Flashcard Supplement to A First Course in Linear Algebra

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Version 3.00-preview(August 19, 2013)

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Edition Version 3.00 Flashcard Supplement August 19, 2013

Publisher

Robert A. Beezer Congruent Press Gig Harbor, Washington, USA

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Definition SLE System of Linear Equations A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

> $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

where the values of a_{ij} , b_i and x_j , $1 \le i \le m$, $1 \le j \le n$, are from the set of complex numbers, $\mathbb{C}.$

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Definition SSLE Solution of a System of Linear Equations

A solution of a system of linear equations in n variables, $x_1, x_2, x_3, \ldots, x_n$ (such as the system given in Definition SLE), is an ordered list of n complex numbers, $s_1, s_2, s_3, \ldots, s_n$ such that if we substitute s_1 for x_1 , s_2 for x_2 , s_3 for x_3 , s_n for x_n , then for every equation of the system the left side will equal the right side, i.e. each equation is true simultaneously.

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Definition SSSLE Solution Set of a System of Linear Equations 3
The solution set of a linear system of equations is the set which contains every solution to the
system, and nothing more.

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Two systems of linear equations are equivalent if their solution sets are equal.	
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Definition EO Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an equation operation.

- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

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Definition M Matrix

An $m \times n$ matrix is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

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Definition CV Column Vector

A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$. Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component of vector \mathbf{v} in location i of the list, we write $[\mathbf{v}]_i$.

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Definition ZCV Zero Column Vector The zero vector of size *m* is the column vector of size *m* where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0\\0\\0\\\vdots\\0 \end{bmatrix}$$

or defined much more compactly, $[\mathbf{0}]_i = 0$ for $1 \le i \le m$.

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 $\begin{aligned} \textbf{Definition CM Coefficient Matrix} \\ \text{For a system of linear equations,} \\ & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{aligned}$ the coefficient matrix is the $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$

Definition VOC Vector of Constants For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

 $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$

the vector of constants is the column vector of size \boldsymbol{m}

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SOLV Solution Vector For a system of linear equations, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ \vdots $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ the solution vector is the column vector of size n $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

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Definition MRLS Matrix Representation of a Linear System

If A is the coefficient matrix of a system of linear equations and **b** is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the matrix representation of the linear system.

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Definition AM Augmented Matrix

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants **b**. Then the augmented matrix of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (n + 1) is the column vector **b**. This matrix will be written as $[A \mid \mathbf{b}]$.

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Definition RO Row Operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a row operation.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows *i* and *j*.
- 2. αR_i : Multiply row *i* by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row *i* by the scalar α and add to row *j*.

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Definition REM Row-Equivalent Matrices

Two matrices, A and B, are row-equivalent if one can be obtained from the other by a sequence of row operations.

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Theorem REMES Row-Equivalent Matrices represent Equivalent Systems 17 Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

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Definition RREF Reduced Row-Echelon Form

A matrix is in reduced row-echelon form if it meets all of the following conditions:

- 1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries will be called a zero row and the leftmost nonzero entry of a nonzero row will be called a leading 1. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a pivot column. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Theorem REMEF Row-Equivalent Matrix in Echelon FormSuppose A is a matrix. Then there is a matrix B so that
1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

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Theorem RREFU Reduced Row-Echelon Form is Unique

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

Definition CS Consistent System 21 A system of linear equations is consistent if it has at least one solution. Otherwise, the system is called inconsistent.

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Definition IDV Independent and Dependent Variables

Suppose A is the augmented matrix of a consistent system of linear equations and B is a rowequivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column). Then the variable x_j is dependent. A variable that is not dependent is called independent or free.

Theorem RCLS Recognizing Consistency of a Linear System

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n + 1 of B.

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Theorem ISRN Inconsistent Systems, r and n

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

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Theorem CSRN Consistent Systems, r and n

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n - r free variables.

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Theorem PSSLS Possible Solution Sets for Linear Systems	27
A system of linear equations has no solutions, a unique solution or infinitely many solution	ns.

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Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions28 Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions.

Definition HS Homogeneous System			29
A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is homogeneous if the vector of constants	is t	he 2	zero
vector, in other words, if $\mathbf{b} = 0$.			

Theorem HSC Homogeneous Systems are Consistent Suppose that a system of linear equations is homogeneous. Then the system is consistent.	30
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Definition TSHSE Trivial Solution to Homogeneous Systems of Equations 31 Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0$, , $x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the trivial solution.

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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 32

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

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Definition NSM	Null Space of a Matrix	33
The null space of a m	natrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions	s to
the homogeneous syst	tem $\mathcal{LS}(A, 0)$.	

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Definition SQM Square Matrix

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A matrix with m rows and n columns is square if m = n. In this case, we say the matrix has size n. To emphasize the situation when a matrix is not square, we will call it rectangular.

Definition NM Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, in other words, the system has only the trivial solution. Then we say that A is a nonsingular matrix. Otherwise we say A is a singular matrix.

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Definition IM Identity Matrix The $m \times m$ identity matrix, I_m , is defined by

$$\left[I_m\right]_{ij} = \begin{cases} 1 & i=j\\ 0 & i\neq j \end{cases} \qquad 1 \le i, \, j \le m$$

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Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 37 Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

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Theorem NMTNS Nonsingular Matrices have Trivial Null Spaces 38 Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

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Theorem NMUS Nonsingular Matrices and Unique Solutions 3	39
Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbb{I})$	b)
has a unique solution for every choice of the constant vector \mathbf{b} .	

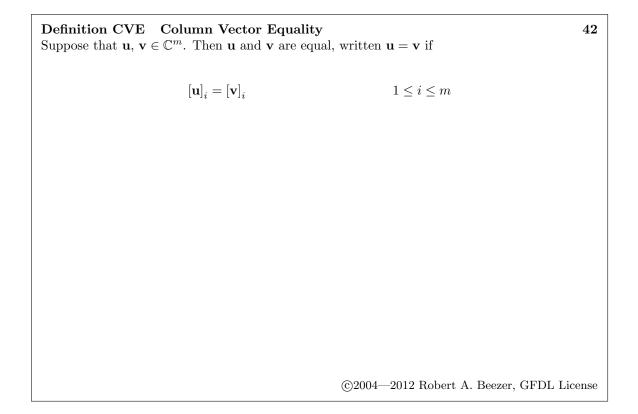
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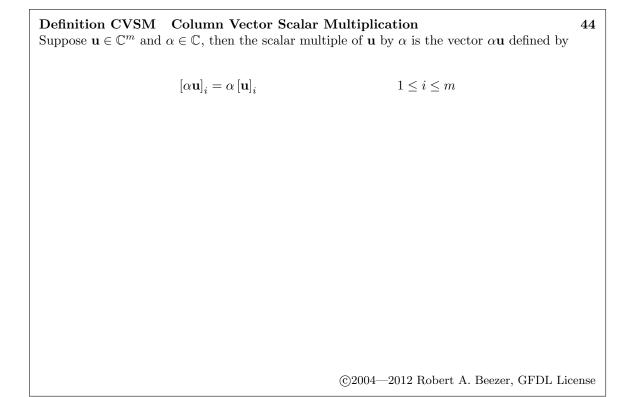
Theorem NME1Nonsingular Matrix Equivalences, Round 1Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .

