transformations is a Linear Transformation   **224**

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \mapsto V$ is a linear transformation.

Definition LTSM   Linear Transformation Scalar Multiplication   **225**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT   Multiple of a Linear Transformation is a Linear Transformation   **226**

Suppose that $T\colon U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \mapsto V$ is a linear transformation.

Theorem VSLT   Vector Space of Linear Transformations

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\text{LT}(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC   Linear Transformation Composition

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Theorem CLTLT   Composition of Linear Transformations is a Linear Transformation   **229**

Suppose that $T \colon U \mapsto V$ and $S \colon V \mapsto W$ are linear transformations. Then $(S \circ T) \colon U \mapsto W$ is a linear transformation.

Definition ILT   Injective Linear Transformation   **230**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Definition NSLT   Null Space of a Linear Transformation   **231**

Suppose $T: U \mapsto V$ is a linear transformation. Then the **null space** of T is the set $\mathcal{N}(T) = \{\, \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \,\}$

Theorem NSLTS   Null Space of a Linear Transformation is a Subspace   **232**

Suppose that $T: U \mapsto V$ is a linear transformation. Then the null space of T, $\mathcal{N}(T)$, is a subspace of U.

Theorem NSPI   Null Space and Pre-Image Suppose $T\colon U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then
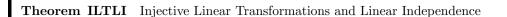
$$T^{-1}(\mathbf{v}) = \{\,\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{N}(T)\} = \mathbf{u} + \mathcal{N}(T)$$

Theorem NSILT   Null Space of an Injective Linear Transformation Suppose that $T\colon U \mapsto V$ is a linear transformation. Then T is injective if and only if the null space of T is trivial, $\mathcal{N}(T) = \{\mathbf{0}\}$.
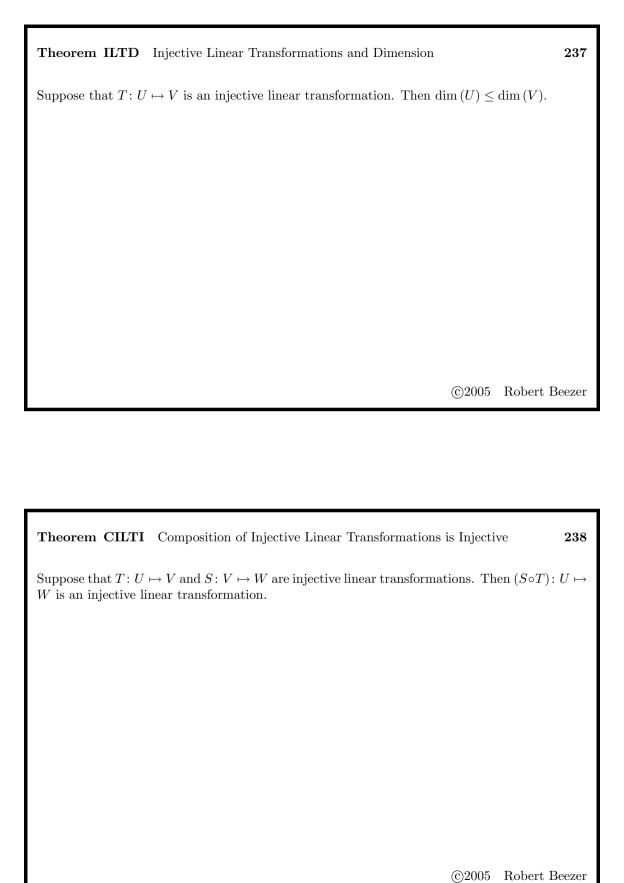
Theorem ILTLI   Injective Linear Transformations and Linear Independence   **235**

Suppose that $T \colon U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.

Theorem ILTB   Injective Linear Transformations and Bases   **236**

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of U. Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V.

Theorem ILTD   Injective Linear Transformations and Dimension   **237**

Suppose that $T\colon U \mapsto V$ is an injective linear transformation. Then $\dim(U) \le \dim(V)$.

Theorem CILTI   Composition of Injective Linear Transformations is Injective   **238**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are injective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is an injective linear transformation.

Definition SLT   Surjective Linear Transformation   **239**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.
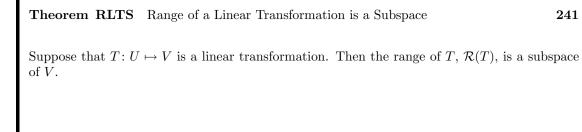
Definition RLT   Range of a Linear Transformation   **240**

Suppose $T\colon U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{\, T(\mathbf{u}) \,|\, \mathbf{u} \in U \,\}$$

Theorem RLTS   Range of a Linear Transformation is a Subspace   **241**

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Theorem RSLT   Range of a Surjective Linear Transformation   **242**

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Suppose that $T \colon U \mapsto V$ is a surjective linear transformation and $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ spans U. Then $R = \{T\left(\mathbf{u}_1\right),\, T\left(\mathbf{u}_2\right),\, T\left(\mathbf{u}_3\right),\, \ldots,\, T\left(\mathbf{u}_t\right)\}$ spans $\mathcal{R}(T)$.