Definition VSCV	Vector Space of Column Vectors	41
The vector space \mathbb{C}^m	is the set of all column vectors (Definition CV) of size m with entries	s from
the set of complex nu	$mbers, \mathbb{C}.$	



Definition CVA Column Vector Addition 43 Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. The sum of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by $[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$ $1 \le i \le m$



Theorem VSPCV Vector Space Properties of Column Vectors

Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- ZC Zero Vector, Column Vectors: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- OC One, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

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Definition LCCV Linear Combination of Column Vectors

Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n$

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Theorem SLSLC Solutions to Linear Systems are Linear Combinations 47 Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}n$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if \mathbf{b} equals the linear combination of the columns of A formed with the entries of \mathbf{x} ,

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \dots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

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Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that $[A | \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n + 1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$ of size n by

$$\begin{split} \left[\mathbf{c} \right]_{i} &= \begin{cases} 0 & \text{if } i \in F \\ \left[B \right]_{k,n+1} & \text{if } i \in D, \, i = d_{k} \end{cases} \\ \left[\mathbf{u}_{j} \right]_{i} &= \begin{cases} 1 & \text{if } i \in F, \, i = f_{j} \\ 0 & \text{if } i \in F, \, i \neq f_{j} \\ -\left[B \right]_{k,f_{j}} & \text{if } i \in D, \, i = d_{k} \end{cases} \end{split}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} | \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$

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Theorem PSPHS Particular Solution Plus Homogeneous Solutions 49 Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$. Then **y** is a solution to $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

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Definition SSCV Span of a Set of Column Vectors 50 Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p | \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n - r vectors \mathbf{z}_j , $1 \le j \le n - r$ of size n as

$$\left[\mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \} \rangle$$

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Definition RLDCV Relation of Linear Dependence for Column Vectors Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is the trivial relation of linear dependence on S.

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Definition LICV Linear Independence of Column Vectors

The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.

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Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems 54

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

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 $\mathbf{53}$

Theorem LIVRN Linearly Independent Vectors, r and n

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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Theorem MVSLD More Vectors than Size implies Linear Dependence 56 Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is the set of vectors in \mathbb{C}^m , and that n > m. Then S is a linearly dependent set.

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Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 57 Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

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 $\mathbf{58}$

Theorem NME2Nonsingular Matrix Equivalences, Round 2Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A form a linearly independent set.

Theorem BNS Basis for Null Spaces

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \leq j \leq n-r$ of size n as

$$\left[\mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Define the set $S = {\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

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Theorem DLDS Dependency in Linearly Dependent Sets

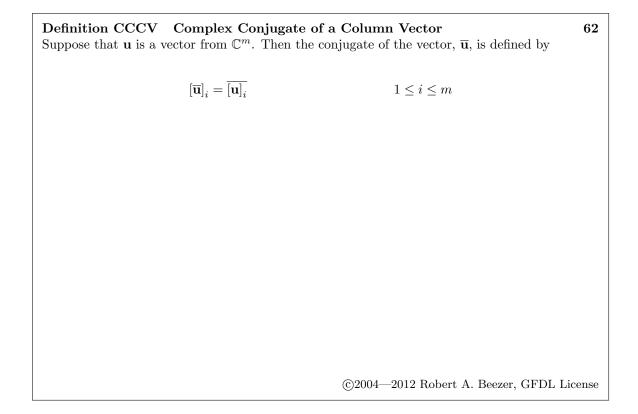
Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index $t, 1 \le t \le n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

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60

Theorem BS Basis of a Span 61 Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = {d_1, d_2, d_3, \dots, d_r}$ the set of column indices corresponding to the pivot columns of B. Then 1. $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}}$ is a linearly independent set.

2. $W = \langle T \rangle$.



63 Theorem CRVA Conjugation Respects Vector Addition Suppose **x** and **y** are two vectors from \mathbb{C}^m . Then $\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$

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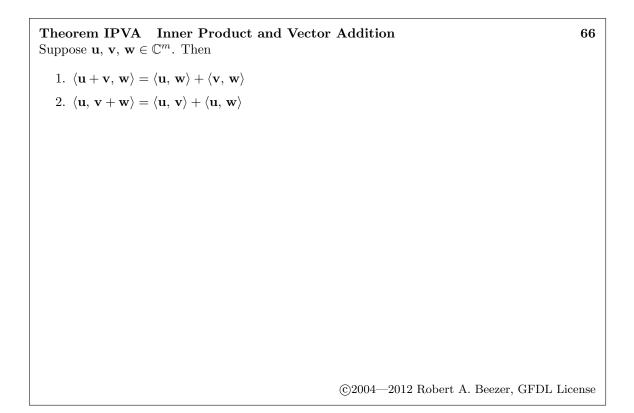


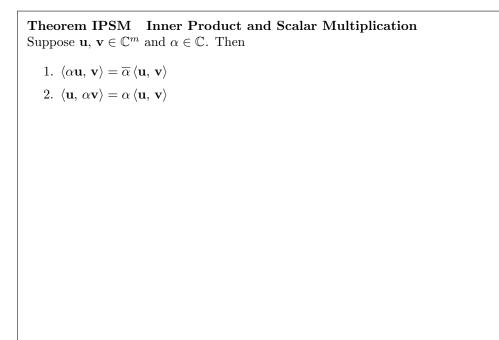
 $\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$

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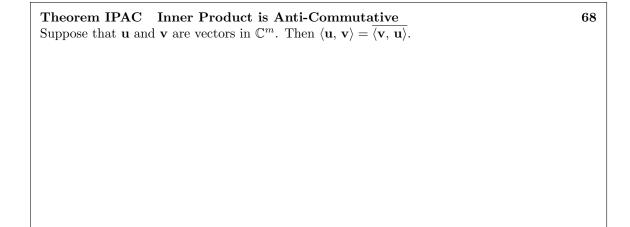
 $\mathbf{64}$

Definition IP Inner Product Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the inner product of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} , $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{[\mathbf{u}]_1} [\mathbf{v}]_1 + \overline{[\mathbf{u}]_2} [\mathbf{v}]_2 + \overline{[\mathbf{u}]_3} [\mathbf{v}]_3 + \dots + \overline{[\mathbf{u}]_m} [\mathbf{v}]_m = \sum_{i=1}^m \overline{[\mathbf{u}]_i} [\mathbf{v}]_i$ (3)





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$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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Theorem IPN Inner Products and Norms	70
Suppose that u is a vector in \mathbb{C}^m . Then $\ \mathbf{u}\ ^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.	

Theorem PIP	Positive Inner	Products	
Suppose that \mathbf{u} is	a vector in \mathbb{C}^m .	Then $\langle \mathbf{u}, \mathbf{u} \rangle$	$\rangle \geq 0$ with equality if and only if $\mathbf{u} = 0$.

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 $\mathbf{72}$

Definition OV Orthogonal Vectors

A pair of vectors, **u** and **v**, from \mathbb{C}^m are orthogonal if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

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Definition OSV Orthogonal Set of Vectors 73 Suppose that $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is a set of vectors from \mathbb{C}^m . Then S is an orthogonal set if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

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Definition SUV Standard Unit Vectors

Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$\left[\mathbf{e}_{j}\right]_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

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Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

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Theorem GSP Gram-Schmidt Procedure 76 Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \le i \le p$ by $(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \mathbf{u})$

$$\mathbf{u}_{i} = \mathbf{v}_{i} - \frac{\langle \mathbf{u}_{1}, \mathbf{v}_{i} \rangle}{\langle \mathbf{u}_{1}, \mathbf{u}_{1} \rangle} \mathbf{u}_{1} - \frac{\langle \mathbf{u}_{2}, \mathbf{v}_{i} \rangle}{\langle \mathbf{u}_{2}, \mathbf{u}_{2} \rangle} \mathbf{u}_{2} - \frac{\langle \mathbf{u}_{3}, \mathbf{v}_{i} \rangle}{\langle \mathbf{u}_{3}, \mathbf{u}_{3} \rangle} \mathbf{u}_{3} - \dots - \frac{\langle \mathbf{u}_{i-1}, \mathbf{v}_{i} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Definition ONS OrthoNormal Set 77 Suppose $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ is an orthogonal set of vectors such that $||\mathbf{u}_i|| = 1$ for all $1 \le i \le n$. Then S is an orthonormal set of vectors.

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Definition VSM Vector Space of $m \times n$ Matrices 78 The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

Definition ME Matrix Equality 79 The $m \times n$ matrices A and B are equal, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

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Definition MA Matrix Addition 80 Given the $m \times n$ matrices A and B, define the sum of A and B as an $m \times n$ matrix, written A + B, according to

 $[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$

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Definition MSM Matrix Scalar Multiplication Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of A is an $m \times n$ matrix, written αA and defined according to $[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, 1 \le j \le n$

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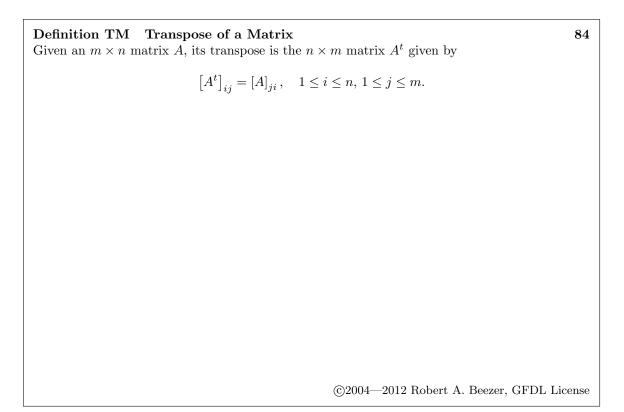
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Theorem VSPM Vector Space Properties of Matrices

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices: If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices: If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices: If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices: If A, B, $C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices: There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices: If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices: If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One, Matrices: If $A \in M_{mn}$, then 1A = A.

Definition ZM Zero Matrix 83 The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$.



Definition SYM Symmetric Matrix The matrix A is symmetric if $A = A^t$.

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Theorem SMS Symmetric Matrices are Square Suppose that A is a symmetric matrix. Then A is square.	86

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Theorem TMA Transpose and Matrix Addition Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

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Theorem TMSM	Transpose and Matrix Scalar Multiplication	
Suppose that $\alpha \in \mathbb{C}$	and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.	

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Theorem TT Transpose of a Transpose

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

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Definition CCMComplex Conjugate of a Matrix90Suppose A is an $m \times n$ matrix. Then the conjugate of A, written \overline{A} is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{\left[A\right]_{ij}}$$

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Theorem CRMA	Conjugation Respects Matrix Addition
Suppose that A and A	B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.	92

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Theorem CCM	Conjugate of the Conjugate of a Matrix
Suppose that A is a	In $m \times n$ matrix. Then $\overline{(A)} = A$.

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Theorem MCT Matrix Conjugation and Transposes	94
Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = (\overline{A})^t$.	

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Definition A Adjoint If A is a matrix, then its adjoint is $A^* = (\overline{A})^t$.

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Theorem AMSM Adjoint and Matrix Scalar Multiplication Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$.

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Theorem AA Adjoint of an Adjoint Suppose that A is a matrix. Then $(A^*)^* = A$.

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Definition MVP Matrix-Vector Product

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Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the matrix-vector product of A with \mathbf{u} is the linear combination

 $A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$

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Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 100 The set of solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ equals the set of solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.

Theorem EMMVP Equal Matrices and Matrix-Vector Products 101 Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then A = B.

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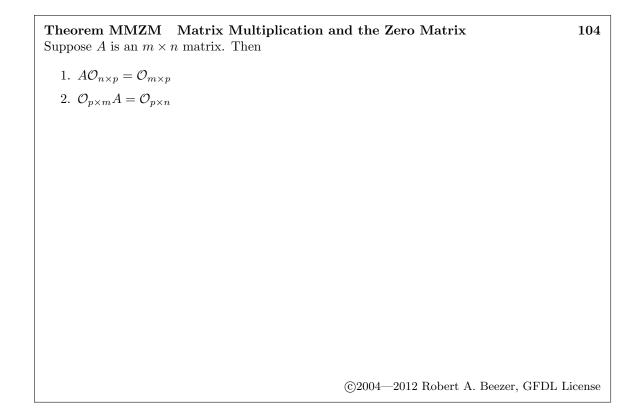
Definition MM Matrix Multiplication

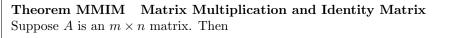
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Suppose A is an $m \times n$ matrix and $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ are the columns of an $n \times p$ matrix B. Then the matrix product of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

 $AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$

Theorem EMP Entries of Matrix Products Suppose *A* is an $m \times n$ matrix and *B* is an $n \times p$ matrix. Then for $1 \le i \le m, 1 \le j \le p$, the individual entries of *AB* are given by $[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$ $= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$

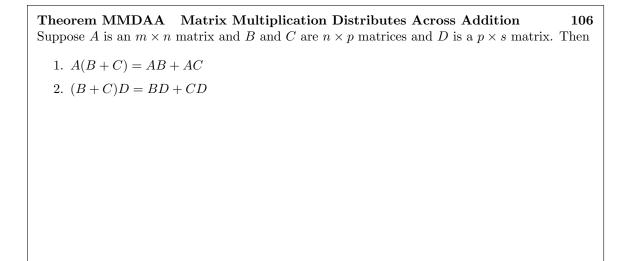




1. $AI_n = A$

2. $I_m A = A$

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 107 Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

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Theorem MMA Matrix Multiplication is Associative 108 Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then A(BD) = (AB)D.

Theorem MMIP Matrix Multiplication and Inner Products 109

 If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then
 $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\mathbf{u}}^t \mathbf{v} = \mathbf{u}^* \mathbf{v}$

Matrix Multiplication and Complex Conjugation n matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \overline{A}\overline{B}$.	110
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Theorem MMT	Matrix Multiplication and Transposes
Suppose A is an m	$\times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

111

Theorem MMAD Matrix Multiplication and Adjoints Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^* = B^*A^*$.	112

Theorem AIP Adjoint and Inner Product Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$.

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Definition HMHermitian MatrixThe square matrix A is Hermitian (or self-adjoint)	if $A = A^*$.
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Theorem HMIP Hermitian Matrices and Inner Products 115 Suppose that A is a square matrix of size n. Then A is Hermitian if and only if $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

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Definition MI Matrix Inverse

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is invertible and B is the inverse of A. In this situation, we write $B = A^{-1}$.

Theorem TTMI Two-by-Two Matrix Inverse Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Theorem CINM Computing the Inverse of a Nonsingular Matrix 118 Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ = I_n$.

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Theorem MIU Matrix Inverse is Unique Suppose the square matrix A has an inverse. Then A^{-1} is unique.

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Theorem SS Socks and Shoes

Suppose A and B are invertible matrices of size n. Then AB is an invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$.

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Theorem MIMI Matrix Inverse of a Matrix Inverse			
Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.			

Theorem MIT Matrix Inverse of a Transpose	122	
Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.		
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Theorem MISM Matrix Inverse of a Scalar Multiple

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

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Theorem NPNT Nonsingular Product has Nonsingular Terms 124Suppose that A and B are square matrices of size n. The product AB is nonsingular if and only if A and B are both nonsingular.

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Theorem OSIS	One-Sided Inverse is Sufficient
Suppose A and B a	are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

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Theorem NI Nonsingularity is Invertibility

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Theorem NME3Nonsingular Matrix Equivalences, Round 3 Suppose that A is a square matrix of size n . The following are equivalent.	127
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	

		Nonsingular Coe		128
Suppose that A is not	onsingular. Ther	the unique solution	to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.	
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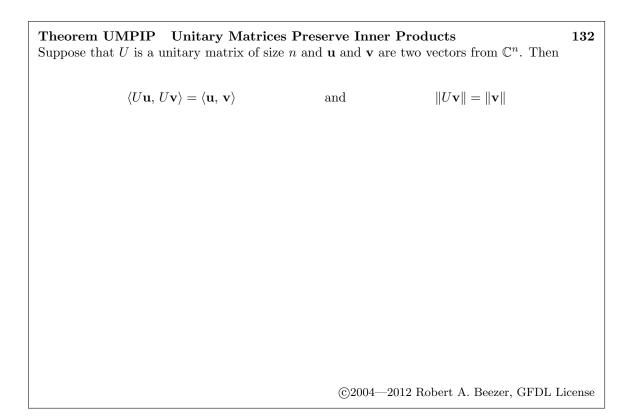
Г

Definition UM	Unitary Matrices	129
Suppose that U is	a square matrix of size n such that $U^*U = I_n$. Then we say U is unitary.	

Theorem UMI Unitary Matrices are Inv	
Suppose that U is a unitary matrix of size n . The second sec	hen U is nonsingular, and $U^{-1} = U^*$.
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	<u> </u>

Γ

Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets 131 Suppose that $S = \{A_1, A_2, A_3, \dots, A_n\}$ is the set of columns of a square matrix A of size n. Then A is a unitary matrix if and only if S is an orthonormal set.