Suppose that $T \colon U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}\left(\mathbf{v}\right) \neq \emptyset$$

Suppose that $T \colon U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T\,(\mathbf{u}_1),\, T\,(\mathbf{u}_2),\, T\,(\mathbf{u}_3),\, \ldots,\, T\,(\mathbf{u}_m)\}$ is a spanning set for V.

Suppose that $T \colon U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective **247**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are surjective linear transformations. Then $(S \circ T)\colon U \mapsto W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation **248**

The **identity linear transformation** on the vector space W is defined as

$$I_W\colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

Definition IVLT   Invertible Linear Transformations   **249**

Suppose that $T \colon U \mapsto V$ is a linear transformation. If there is a function $S \colon V \mapsto U$ such that

$$S \circ T = I_U \qquad\qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT   Invese of a Linear Transformation is a Linear Transformation   **250**

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1} \colon V \mapsto U$ is a linear transformation.

Theorem IILT   Inverse of an Invertible Linear Transformation   **251**

Suppose that $T \colon U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Theorem ILTIS   Invertible Linear Transformations are Injective and Surjective   **252**

Suppose $T \colon U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT   Composition of Invertible Linear Transformations   **253**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T)\colon U \mapsto W$ is an invertible linear transformation.

Theorem ICLT   Inverse of a Composition of Linear Transformations   **254**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Definition IVS   Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T: U \mapsto V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V.

Theorem IVSED   Isomorphic Vector Spaces have Equal Dimension

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Suppose that $T : U \mapsto V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,

$$r(T) = \dim(\mathcal{R}(T))$$

Suppose that $T : U \mapsto V$ is a linear transformation. Then the **nullity** of T, $n(T)$, is the dimension of the null space of T,

$$n(T) = \dim(\mathcal{N}(T))$$

Theorem ROSLT   Rank Of a Surjective Linear Transformation

Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Theorem NOILT   Nullity Of an Injective Linear Transformation

Suppose that $T: U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Theorem RPNDD   Rank Plus Nullity is Domain Dimension   **261**

Suppose that $T: U \mapsto V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

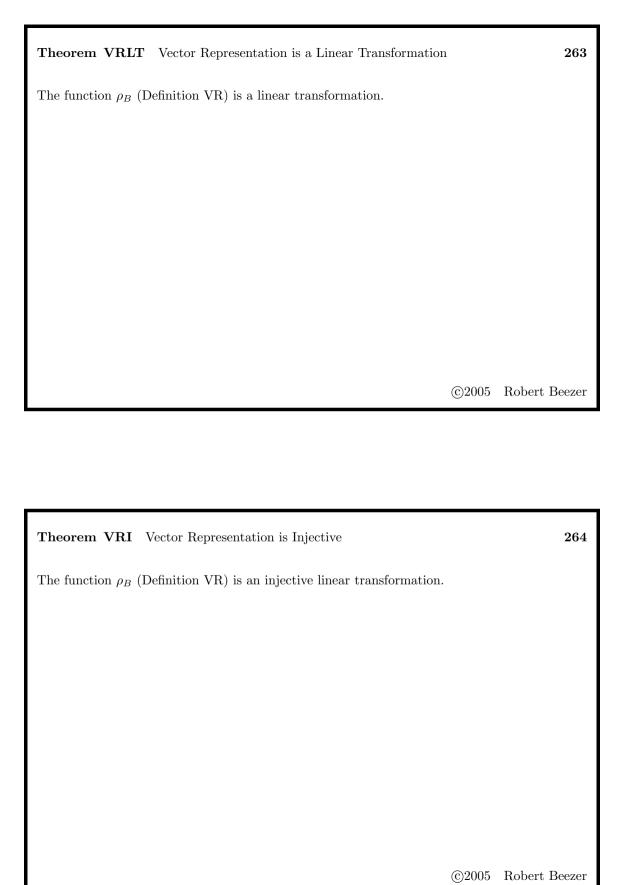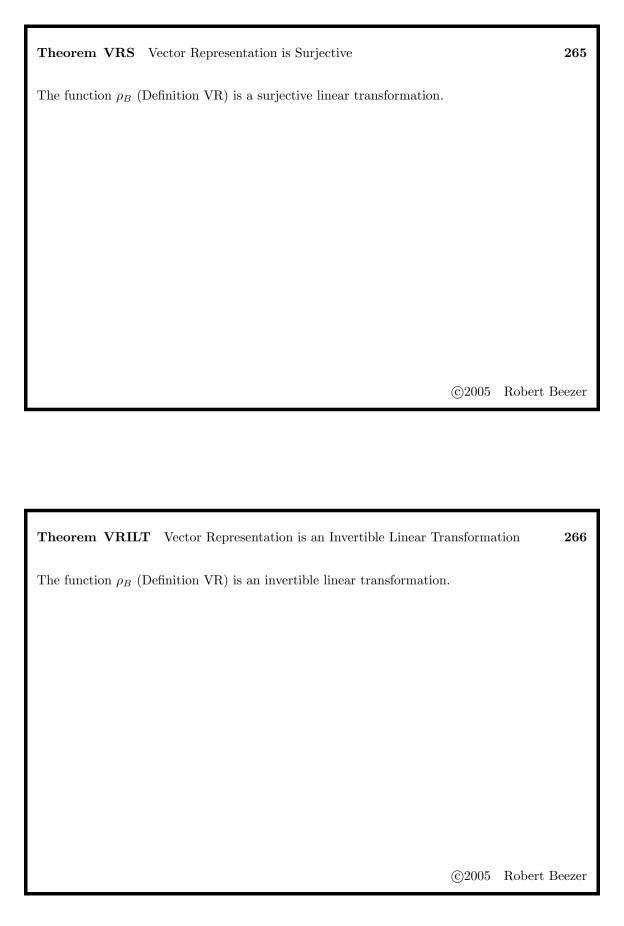**Definition VR**   Vector Representation   **262**