Definition CSM Column Space of a Matrix 133Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$. Then the column space of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

 $\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$

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Theorem CSCSColumn Spaces and Consistent Systems134Suppose A is an
$$m \times n$$
 matrix and **b** is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

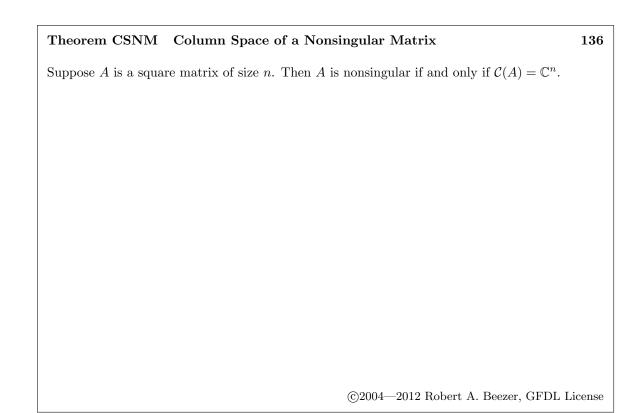
Theorem BCS Basis of the Column Space

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.

2. $\mathcal{C}(A) = \langle T \rangle$.

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Theorem NME4 Nonsingular Matrix Equivalences, Round 4 Suppose that A is a square matrix of size n . The following are equivalent.	137
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}.$	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of b .	
5. The columns of A are a linearly independent set.	
6. A is invertible.	
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.	

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Definition RSM Row Space of a Matrix 138 Suppose A is an $m \times n$ matrix. Then the row space of A, $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

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Theorem REMRS Row-Equivalent Matrices have equal Row Spaces	139
Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.	

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Theorem BRSBasis for the Row Space140Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let
S be the set of nonzero columns of B^t . Then140

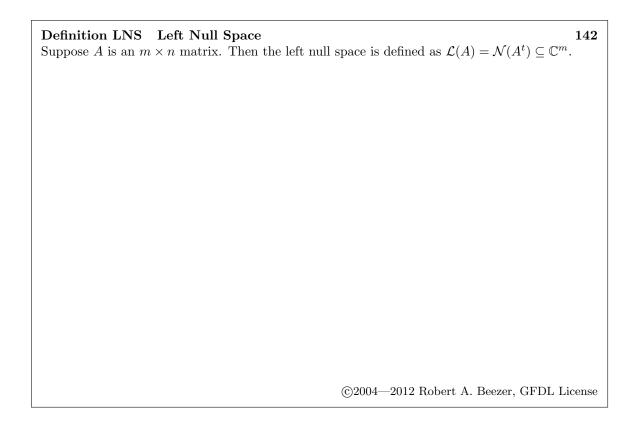
1. $\mathcal{R}(A) = \langle S \rangle$.

2. S is a linearly independent set.

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Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

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Definition EEF Extended Echelon Form

Suppose A is an $m \times n$ matrix. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the extended reduced row-echelon form of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m - r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$

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Theorem PEEF Properties of Extended Echelon Form144Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then1. J is nonsingular.1. J is nonsingular.2. B = JA.3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.5. L is in reduced row-echelon form, has no zero rows and has m - r pivot columns.

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Theorem FS Four Subsets

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m - r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

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Definition VS Vector Space

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Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space over \mathbb{C} if the following ten properties hold.

- AC Additive Closure: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- Z Zero Vector: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses: If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One: If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

Theorem ZVU Zero Vector is Unique Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

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Theorem AIU Additive Inverses are Unique	148
Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.	
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Theorem ZSSM	Zero Scalar in Scalar Multiplication	L
Suppose that V is a	vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = 0$.	

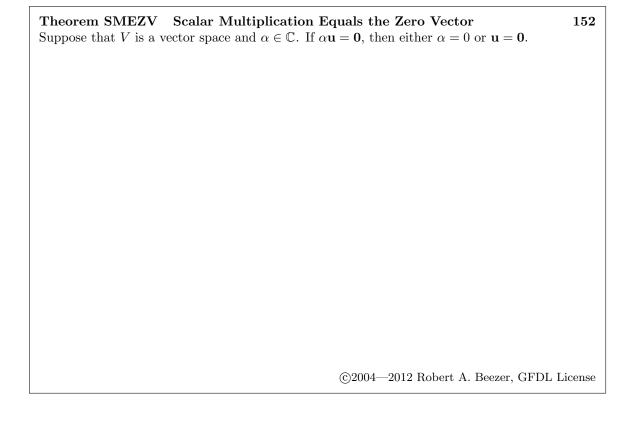
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Theorem ZVSM Zero Vector in Scalar Multiplication Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha 0 = 0$.	150

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Theorem AISM	Additive Inverses from Scalar Multiplication
Suppose that V is a	vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

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Definition S Subspace

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Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of $V, W \subseteq V$. Then W is a subspace of V.

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Theorem TSS Testing Subsets for Subspaces

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

Definition TS	Trivial	Subspaces	
Given the vector	space V ,	the subspaces V and $\{0\}$ are each called a trivial subspace.	

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Theorem NSMS Null Space of a Matrix is a Subspace 15 Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of \mathbb{C}^n . 15	6
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Definition LCLinear Combination157Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$ and n scalars
 $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$, their linear combination is the vector $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$.

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Definition SS Span of a Set 158 Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

Theorem SSS Span of a Set is a Subspace	159
Suppose V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t} \subseteq V$,	their span,
$\langle S \rangle$, is a subspace.	

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Theorem CSMS Column Space of a Matrix is a Subspace Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .	160

Theorem RSMS	Row Space of a Matrix is a Subspace	
Suppose that A is an	$m m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .	

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	Null Space of a Matrix is a Subspace <i>n</i> matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .	162

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Definition RLD Relation of Linear Dependence 163 Suppose that V is a vector space. Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$, an equation of the form

 $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$

is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is a trivial relation of linear dependence on S.

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Definition LI Linear Independence

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Suppose that V is a vector space. The set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ from V is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.

Definition SSVS Spanning Set of a Vector Space 165Suppose V is a vector space. A subset S of V is a spanning set of V if $\langle S \rangle = V$. In this case, we also frequently say S spans V.

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Theorem VRRB Vector Representation Relative to a Basis 166 Suppose that V is a vector space and $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ is a linearly independent set that spans V. Let **w** be any vector in V. Then there exist unique scalars $a_1, a_2, a_3, \ldots, a_m$ such that

 $\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$

Definition B Basis

Suppose V is a vector space. Then a subset $S \subseteq V$ is a basis of V if it is linearly independent and spans V.

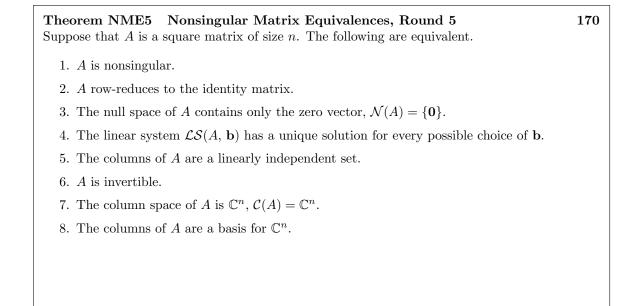
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Theorem SUVB Standard Unit Vectors are a Basis 168 The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_i | 1 \le i \le m\}$ is a basis for the vector space \mathbb{C}^m .

Theorem CNMBColumns of Nonsingular Matrix are a Basis169Suppose that A is a square matrix of size m. Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

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Theorem COB Coordinates and Orthonormal Bases 171 Suppose that $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{v}_1, \, \mathbf{w} \rangle \, \mathbf{v}_1 + \langle \mathbf{v}_2, \, \mathbf{w} \rangle \, \mathbf{v}_2 + \langle \mathbf{v}_3, \, \mathbf{w} \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{v}_p, \, \mathbf{w} \rangle \, \mathbf{v}_p$$

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Theorem UMCOB Unitary Matrices Convert Orthonormal Bases 172 Let A be an $n \times n$ matrix and $B = {\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n}$ be an orthonormal basis of \mathbb{C}^n . Define

 $C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$

Then A is a unitary matrix if and only if C is an orthonormal basis of \mathbb{C}^n .

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Definition D Dimension

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the dimension of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

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Theorem SSLD Spanning Sets and Linear Dependence 174 Suppose that $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t}$ is a finite set of vectors which spans the vector space V. Then any set of t + 1 or more vectors from V is linearly dependent.

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Theorem BIS Bases have Identical Sizes	175
Suppose that V is a vector space with a finite basis B and a second basis C . Then B and C	7 have
the same size.	

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Theorem	DCM	Dimension	of \mathbb{C}^m
The dimen	sion of \mathbb{C}	\mathbb{C}^m (Example)	VSCV) is m .

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Theorem DP Dimension of P_n The dimension of P_n (Example VSP) is n + 1.

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Theorem DM Dimension of M_{mn} The dimension of M_{mn} (Example VSM) is mn.

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Definition NOM Nullity Of a Matrix 179 Suppose that A is an $m \times n$ matrix. Then the nullity of A is the dimension of the null space of $A, n(A) = \dim(\mathcal{N}(A)).$

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Definition ROM Rank Of a Matrix 180 Suppose that A is an $m \times n$ matrix. Then the rank of A is the dimension of the column space of $A, r(A) = \dim (\mathcal{C}(A)).$

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Theorem CRN Computing Rank and Nullity

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

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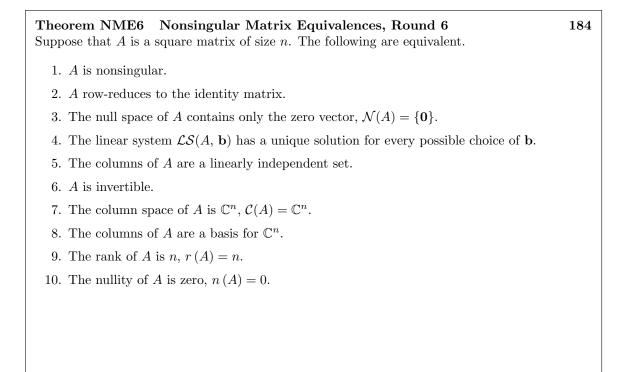
Theorem RPNC Rank Plus Nullity is Columns Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.	182
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Theorem RNNM Rank and Nullity of a Nonsingular Matrix

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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Theorem ELIS Extending Linearly Independent Sets 185 Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose w is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

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Theorem G Goldilocks

Suppose that V is a vector space of dimension t. Let $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$ be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

Theorem PSSD Proper Subspaces have Smaller Dimension

Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then dim $(U) < \dim(V)$.

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Theorem EDYES Equal Dimensions Yields Equal Subspaces 188 Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then U = V.

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Theorem RMRT Rank of a Matrix is the Rank of the Transpose Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

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Theorem DFS Dimensions of Four Subspaces

Suppose that A is an $m\times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

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Definition ELEM Elementary Matrices

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$\left[E_{i}\left(\alpha\right)\right]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

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Theorem EMDRO Elementary Matrices Do Row Operations

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row *i* by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha) A$.

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Theorem EMN	Elementary Matrices are Nonsingula	ır
If E is an elementar	y matrix, then E is nonsingular.	

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Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices194 Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \ldots, E_t$ so that $A = E_1 E_2 E_3 \ldots E_t$.

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Definition SM SubMatrix Suppose that A is an $m \times n$ matrix. Then the submatrix A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

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Definition DM Determinant of a Matrix 196 Suppose A is a square matrix. Then its determinant, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

1. If A is a 1×1 matrix, then det $(A) = [A]_{11}$.

2. If A is a matrix of size n with $n \ge 2$, then

$$det (A) = [A]_{11} det (A (1|1)) - [A]_{12} det (A (1|2)) + [A]_{13} det (A (1|3)) - [A]_{14} det (A (1|4)) + \dots + (-1)^{n+1} [A]_{1n} det (A (1|n))$$

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Theorem DMST Determinant of Matrices of Size Two 197 Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then det (A) = ad - bc.

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Theorem DER Determinant Expansion about Rows198Suppose that A is a square matrix of size n. Then for $1 \le i \le n$ $det (A) = (-1)^{i+1} [A]_{i1} det (A(i|1)) + (-1)^{i+2} [A]_{i2} det (A(i|2)) + (-1)^{i+3} [A]_{i3} det (A(i|3)) + \dots + (-1)^{i+n} [A]_{in} det (A(i|n))$ which is known as expansion about row i.

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Theorem DT Determinant of the Transpose Suppose that A is a square matrix. Then $det(A^t) = det(A)$.

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Theorem DEC Determinant Expansion about Columns200Suppose that A is a square matrix of size n. Then for $1 \le j \le n$ $det (A) = (-1)^{1+j} [A]_{1j} det (A(1|j)) + (-1)^{2+j} [A]_{2j} det (A(2|j)) + (-1)^{3+j} [A]_{3j} det (A(3|j)) + \dots + (-1)^{n+j} [A]_{nj} det (A(n|j))$ which is known as expansion about column j.

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Theorem DZRC Determinant with Zero Row or Column 201 Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

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Theorem DRCS Determinant for Row or Column Swap

202 Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Theorem DRCM Determinant for Row or Column Multiples	203
Suppose that A is a square matrix. Let B be the square matrix obtained from A be	by multiplying
a single row by the scalar α , or by multiplying a single column by the scalar α . The	hen $\det(B) =$
$\alpha \det(A).$	

Theorem DERC Determinant with Equations Suppose that A is a square matrix with two equations	al Rows or Columns 204 al rows, or two equal columns. Then $det(A) = 0$.
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Theorem DRCMA Determinant for Row or Column Multiples and Addition 205 Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then det $(B) = \det(A)$.

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Theorem DIM Determinant of the Identity Matrix

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For every $n \ge 1$, det $(I_n) = 1$.

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 Theorem DEM
 Determinants of Elementary Matrices
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 For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,
 and an elementary matrix (Definition ELEM) we have the determinants,

1. det $(E_{i,j}) = -1$ 2. det $(E_i(\alpha)) = \alpha$

3. det $(E_{i,j}(\alpha)) = 1$

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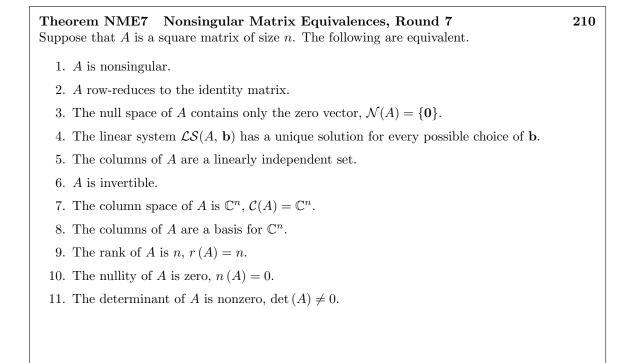
Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication208 Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

 $\det (EA) = \det (E) \det (A)$

Theorem SMZD Singular Matrices have Zero Determinants

Let A be a square matrix. Then A is singular if and only if det(A) = 0.

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Theorem DRMM	Determinant Respects Matrix Multiplication	211
Suppose that A and B	B are square matrices of the same size. Then $\det(AB) = \det(A) \det(A)$	(B).

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Definition EEMEigenvalues and Eigenvectors of a Matrix212Suppose that A is a square matrix of size $n, \mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Thenwe say \mathbf{x} is an eigenvector of A with eigenvalue λ if

 $A\mathbf{x} = \lambda \mathbf{x}$

Theorem EMHE	Every Matrix Has an Eigenvalue	
Suppose A is a square	re matrix. Then A has at least one eigenvalue.	

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Definition CPCharacteristic Polynomial214Suppose that A is a square matrix of size n. Then the characteristic polynomial of A is the
polynomial $p_A(x)$ defined by214

 $p_A(x) = \det\left(A - xI_n\right)$

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Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials 215

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

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Definition EM Eigenspace of a Matrix

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Suppose that A is a square matrix and λ is an eigenvalue of A. Then the eigenspace of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.

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Theorem EMSEigenspace for a Matrix is a Subspace217Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $\mathcal{E}_A(\lambda)$ is a subspace of the vector space \mathbb{C}^n .

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Theorem EMNS Eigenspace of a Matrix is a Null Space Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

 $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$

Definition AME Algebraic Multiplicity of an Eigenvalue 219 Suppose that A is a square matrix and λ is an eigenvalue of A. Then the algebraic multiplicity of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

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Definition GME Geometric Multiplicity of an Eigenvalue

Suppose that A is a square matrix and λ is an eigenvalue of A. Then the geometric multiplicity of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 221

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

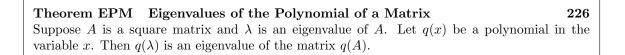
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Theorem NME8 Nonsingular Matrix Equivalences, Round 8 223Suppose that A is a square matrix of size n. The following are equivalent. 1. A is nonsingular. 2. A row-reduces to the identity matrix. 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$ 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} . 5. The columns of A are a linearly independent set. 6. A is invertible. 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$. 8. The columns of A are a basis for \mathbb{C}^n . 9. The rank of A is n, r(A) = n. 10. The nullity of A is zero, n(A) = 0. 11. The determinant of A is nonzero, $\det(A) \neq 0$. 12. $\lambda = 0$ is not an eigenvalue of A. ©2004—2012 Robert A. Beezer, GFDL License

Theorem ESMM	Eigenvalues of a Scalar Multiple of a Matrix 22	4
Suppose A is a square	re matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of αA .	
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Theorem EOMPEigenvalues Of Matrix Powers225Suppose A is a square matrix, λ is an eigenvalue of A, and $s \ge 0$ is an integer. Then λ^s is an eigenvalue of A^s .

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Theorem EIMEigenvalues of the Inverse of a Matrix227Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then λ^{-1} is an eigenvalue of the matrix A^{-1} .

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Theorem ETMEigenvalues of the Transpose of a Matrix228Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t .

Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs 229

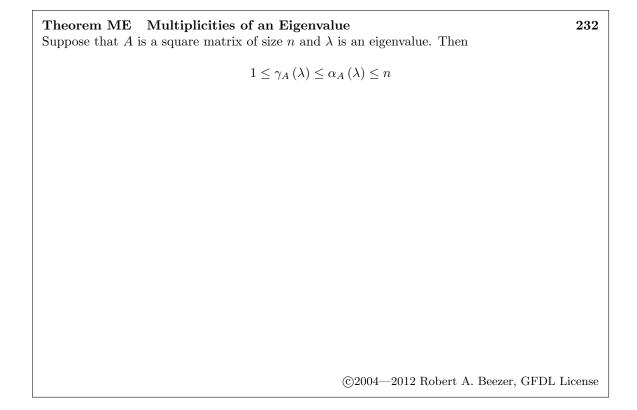
Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue λ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

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Theorem DCPDegree of the Characteristic Polynomial230Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$,
has degree n.

Theorem NEMNumber of Eigenvalues of a Matrix231Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ are the distinct eigenvalues of a square matrix A of size n. Then

$$\sum_{i=1}^{k} \alpha_A\left(\lambda_i\right) = n$$



Theorem MNEM	Maximum Numbe	r of Eigenvalues of a Matrix	233
Suppose that A is a sq	uare matrix of size n .	Then A cannot have more than n distinct eig	genvalues.

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Theorem HMRE Hermitian Matrices have Real EigenvaluesSuppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.	234
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Theorem HMOEHermitian Matrices have Orthogonal Eigenvectors235Suppose that A is a Hermitian matrix and x and y are two eigenvectors of A for different
eigenvalues. Then x and y are orthogonal vectors.235

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Definition SIM Similar Matrices

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Suppose A and B are two square matrices of size n. Then A and B are similar if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$.

Theorem SER Similarity is an Equivalence Relation Suppose A , B and C are square matrices of size n . Then	237
1. A is similar to A . (Reflexive)	
2. If A is similar to B , then B is similar to A. (Symmetric)	
3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)	

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Theorem SMEESimilar Matrices have Equal Eigenvalues238Suppose A and B are similar matrices.Then the characteristic polynomials of A and B areequal, that is, $p_A(x) = p_B(x)$.

Definition DIM	Diagonal Matri	ix		239
Suppose that A is a	square matrix. T	Then A is a diagona	l matrix if $[A]_{ij} =$	0 whenever $i \neq j$.

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Definition DZMDiagonalizable Matrix 24 Suppose A is a square matrix. Then A is diagonalizable if A is similar to a diagonal matrix.	40
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Theorem DC Diagonalization Characterization	241
Suppose A is a square matrix of size n . Then A is diagonalizable if and only if there exis	sts a
linearly independent set S that contains n eigenvectors of A .	

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Theorem DMFE Diagonalizable Matrices have Full Eigenspaces 242 Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Theorem DEDDistinct Eigenvalues implies Diagonalizable243Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

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Definition LT Linear Transformation

A linear transformation, $T: U \to V$, is a function that carries elements of the vector space U (called the domain) to the vector space V (called the codomain), and which has two additional properties

1. $T(\mathbf{u}_{1} + \mathbf{u}_{2}) = T(\mathbf{u}_{1}) + T(\mathbf{u}_{2})$ for all $\mathbf{u}_{1}, \mathbf{u}_{2} \in U$

2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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Theorem LTTZZ	Linear	Transformation	ns Take Zero to Ze	ero
Suppose $T: U \to V$ is	s a linear	${\it transformation}.$	Then $T(0) = 0$.	

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Theorem MBLT Matrices Build Linear Transformations 246 Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \to \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors

Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

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Theorem LTLC Linear Transformations and Linear Combinations

Suppose that $T: U \to V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

 $T (a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_t \mathbf{u}_t) = a_1 T (\mathbf{u}_1) + a_2 T (\mathbf{u}_2) + a_3 T (\mathbf{u}_3) + \dots + a_t T (\mathbf{u}_t)$

Theorem LTDB Linear Transformation Defined on a Basis

Suppose U is a vector space with basis $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ and the vector space V contains the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ (which may not be distinct). Then there is a unique linear transformation, $T: U \to V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i, 1 \le i \le n$.

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Definition PI Pre-Image 250 Suppose that $T: U \to V$ is a linear transformation. For each **v**, define the pre-image of **v** to be the subset of U given by

 $T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{v} \}$

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Definition LTA Linear Transformation Addition

Suppose that $T: U \to V$ and $S: U \to V$ are two linear transformations with the same domain and codomain. Then their sum is the function $T + S: U \to V$ whose outputs are defined by

 $(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$

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Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 252 Suppose that $T: U \to V$ and $S: U \to V$ are two linear transformations with the same domain and codomain. Then $T + S: U \to V$ is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication 253 Suppose that $T: U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the scalar multiple is the function $\alpha T: U \to V$ whose outputs are defined by $(\alpha T) (\mathbf{u}) = \alpha T (\mathbf{u})$

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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 254 $C_{1} = C_{1} = C_{2} = C_{1} = C_{2} = C_{$

Suppose that $T: U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T): U \to V$ is a linear transformation.

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Theorem VSLT Vector Space of Linear Transformations

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{L}T(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

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Definition LTC Linear Transformation Composition 256 Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations. Then the composition of S and T is the function $(S \circ T): U \to W$ whose outputs are defined by

 $(S \circ T) (\mathbf{u}) = S (T (\mathbf{u}))$

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Theorem CLTLTComposition of Linear Transformations is a Linear Transformation257Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations. Then $(S \circ T): U \to W$ is a

linear transformation.

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Definition ILT Injective Linear Transformation 258 Suppose $T: U \to V$ is a linear transformation. Then T is injective if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

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Definition KLT Kernel of a Linear Transformation Suppose $T: U \to V$ is a linear transformation. Then the kernel of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0} \}$$

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Theorem KLTS Kernel of a Linear Transformation is a Subspace 260 Suppose that $T: U \to V$ is a linear transformation. Then the kernel of $T, \mathcal{K}(T)$, is a subspace of U.

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Theorem KPI Kernel and Pre-Image

Suppose $T: U \to V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$

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Theorem KILT Kernel of an Injective Linear Transformation 262 Suppose that $T: U \to V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}.$

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Theorem ILTLI Injective Linear Transformations and Linear Independence 263 Suppose that $T: U \to V$ is an injective linear transformation and

$$S = {\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_t}$$

is a linearly independent subset of U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

is a linearly independent subset of V.

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Theorem ILTB Injective Linear Transformations and Bases Suppose that $T: U \to V$ is a linear transformation and

 $B = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_m\}$

is a basis of U. Then T is injective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}\$$

is a linearly independent subset of V.

Theorem ILTDInjective Linear Transformations and Dimension265Suppose that $T: U \to V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

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Theorem CILTI Composition of Injective Linear Transformations is Injective 266 Suppose that $T: U \to V$ and $S: V \to W$ are injective linear transformations. Then $(S \circ T): U \to W$ is an injective linear transformation.

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Definition SLT Surjective Linear Transformation 267 Suppose $T: U \to V$ is a linear transformation. Then T is surjective if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

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Definition RLT Range of a Linear Transformation Suppose $T: U \to V$ is a linear transformation. Then the range of T is the set

 $\mathcal{R}(T) = \{ T(\mathbf{u}) | \mathbf{u} \in U \}$

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Theorem RLTS Range of a Linear Transformation is a Subspace 269 Suppose that $T: U \to V$ is a linear transformation. Then the range of $T, \mathcal{R}(T)$, is a subspace of V.

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Theorem RSLT Range of a Surjective Linear Transformation 270 Suppose that $T: U \to V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

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Theorem SSRLT Spanning Set for Range of a Linear Transformation Suppose that $T: U \to V$ is a linear transformation and

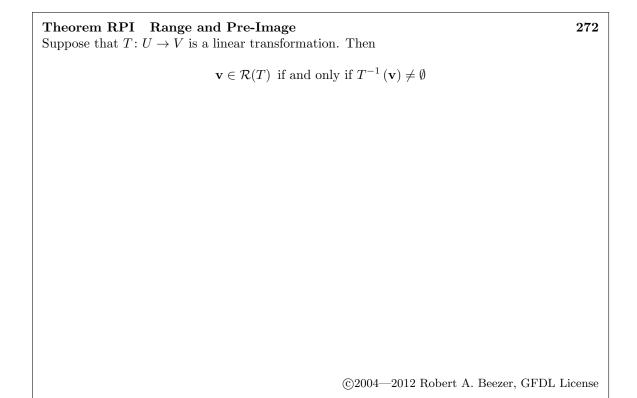
$$S = {\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_t}$$

spans U. Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}\$$

spans $\mathcal{R}(T)$.

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Theorem SLTB Surjective Linear Transformations and Bases Suppose that $T: U \to V$ is a linear transformation and

$$B = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_m\}$$

is a basis of U. Then T is surjective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}\$$

is a spanning set for V.

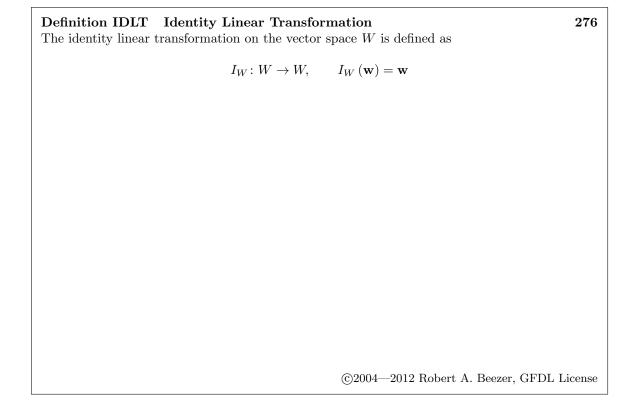
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Theorem SLTD Surjective Linear Trans Suppose that $T: U \to V$ is a surjective linear tr		274
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Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 275

Suppose that $T: U \to V$ and $S: V \to W$ are surjective linear transformations. Then $(S \circ T): U \to W$ is a surjective linear transformation.



Definition IVLTInvertible Linear Transformations277Suppose that $T: U \to V$ is a linear transformation. If there is a function $S: V \to U$ such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is invertible. In this case, we call S the inverse of T and write $S = T^{-1}$.

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Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation278 Suppose that $T: U \to V$ is an invertible linear transformation. Then the function $T^{-1}: V \to U$ is a linear transformation.

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Theorem IILT Inverse of an Invertible Linear Transformation 279 Suppose that $T: U \to V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$.

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Theorem ILTIS Invertible Linear Transformations are Injective and Surjective280 Suppose $T: U \to V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Theorem CIVLT Composition of Invertible Linear Transformations 281 Suppose that $T: U \to V$ and $S: V \to W$ are invertible linear transformations. Then the composition, $(S \circ T): U \to W$ is an invertible linear transformation.

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Theorem ICLT Inverse of a Composition of Linear Transformations 282 Suppose that $T: U \to V$ and $S: V \to W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

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Definition IVS Isomorphic Vector Spaces

Two vector spaces U and V are isomorphic if there exists an invertible linear transformation T with domain U and codomain $V, T: U \to V$. In this case, we write $U \cong V$, and the linear transformation T is known as an isomorphism between U and V.

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Theorem IVSED Isomorphic Vector Spaces have Equal Dimension Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.	284

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Definition ROLT Rank Of a Linear Transformation

Suppose that $T: U \to V$ is a linear transformation. Then the rank of T, r(T), is the dimension of the range of T,

 $r(T) = \dim\left(\mathcal{R}(T)\right)$

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Definition NOLTNullity Of a Linear Transformation286Suppose that $T: U \to V$ is a linear transformation. Then the nullity of T, n(T), is the dimension of the kernel of T,

 $n(T) = \dim\left(\mathcal{K}(T)\right)$

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Theorem ROSLT Rank Of a Surjective Linear Transformation 287 Suppose that $T: U \to V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

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Theorem NOILT Nullity Of an Injective Linear Transformation 288 Suppose that $T: U \to V$ is a linear transformation. Then the nullity of T is zero, n(T) = 0, if and only if T is injective.

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Theorem RPNDD Rank Plus Nullity is Domain Dimension Suppose that $T: U \to V$ is a linear transformation. Then

 $r(T) + n(T) = \dim(U)$

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Definition VR Vector Representation Suppose that *V* is a vector space with a basis $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n}$. Define a function $\rho_B: V \to \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by $\mathbf{w} = [\rho_B(\mathbf{w})]_1 \mathbf{v}_1 + [\rho_B(\mathbf{w})]_2 \mathbf{v}_2 + [\rho_B(\mathbf{w})]_3 \mathbf{v}_3 + \dots + [\rho_B(\mathbf{w})]_n \mathbf{v}_n$

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Theorem VRLT	Vector Representation is a Linear Transformation
The function ρ_B (2)	Definition VR) is a linear transformation.

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Theorem VRI Vector Representation is Injective	$\boldsymbol{292}$
The function ρ_B (Definition VR) is an injective linear transformation.	

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Theorem VRS	Vector Representation is Surjective
The function ρ_B	(Definition VR) is a surjective linear transformation.

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Theorem VRILT Vector Representation is an Invertible Linear Transformation 294 The function ρ_B (Definition VR) is an invertible linear transformation.

Theorem CFDVSCharacterization of Finite Dimensional Vector Spaces295Suppose that V is a vector space with dimension n. Then V is isomorphic to \mathbb{C}^n .

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Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces 296 Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if dim $(U) = \dim(V)$.

Theorem CLI Coordinatization and Linear Independence Suppose that U is a vector space with a basis B of size n. Then

$$S = {\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_k}$$

is a linearly independent subset of U if and only if

$$R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}\$$

is a linearly independent subset of \mathbb{C}^n .

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Theorem CSS Coordinatization and Spanning Sets Suppose that U is a vector space with a basis B of size n. Then

$$\mathbf{u} \in \langle \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \ldots, \, \mathbf{u}_k \} \rangle$$

if and only if

$$\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$$

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Definition MR Matrix Representation Suppose that $T: U \to V$ is a linear transformation $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_3$

Suppose that $T: U \to V$ is a linear transformation, $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$ is a basis for U of size n, and C is a basis for V of size m. Then the matrix representation of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$$

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Theorem FTMR Fundamental Theorem of Matrix Representation 300 Suppose that $T: U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_{C}\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 301 Suppose that $T: U \to V$ and $S: U \to V$ are linear transformations, B is a basis of U and C is a basis of V. Then

 $M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 302

Suppose that $T: U \to V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 303

Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

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Theorem KNSI Kernel and Null Space Isomorphism

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

 $\mathcal{K}(T) \cong \mathcal{N}\left(M_{B,C}^T\right)$

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Theorem RCSI Range and Column Space Isomorphism

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}\big(M_{B,C}^T\big)$$

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Theorem IMR Invertible Matrix Representations

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix. When T is invertible,

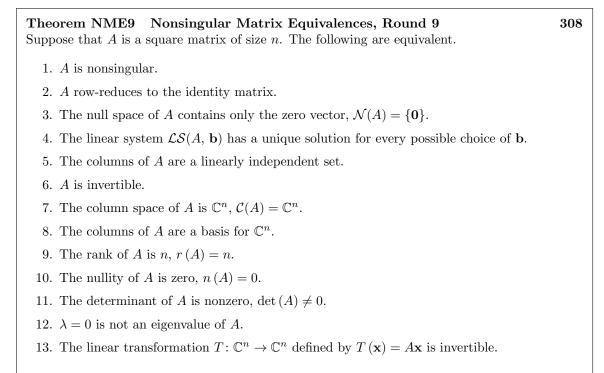
$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^{T}\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation 307 Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation 309 Suppose that $T: V \to V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

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Definition CBM Change-of-Basis Matrix 310 Suppose that V is a vector space, and $I_V: V \to V$ is the identity linear transformation on V. Let $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ and C be two bases of V. Then the change-of-basis matrix from B to C is the matrix representation of I_V relative to B and C,

> $C_{B,C} = M_{B,C}^{I_V}$ = $[\rho_C (I_V (\mathbf{v}_1)) | \rho_C (I_V (\mathbf{v}_2)) | \rho_C (I_V (\mathbf{v}_3)) | \dots | \rho_C (I_V (\mathbf{v}_n))]$ = $[\rho_C (\mathbf{v}_1) | \rho_C (\mathbf{v}_2) | \rho_C (\mathbf{v}_3) | \dots | \rho_C (\mathbf{v}_n)]$

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Theorem CB Change-of-Basis Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

 $\rho_{C}\left(\mathbf{v}\right) = C_{B,C}\rho_{B}\left(\mathbf{v}\right)$

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Theorem ICBMInverse of Change-of-Basis Matrix312Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

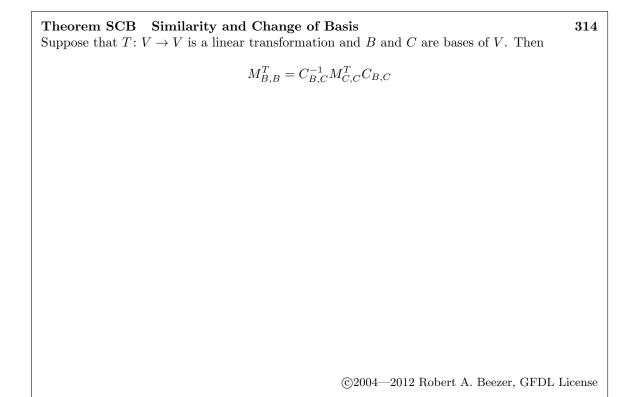
$$C_{B,C}^{-1} = C_{C,B}$$

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Theorem MRCB Matrix Representation and Change of Basis 313 Suppose that $T: U \to V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$



Theorem EER Eigenvalues, Eigenvectors, Representations 315 Suppose that $T: V \to V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

	Upper Triangular Matrix		316
The $n \times n$ square m	natrix A is upper triangular if	$[A]_{ij} = 0$ whenever $i > j$.	
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Definition LTM	Lower Triangular Matrix
The $n \times n$ square	matrix A is lower triangular if $[A]_{ij} = 0$ whenever $i < j$.

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Theorem PTMT Product of Triangular Matrices is Triangular $\mathbf{318}$ Suppose that A and B are square matrices of size n that are triangular of the same type. Then AB is also triangular of that type.

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Theorem ITMT Inverse of a Triangular Matrix is Triangular

Suppose that A is a nonsingular matrix of size n that is triangular. Then the inverse of A, A^{-1} , is triangular of the same type. Furthermore, the diagonal entries of A^{-1} are the reciprocals of the corresponding diagonal entries of A. More precisely, $[A^{-1}]_{ii} = [A]_{ii}^{-1}$.

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Theorem UTMR Upper Triangular Matrix Representation

320 Suppose that $T \colon V \to V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T relative to B, $M_{B,B}^T$, is an upper triangular matrix. Each diagonal entry is an eigenvalue of T, and if λ is an eigenvalue of T, then λ occurs $\alpha_T(\lambda)$ times on the diagonal.

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Theorem OBUTR Orthonormal Basis for Upper Triangular Representation 321 Suppose that A is a square matrix. Then there is a unitary matrix U, and an upper triangular matrix T, such that

 $U^*AU = T$

and T has the eigenvalues of A as the entries of the diagonal.

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Definition NRML Normal Matrix The square matrix A is normal if $A^*A = AA^*$.

Theorem OD Orthonormal Diagonalization

Suppose that A is a square matrix. Then there is a unitary matrix U and a diagonal matrix D, with diagonal entries equal to the eigenvalues of A, such that $U^*AU = D$ if and only if A is a normal matrix.

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Theorem OBNMOrthonormal Bases and Normal Matrices324Suppose that A is a normal matrix of size n. Then there is an orthonormal basis of \mathbb{C}^n composed
of eigenvectors of A.

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Definition CNE Complex Number Equality 325 The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are equal, denoted $\alpha = \beta$, if a = c and b = d.

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Definition CNA Complex Number Addition 326 The sum of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is (a + c) + (b + d)i. **Definition CNM** Complex Number Multiplication 327 The product of the complex numbers $\alpha = a+bi$ and $\beta = c+di$, denoted $\alpha\beta$, is (ac-bd)+(ad+bc)i.

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Theorem PCNA Properties of Complex Number Arithmetic

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The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers: For any $\alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers: For any $\alpha, \beta \in \mathbb{C}, \alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers: For any α , β , $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers: There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers: There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

Definition CCN	Conjugate of a Complex Number	329
The conjugate of the	e complex number $\alpha = a + bi \in \mathbb{C}$ is the complex number $\overline{\alpha} = a - bi$.	

Theorem CCRA Complex Conjugation Respects Addition Suppose that α and β are complex numbers. Then $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.	330
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Theorem CCRM	Complex Conjugation Respects Multiplication
Suppose that α and	β are complex numbers. Then $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.

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Theorem CCT Complex Conjugation Twice Suppose that α is a complex number. Then $\overline{\overline{\alpha}} = \alpha$.	332
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 Definition MCN
 Modulus of a Complex Number
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 The modulus of the complex number $\alpha = a + bi \in \mathbb{C}$, is the nonnegative real number
 $|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2 + b^2}$.

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Definition SET Set

A set is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.

Definition S	SET	\mathbf{Subset}	:	335
If S and T are	e two se	ts, then	S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in S$	T.

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Definition ES Empty Set	
The empty set is the set with no element	s. It is denoted by \emptyset .

Definition SE Set Equality Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write S = T.

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Definition C Cardinality

Suppose S is a finite set. Then the number of elements in S is called the cardinality or size of S, and is denoted |S|.

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Definition SU Set Union

Suppose S and T are sets. Then the union of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$

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Definition SI Set Intersection

Suppose S and T are sets. Then the intersection of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

 $x \in S \cap T$ if and only if $x \in S$ and $x \in T$

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Definition SC Set Complement

Suppose S is a set that is a subset of a universal set U. Then the complement of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x\in\overline{S}$ if and only if $x\in U$ and $x\not\in S$

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