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define a function $\rho_B : V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find some scalars $a_1, a_2, a_3, \ldots, a_n$ so that

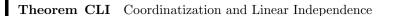$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$\rho_B(\mathbf{w}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Theorem VRLT   Vector Representation is a Linear Transformation   **263**

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI   Vector Representation is Injective   **264**

The function ρ_B (Definition VR) is an injective linear transformation.

Theorem VRS Vector Representation is Surjective

The function ρ_B (Definition VR) is a surjective linear transformation.

Theorem VRILT Vector Representation is an Invertible Linear Transformation

The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces **267**

Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces **268**

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Theorem CLI   Coordinatization and Linear Independence   **269**

Suppose that U is a vector space with a basis B of size n. Then $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of $\mathbb{C}^n$.

Theorem CSS   Coordinatization and Spanning Sets   **270**

Suppose that U is a vector space with a basis B of size n. Then $\mathbf{u} \in \mathcal{S}p(\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\})$ if and only if $\rho_B(\mathbf{u}) \in \mathcal{S}p(\{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\})$.

Definition MR   Matrix Representation   **271**

Suppose that $T\colon U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. The the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = \left[\, \rho_C\left(T\left(\mathbf{u}_1\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_2\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_3\right)\right)\middle|\ldots\middle|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\,\right]$$

Theorem FTMR   Fundamental Theorem of Matrix Representation   **272**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

Theorem MRSLT   Matrix Representation of a Sum of Linear Transformations

Suppose that $T\colon U \mapsto V$ and $S\colon U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

Theorem MRMLT   Matrix Representation of a Multiple of a Linear Transformation

Suppose that $T\colon U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

Theorem MRCLT   Matrix Representation of a Composition of Linear Transformations **275**

Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^{S} M_{B,C}^{T}$$

Theorem INS   Isomorphic Null Spaces   **276**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the null space of T is isomorphic to the null space of $M_{B,C}^{T}$,

$$\mathcal{N}(T) \cong \mathcal{N}\!\left(M_{B,C}^{T}\right)$$

Theorem IR   Isomorphic Ranges   **277**

Suppose that $T\colon U \mapsto V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the range of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{R}\big(M_{B,C}^T\big)$$

Theorem IMR   Invertible Matrix Representations   **278**

Suppose that $T\colon U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \big(M_{B,C}^T\big)^{-1}$$

Definition EELT   Eigenvalue and Eigenvector of a Linear Transformation   **279**

Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Definition CBM   Change-of-Basis Matrix   **280**

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M_{B,C}^{I_V} \\
&= [\rho_C(I_V(\mathbf{v}_1))|\, \rho_C(I_V(\mathbf{v}_2))|\, \rho_C(I_V(\mathbf{v}_3))|\ldots|\rho_C(I_V(\mathbf{v}_n))] \\
&= [\rho_C(\mathbf{u}_1)|\, \rho_C(\mathbf{u}_2)|\, \rho_C(\mathbf{u}_3)|\ldots|\rho_C(\mathbf{u}_n)]
\end{aligned}
$$

Suppose that $\mathbf{u}$ is a vector in the vector space V and B and C are bases of V. Then

$$C_{B,C}\rho_B\left(\mathbf{v}\right) = \rho_C\left(\mathbf{v}\right)$$

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Theorem MRCB   Matrix Representation and Change of Basis   **283**

Suppose that $T: U \mapsto V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^{T} = C_{E,D} M_{C,E}^{T} C_{B,C}$$

Theorem SCB   Similarity and Change of Basis   **284**

Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^{T} = C_{B,C}^{-1} M_{C,C}^{T} C_{B,C}$$

Suppose that $T\colon V \mapsto V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ.